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Linear and nonlinear stability analyses of double-diffusive convection in a porous medium layer saturated in a Maxwell nanofluid with variable viscosity and conductivity

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ABSTRACT

The double diffusive convection in a horizontal layer of a porous medium saturated with a Maxwell nanofluid is studied using linear and non-linear stability analyses. The modified Darcy-Maxwell model is used for the momentum equation. The model used for the Maxwell nanofluid incorporates the effects of Brownian motion and thermophoresis. The thermal energy equations include the diffusion and cross diffusion terms. The linear theory depends on normal mode technique and the onset criterion for stationary and oscillatory convection is derived analytically. The non-linear theory based on the representation of Fourier series method is used to find the transient behavior of heat and mass transports. We observe that the transient Nusselt numbers show oscillatory motion when time is small. However, when time becomes very large all the three transient Nusselt values approaches to their steady-state values.

Introduction

Choi [1] introduced the term nanofluid. The term nanofluid refers to the suspensions of nanoscale particles in the base fluid. Due to their potential for high rate of heat exchange incurring either little or no penalty in the pressure drop, they have attracted enormous interests from researchers. The convective heat transfer characteristics of nanofluids depends on the thermo-physical properties of the base fluid and the ultra fine particles, the flow pattern and flow structure, the volume fraction of the suspended particles, the dimensions and the shape of these particles. Kumar et al. [2] established the utility of a particular nanofluid for a heat transfer application by suitably modeling the convective transport in the nanofluid. The particles are different from conventional particles (millimeter or micro-scale) in that they stay in suspension in the fluid and no sedimentation occurs.

On the “magic” power of nanoparticles, many exciting new phenomena and results have been reported since the dawn of nanotechnology in the early 1990s of last century. Similar excitement can be seen in the area of nanofluids. The abnormal thermal conduction of a fluid comprising nanoparticles, effective diffusion of nanoparticle have been reported through the cells of live bodies, super magnetism and quantum dots effect of nanoparticles.

During the last few decades, because of the wide range of applications of the double diffusive convection in porous media has attracted considerable interest from the solidification of binary mixtures to the migration of solutes in water-saturated soils. Geophysical systems, electro-chemistry, and the migration of moisture through air contained in fibrous insulation are some other important areas of applications.

Instabilities can occur only if one of the components are destabilizing in a system where two diffusing properties are present. The relation between the fluxes and the driving potentials are of more intricate in nature, when the heat and mass transfers occur simultaneously in a moving fluid. It can be found that an energy flux is generated not only by temperature gradient but also by composition gradients. Dufour or diffusion-thermo effect may be referred to as the energy flux caused by a composition gradient, while the mass fluxes created by temperature gradients are termed as Soret or thermal-diffusion effect. However, the situation will be quite different if the cross-diffusion terms are included in the species transport equations. Each property gradient has a significant influence on the flux of the other property, owing to the cross-diffusion effects.

On the phenomena of double diffusive convection in porous media, the earlier studies were all mainly concerned with the problem of convective instability in a horizontal layer heated and salted from below. On the basis of linear stability theory for various thermal and solutal boundary conditions, Nield [3] was the first person who undertook the study of double diffusive convection in porous media. Nield’s analysis was by extended by Taunton et al. [4] and they considered the salt-fingering convection case in a porous layer. The linear and nonlinear stability analysis was applied by Rudraiah et al. [5] and they found that the subcritical instabilities would be possible in the case of a two-component fluid in a porous medium. Poulikakos [6] carried out the linear stability analysis of the thermosolutal convection using the Darcy-Brinkman model. The onset of double diffusive convection in a binary viscoelastic fluid layer was investigated by Malashetty and Swamy [7]. The onset of double diffusive convection in a binary Maxwell fluid saturated porous layer with cross-diffusion effects was studied by Malashetty and Biradar [8]. The double diffusive convection in a rotating anisotropic porous layer saturated with a viscoelastic fluid was studied by Malashetty et al. [9].

For nanofluid, the double-diffusive convection in a horizontal porous medium was studied by Kuznetsov and Nield [10], Yadav et al. [11] and Agarwal et al. [12]. In all those studies, the thermo physical properties of nanofluids such as viscosity, thermal

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conductivity and specific heat were taken as a constant, but the experimental results have shown that these quantities are not constant and strongly depend on the volume fraction of nanoparticles [13]. The effect of the variation of these quantities with volumetric fraction of nanoparticles on the onset of convection has been studied by only few researchers. Nield and Kuznetsov [14] and Yadav et al. [15] studied the onset of convection in a layer of a porous medium saturated by a nanofluid under the effect of conductivity and viscosity variations. The consequence of these factors is to increase the critical value of the Rayleigh number. The extension to the double diffusive case was made by Yadav et al [16].

Quite recently non-Newtonian fluids housed in fluid-based systems, with and without porous matrix, have been extensively used in application situations and hence warranted the attention they have been duly attracted. It is well known now that viscoelastic behavior is an important rheological process in the asthenosphere and the deeper mantle. Flow through composites, timber wood, snow system and rheology of food transport are some other application areas of viscoelastic fluid saturated porous media.

Although the problem of double diffusive convection has been extensively investigated for Newtonian nanofluids, but with growing importance of non-Newtonian fluids with suspended particles in modern technology and industries, the investigations of such fluids are desirable. The study of such fluids has applications in a number of processes that occur in industry, such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, exotic lubrication, and colloidal and suspension solutions. Also, the flow of non-Newtonian fluids in a porous medium is of considerable importance in various areas of science, engineering, and technology, for example, in the material processing, petroleum, chemical engineering, nuclear industries, geophysics, polymer engineering, oil reservoirs engineering, and bio-rheology [17]. The above literature review reveals that double-diffusive convection in a porous medium layer saturated in a Maxwell nanofluid with variable viscosity and conductivity is unsolved. Therefore the main objective of the present work is to study the effect of thermal conductivity and viscosity on the onset of double-diffusive convection in a horizontal porous medium saturated with a Maxwell nanofluid.

Analysis

Conservation equations

We select a coordinate frame in which the *z*-axis is aligned vertically upwards. We consider a horizontal layer of fluid confined between the planes z*= 0 and z* = *H*. Asterisks are used to denote dimensional variables. Each boundary wall is assumed to be perfectly thermally conducting. The temperatures and concentrations at the lower and upper boundaries are taken to be $T_0^* + \Delta T^*$, $C_0^* + \Delta C^*$, T_0^*, and C_0^*, respectively. The Oberbeck Boussinesq approximation is employed. In the linear stability theory being applied here, the temperature change in the fluid is assumed to be small in comparison with T_0^*. The mass conservation equation takes the form

$$\nabla^* \cdot \mathbf{v}_D^* = 0 \tag{1}$$

Here $\mathbf{v}_D^* = \left(u^*, v^*, w^*\right)$ is the nanofluid Darcy velocity.

In the presence of thermophoresis, the conservation equation for the nanoparticles, in the absence of chemical reactions, takes the form

$$\frac{\partial \phi^*}{\partial t^*} + \frac{1}{\varepsilon}\mathbf{v}_D^* \cdot \nabla \phi^* = \nabla^* \cdot \left[D_B \nabla^* \phi^* + D_T \frac{\nabla^* T^*}{T^*} \right], \tag{2}$$

where ϕ^* is the nanoparticle volume fraction, ε is the porosity, T^* is the temperature, D_B is the Brownian diffusion coefficient, and D_T is the thermophoretic diffusion coefficient.

If one introduces a buoyancy force, adopts the Boussinesq approximation, and uses the Darcy model for a porous medium, then the momentum equation can be written as

$$\left(1 + \lambda_1 \frac{\partial}{\partial t^*}\right)\left(\frac{\rho}{\varepsilon}\frac{\partial \mathbf{v}_D^*}{\partial t^*}\right) = \left(1 + \lambda_1 \frac{\partial}{\partial t^*}\right)\left(-\nabla^* p^* + \rho \mathrm{g}\right) - \frac{\mu_{eff}}{K}\mathbf{v}_{\mathrm{D}}^* . \tag{3}$$

Here ρ is the overall density of the nanofluid, which is given by

$$\rho = \phi^* \rho_{\mathrm{p}} + \left(1 - \phi^*\right)\rho_0 \left[1 - \beta_{\mathrm{T}}\left(T^* - T_0^*\right) - \beta_C\left(C^* - C_0^*\right)\right], \tag{4}$$

where C is the concentration, ρ_{p} is the particle density, ρ_0 is a reference density for the fluid, β_{T} is the thermal volumetric expansion coefficient and β_C is the analogous solutal coefficient. The thermal energy equation for a nanofluid can be written as

$$(\rho c)_m \frac{\partial T^*}{\partial t^*} + (\rho c)_f \mathbf{v}_{\mathrm{D}}^* \cdot \nabla^* T^* = k_m \nabla^{*2} T^* + \varepsilon (\rho c)_{\mathrm{p}} \left[D_{\mathrm{B}} \nabla^* \phi^* \cdot \nabla^* T^* + D_{\mathrm{T}} \frac{\nabla^* T^* \cdot \nabla^* T^*}{T_0^*} \right], \tag{5}$$
$$+ \rho c \, D_{TC} \nabla^{*2} C^*$$

where *c* is the fluid specific heat (at constant pressure), k_m is the overall thermal conductivity of the porous medium saturated by the nanofluid, c_p is the nanoparticle specific heat of the material constituting the nanoparticles, D_{TC} is a diffusivity of Dufour type.

To this we add a conservation equation for the solute of the form

$$\frac{\partial C^*}{\partial t^*}+\frac{1}{\varepsilon}\mathbf{v}_D^*\cdot\nabla^* C^*=D_{Sm}\nabla^{*2}C^*+D_{CT}\nabla^{*2}T^*, \qquad (6)$$

where D_{Sm} is the solutal diffusivity for the porous medium and D_{CT} is a diffusivity of Soret type. It has been assumed that the nanoparticles do not affect the transport of the solute.

Thus,

$$k_m=\varepsilon k_{eff}+(1-\varepsilon)k_s, \qquad (7)$$

where ε is the porosity, k_{eff} is the effective conductivity of the nanofluid (fluid plus nanoparticles) and k_s is the conductivity of the solid material forming the matrix of the porous medium.

Following Tiwari and Das [18], we adopt the formulas for the viscosity and the conductivity with dependence on nanoparticle fraction, based on the theory of mixtures, as:

$$\frac{\mu_{eff}}{\mu_f}=\frac{1}{(1-\phi^*)^{2.5}}, \qquad (8)$$

$$\frac{k_{eff}}{k_f}=\frac{(k_p+2k_f)-2\phi^*(k_f-k_p)}{(k_p+2k_f)+\phi^*(k_f-k_p)}. \qquad (9)$$

Here k_f and k_p are the thermal conductivities of the fluid and the nanoparticles, respectively.

We note that Eq. (8) has been obtained by Brinkman [19] and Eq. (9) is the Maxwell-Garnett formula for a suspension of spherical particles that dates back to Maxwell [20]. In the case that ϕ^* is small compared with unity, we can approximate these formulas by:

$$\frac{\mu_{eff}}{\mu_f}=1+2.5\phi^* \quad \text{and} \quad \frac{k_{eff}}{k_f}=\frac{(k_p+2k_f)-2\phi^*(k_f-k_p)}{(k_p+2k_f)+\phi^*(k_f-k_p)}=1+3\phi^*\frac{(k_p-k_f)}{(k_p+2k_f)}. \qquad (10)$$

We assume that the temperature and the volumetric fraction of the nanoparticles are constant on the boundaries. Thus the boundaries conditions are

$$w^*=0,\ T^*=T_0^*+\Delta T^*,\ C^*=C_0^*+\Delta C^*,\ \phi^*=\phi_0^* \text{ at } z^*=0, \qquad (11a)$$

$$w^*=0,\ T^*=T_0^*,\ C^*=C_0^*,\ \phi^*=\phi_1^* \text{ at } z^*=H. \qquad (11b)$$

The dimensionless variables are introduced as follows:

$$(x,y,z)=(x^*,y^*,z^*)/H,\ t=t^*\alpha_m/\sigma H^2,\ (u,v,w)=(u^*,v^*,w^*)H/\alpha_m,\quad p=p^*K/\mu_f\alpha_m,$$

$$\phi=\frac{\phi^*-\phi_0^*}{\phi_1^*-\phi_0^*},\ T=\frac{T^*-T_0^*}{\Delta T^*},\ C=\frac{C^*-C_0^*}{\Delta C^*}, \qquad (12)$$

where $\alpha_m=\frac{k_m}{(\rho c_p)_f},\ \sigma=\frac{(\rho c_p)_m}{(\rho c_p)_f}$. We also define

$$\tilde{\mu}=\frac{\mu_{eff}}{\mu_f},\ k=\frac{k_p}{k_f},\ \tilde{k}_s=\frac{k_s}{k_f},\ \tilde{k}=\frac{k_m}{k_f}. \qquad (13)$$

From Eqs. (7), (10) and (13), we have:

$$\tilde{\mu}=1+2.5[\phi_0^*+\phi(\phi_1^*-\phi_0^*)],\quad \tilde{k}=\varepsilon\left\{1+3[\phi_0^*+\phi(\phi_1^*-\phi_0^*)]\frac{\tilde{k}_p-1}{\tilde{k}_p+2}\right\}+(1-\varepsilon)\tilde{k}_s. \qquad (14)$$

Then Eqs. (1) and (3) together with Eqs. (2), (4), (5) and (11) take the form:

$$\nabla\cdot\mathbf{v}=0 \qquad (15)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left(\gamma_a\frac{\partial\mathbf{v}}{\partial t}+\nabla p+Rme_z-Ra_TTe_z-\left(Rs/Le\right)Ce_z+Rn\phi e_z\right)+\tilde{\mu}\,\mathbf{v}=0 \qquad (16)$$

$$\frac{\partial T}{\partial t}+\mathbf{v}\cdot\nabla T=\tilde{k}\nabla^2T+\frac{N_B}{Ln}\nabla\phi\cdot\nabla T+\frac{N_AN_B}{Ln}\nabla T\cdot\nabla T+N_{TC}\nabla^2C \qquad (17)$$

$$\frac{1}{\sigma}\frac{\partial C}{\partial t}+\frac{1}{\varepsilon}\mathbf{v}\cdot\nabla C=\frac{1}{Le}\nabla^2C+N_{CT}\nabla^2T \qquad (18)$$

$$\frac{1}{\sigma}\frac{\partial\phi}{\partial t}+\frac{1}{\varepsilon}\mathbf{v}\cdot\nabla\phi=\frac{1}{Ln}\nabla^2\phi+\frac{N_A}{Ln}\nabla^2T \qquad (19)$$

$$w=0,\ T=1,\ C=1,\ \phi=0\ \ at\ \ z=0,\ w=0,\ T=0,\ C=0,\ \phi=1 \text{ at } z=1 \tag{20}$$

Here $\lambda=\dfrac{\lambda_1\alpha_m}{\sigma H^2}$, $\gamma_a=\dfrac{\varepsilon}{\sigma Va}$, $Ln=\dfrac{\alpha_m}{D_B}$, $Va=\dfrac{\varepsilon^2 \Pr}{Da}$, $Ra_T=\dfrac{\rho g\beta KH\Delta T^*}{\mu_f\alpha_m}$, $\Pr=\dfrac{\mu_f}{\rho\,\alpha_m}$, $Da=\dfrac{K}{H^2}$,

$Rs=\dfrac{\rho g\beta_C KH\Delta C^*}{\mu_f D_{Sm}}$, $Rm=\dfrac{\left[\rho_p\phi_0^*+\rho(1-\phi_0^*)\right]gKH}{\mu_f\alpha_m}$, $Rn=\dfrac{(\rho_p-\rho)(\phi_1^*-\phi_0^*)gKH}{\mu_f\alpha_m}$, $Le=\dfrac{\alpha_f}{D_S}$,

$N_A=\dfrac{D_T\Delta T^*}{D_B T_c^*(\phi_1^*-\phi_0^*)}$, $N_B=\dfrac{\varepsilon(\rho c)_p(\phi_1^*-\phi_0^*)}{(\rho c)_f}$, $N_{TC}=\dfrac{D_{TC}\Delta C^*}{\alpha_m\Delta T^*}$, $N_{CT}=\dfrac{D_{CT}\Delta T^*}{\alpha_m\Delta C^*}$.

The parameter λ is the relaxation parameter (which is also known as the Deborah number), γ_a is the non-dimensional acceleration coefficient, Ln is a thermo-nanofluid Lewis number, Va is the Vadász number, Ra_T is the familiar thermal Rayleigh-Darcy number, *Pr* is the Prandtl number, *Da* is the Darcy number, *Rs* is the thermo-solutal Rayleigh number. The new parameters *Rm* and *Rn* may be regarded as a basic-density Rayleigh number and a concentration Rayleigh number, respectively. The parameter N_A is a modified diffusivity ratio, N_B is a modified particle-density increment and *Le* is the familiar thermo-solutal Lewis number. The Deborah number is a dimensionless number used in rheology to characterize how fluid and material will be. The smaller the Deborah number, the more fluid the material appears. The parameter λ that relates to the relaxation time to the thermal diffusion time is of order one for most viscoelastic fluids. The value for Deborah number for dilute polymeric solution falls most likely in the range [0.1, 2]. At present, there are no experimental data available for comparison and, therefore, we have considered a wide range of values for the parameters. The Prandtl number affects the stability of the porous system through the combined dimensionless group known as Vadász number. The Vadász number is also known as Darcy-Prandtl number in the literature.

In the spirit of the Oberbeck–Boussinesq approximation, Eq. (16) has been linearized by the neglect of a term proportional to the product of ϕ and *T*. This assumption is likely to be valid in the case of small temperature gradients in a dilute suspension of nanoparticles.

Basic solution

We seek a time-independent quiescent solution of Eqs. (15)-(20) with temperature, concentration and nanoparticle volume fraction varying in the z-direction only, that is a solution of the form:

$$\mathbf{v}=0,\ p=p_b(z),\ T=T_b(z),\ C=C_b(z),\ \phi=\phi_b(z) \tag{21}$$

Eqs. (16)-(19) reduce to:

$$0=-\frac{dp_b}{dz}-Rm+Ra_T\,T_b+\left(Rs/Le\right)C_b-Rn\,\phi_b, \tag{22}$$

$$\tilde{k}\frac{d^2T_b}{dz^2}+\frac{N_B}{Ln}\frac{d\phi_b}{dz}\frac{dT_b}{dz}+\frac{N_AN_B}{Ln}\left(\frac{dT_b}{dz}\right)^2+N_{TC}\frac{d^2C_b}{dz^2}=0, \tag{23}$$

$$\frac{1}{Le}\frac{d^2C_b}{dz^2}+N_{CT}\frac{d^2T_b}{dz^2}=0, \tag{24}$$

$$\frac{d^2\phi_b}{dz^2}+N_A\frac{d^2T_b}{dz^2}=0. \tag{25}$$

According to Buongiorno [21], for most nanofluids investigated so far $Ln/\left(\phi_1^*-\phi_0^*\right)$ is large, of order 10^5-10^6, and since the nanoparticle fraction decrement is typically no smaller than 10^3, this means that *Ln* is large, of order 10^2-10^3, while N_A is no greater than about 10. Using this approximation, the basic solution is found to be:

$$T_b=1-z,\ C_b=1-z \text{ and so } \phi_b=z \tag{26}$$

Perturbation solution

We now superimpose perturbations on the basic solution as:

$$\mathbf{v}=\mathbf{v}',\ p=p_b+p',\ T=T_b+T',\ C=C_b+C',\ \phi=\phi_b+\phi'. \tag{27}$$

On substituting Eq. (27) into Eqs. (13)-(19), and linearized by neglecting products of primed quantities. The following equations are obtained when Eq. (26) is used.

$$\nabla\cdot\mathbf{v}'=0 \tag{28}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left(\nabla p'+\gamma_a\frac{\partial\mathbf{v}'}{\partial t}-Ra_T\,T'e_z-\left(Rs/Le\right)C'e_z+Rn\,\phi'e_z\right)+\tilde{\mu}\,\mathbf{v}'=0 \tag{29}$$

$$\frac{\partial T'}{\partial t}-w'=\tilde{k}\nabla^2 T'+\frac{N_B}{Ln}\left(\frac{\partial T'}{\partial z}-\frac{\partial \phi'}{\partial z}\right)-\frac{2N_A N_B}{Ln}\frac{\partial T'}{\partial z}+N_{TC}\nabla^2 C' \tag{30}$$

$$\frac{1}{\sigma}\frac{\partial C'}{\partial t}-\frac{1}{\varepsilon}w'=\frac{1}{Le}\nabla^2 C'+N_{CT}\nabla^2 T' \tag{31}$$

$$\frac{1}{\sigma}\frac{\partial \phi'}{\partial t}+\frac{1}{\varepsilon}w'=\frac{1}{Ln}\nabla^2\phi'+\frac{N_A}{Ln}\nabla^2 T' \tag{32}$$

$$w'=0,\ T'=0,\ C'=0,\ \phi'=0 \text{ at } z=0 \text{ and } z=1. \tag{33}$$

Now we can approximate the viscosity and conductivity distributions by substituting the basic solution expression for ϕ, namely that given by Eq. (26) into Eq. (14), and obtain

$$\tilde{\mu}(z)=1+2.5\left[\phi_0^*+(\phi_1^*-\phi_0^*)\,z\right],\quad \tilde{k}(z)=\varepsilon\left\{1+3\left[\phi_0^*+\left(\phi_1^*-\phi_0^*\right)z\right]\frac{\tilde{k}_p-1}{\tilde{k}_p+2}\right\}+(1-\varepsilon)\,\tilde{k}_s. \tag{34}$$

It will be noted that the parameter Rm is just a measure of the basic static pressure gradient and is not involved in the present and subsequent equations.

We now recognize that we have a situation where properties are heterogeneous. These are now the viscosity and conductivity (rather that the more usual ones, namely permeability and conductivity) and we can now proceed as in a number of papers surveyed by Nield [22]. We assume that the heterogeneity is weak in the sense that the maximum variation of a property over the domain considered is small compared with the mean value of that property.

The seven unknowns $\mathrm{u}', \mathrm{v}', \mathrm{w}', \mathrm{p}', \mathrm{T}', C'$ and ϕ' can be reduced to four by operating dot product of curl curl of Eq. (29) with e_z and using the identity $\nabla\times\nabla(\ )=\nabla\nabla\cdot(\ )-\nabla^2(\ )$ together with Eq. (28) and the weak heterogeneity approximation. The result is

$$\left\{\left(1+\lambda s\right)s\gamma_a+\tilde{\mu}(z)\right\}\nabla^2 w'=\left(1+\lambda s\right)\left\{Ra_T\nabla_H^2+\left(Rs/Le\right)\nabla_H^2 C'-Rn\nabla_H^2\phi'\right\}, \tag{35}$$

where $\nabla_H^2=\nabla^2-\partial^2/\partial z^2$ is the horizontal Laplacian operator and *s* will be given later.

The thermal convection system composed of Eqs. (35), (29)-(32) and boundary conditions (33) constitutes a boundary-value problem that can be solved using the method of normal modes.

We write

$$\left(w',\ T',\ C',\ \phi'\right)=\left[W(z),\Theta(z),\Sigma(z),\Phi(z)\right]\exp\left(st+ilx+imy\right) \tag{36}$$

and substitute above into the thermal convection system to obtain

$$\begin{aligned}&\left(\tilde{\mu}(z)+\gamma_a s\left(1+\lambda s\right)\right)\left(D^2-\alpha^2\right)W+\left(1+\lambda s\right)Ra_T\,\alpha^2\Theta+\left(1+\lambda s\right)\left(Rs/Le\right)\alpha^2\,\Sigma\\&\qquad-\left(1+\lambda s\right)Rn\,\alpha^2\,\Phi=0,\end{aligned} \tag{37}$$

$$W+\left(\tilde{k}(z)\left(D^2-\alpha^2\right)+\frac{N_B}{Ln}D-\frac{2N_AN_B}{Ln}D-s\right)\Theta-\frac{N_B}{Ln}D\Phi+N_{TC}\left(D^2-\alpha^2\right)\Sigma=0, \tag{38}$$

$$\frac{1}{\varepsilon}W+\frac{1}{Le}\left(D^2-\alpha^2-\frac{1}{\sigma}s\right)\Sigma+N_{CT}\left(D^2-\alpha^2\right)\Theta=0, \tag{39}$$

$$\frac{1}{\varepsilon}W-\frac{N_A}{Ln}\left(D^2-\alpha^2\right)\Theta-\left(\frac{1}{Ln}\left(D^2-\alpha^2\right)-\frac{1}{\sigma}s\right)\Phi=0, \tag{40}$$

$$W=0,\ \Theta=0,\ \Sigma=0,\ \Phi=0 \text{ at } z=0 \text{ and } z=1, \tag{41}$$

where *s* is perturbation growth rate which is in general complex,

$D\equiv\dfrac{d}{dz}$ and $\alpha=(l^2+m^2)^{1/2}$ is the single horizontal wave number. (42)

We now employ a Galerkin-type weighted residuals method to obtain an approximate solution to the system of Eqs. (37)-(41). We choose the trial functions (satisfying the boundary conditions) $W_p,\ \Theta_p,\ \Sigma_p,\ \Phi_p;\ p=1,2,3\ldots$ and write

$$W=\sum_{p=1}^{N}A_pW_p,\quad \Theta=\sum_{p=1}^{N}B_p\Theta_P,\quad \Sigma=\sum_{p=1}^{N}C_p\Sigma_p,\quad \Phi=\sum_{p=1}^{N}D_p\Phi_P \tag{43}$$

Substituting Eq. (43) into Eqs. (37)-(41) and making the expressions on the left-hand sides of those equations (the residuals) orthogonal to the trial functions, thereby we obtain a system of 4*N* linear algebraic equations in the 4*N* unknowns $A_p,\ B_p,\ C_p$ and

D_p, p =1, 2, . . . N. The vanishing of the determinant of coefficients produces the eigenvalue equation for the system. One can regard Ra_T as the eigenvalue. This enables us to find Ra_T in terms of the other parameters.

Trial functions satisfying the boundary condition (41) can be chosen as

$$W_p = \Theta_p = \Phi_p = \Sigma_p = \sin p\pi z \,;\; p = 1, 2, 3, \ldots N. \tag{44}$$

The eigenvalue equation is

$$\det M = 0 \tag{45}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \tag{46}$$

and for i, j = 1, 2, …, N

$$(M_{11})_{ij} = -\left\langle\left(\tilde{\mu}(z)+\gamma_a s(1+\lambda s)\right)W_j D^2 W_i\right\rangle + \alpha^2\left\langle\left(\tilde{\mu}(z)+\gamma_a s(1+\lambda s)\right)W_j W_i\right\rangle,$$

$$(M_{12})_{ij} = -(1+\lambda s)\,Ra_T\alpha^2\left\langle W_j\Theta_i\right\rangle,$$

$$(M_{13})_{ij} = -(1+\lambda s)(Rs/Le)\alpha^2\left\langle W_j\Sigma_i\right\rangle,$$

$$(M_{14})_{ij} = (1+\lambda s)Rn\alpha^2\left\langle W_j\Phi_i\right\rangle,$$

$$(M_{21})_{ij} = -\left\langle\Theta_j W_i\right\rangle,$$

$$(M_{22})_{ij} = -\left\langle\tilde{k}(z)\Theta_j D^2\Theta_i\right\rangle + \alpha^2\left\langle\tilde{k}(z)\Theta_j\Theta_i\right\rangle + s\left\langle\Theta_j\Theta_i\right\rangle + \left(\frac{2N_A N_B}{Ln} - \frac{N_B}{Ln}\right)\left\langle\Theta_j D\Theta_i\right\rangle,$$

$$(M_{23})_{ij} = -\left\langle N_{TC}\Theta_j D^2\Sigma_i\right\rangle + \alpha^2\left\langle N_{TC}\Theta_j\Sigma_i\right\rangle,$$

$$(M_{24})_{ij} = \frac{N_B}{Ln}\left\langle\Theta_j D\Phi_i\right\rangle,$$

$$(M_{31})_{ij} = -\frac{1}{\varepsilon}\left\langle\Sigma_j W_i\right\rangle,$$

$$(M_{32})_{ij} = -\left\langle N_{CT}\Theta_i D^2\Sigma_j\right\rangle + \alpha^2\left\langle N_{CT}\Theta_i\Sigma_j\right\rangle,$$

$$(M_{33})_{ij} = \frac{1}{Le}\left(-\left\langle\Sigma_j D^2\Sigma_i\right\rangle + \alpha^2\left\langle\Sigma_j\Sigma_i\right\rangle + \frac{s}{\sigma}\left\langle\Sigma_j\Sigma_i\right\rangle\right),$$

$$(M_{34})_{ij} = 0,$$

$$(M_{41})_{ij} = -\frac{1}{\varepsilon}\left\langle\Phi_j W_i\right\rangle,$$

$$(M_{42})_{ij} = \frac{N_A}{Ln}\left(\left\langle\Phi_j D^2\Theta_i\right\rangle - \alpha^2\left\langle\Phi_j\Theta_i\right\rangle\right),$$

$$(M_{43})_{ij} = 0,$$

$$(M_{44})_{ij} = \frac{1}{Ln}\left(\left\langle\Phi_j D^2\Phi_i\right\rangle - \alpha^2\left\langle\Phi_j\Phi_i\right\rangle\right) - \frac{s}{\sigma}\left\langle\Phi_j\Phi_i\right\rangle,$$

$$(M_{32})_{ij} = -\left\langle N_{CT}\Sigma_j D^2\Theta_i\right\rangle + \alpha^2\left\langle N_{CT}\Theta_i\Sigma_j\right\rangle. \text{ Here } \left\langle f(z)\right\rangle \equiv \int_0^1 f(z)\,dz. \tag{47}$$

In the present case, where viscosity and conductivity variations are incorporated, the critical wave number is unchanged and the stability boundary becomes

$$Ra_T = \frac{1}{(1+\lambda s)\alpha^2}\left(\frac{JN_{TC}}{\varepsilon}\left(\frac{J}{Ln}+\frac{s}{\sigma}\right)-\frac{1}{Le}\left(J+\frac{s}{\sigma}\right)\left(\frac{J}{Ln}+\frac{s}{\sigma}\right)\right)^{-1} \tag{48}$$

$$\left[\left((-J)\left(\frac{J\eta+s}{Le}\right)\left(\frac{J}{Ln}+\frac{s}{\sigma}\right)\left(J+\frac{s}{\sigma}\right)+J^3 N_{TC}N_{CT}\left(\frac{J}{Ln}+\frac{s}{\sigma}\right)\right)\left(\nu+s\gamma_a(1+\lambda s)\right)+\right.$$

$$(1+\lambda s)\frac{Rs\,\alpha^2}{Le}\left(-JN_{CT}\left(\frac{J}{Ln}+\frac{s}{\sigma}\right)+\left(\frac{J\eta+s}{\varepsilon}\right)\left(\frac{J}{Ln}+\frac{s}{\sigma}\right)\right)+$$

$$\left.(1+\lambda s)Rn\,\alpha^2\left(\frac{N_A J}{Le\,Ln}\left(J+\frac{s}{\sigma}\right)+\left(\frac{J\eta+s}{Le\,\varepsilon}\right)\left(J+\frac{s}{\sigma}\right)-JN_{TC}\left(\frac{N_A J}{\varepsilon\,Ln}+\frac{JN_{CT}}{\varepsilon}\right)\right)\right],$$

where

$$J=(\pi^2+\alpha^2),\ \nu=1+1.25\left(\phi_1^*+\phi_0^*\right)\quad \eta=\varepsilon+(1-\varepsilon)\tilde{k}_s+\frac{3\left(\phi_1^*+\phi_0^*\right)\varepsilon}{2}\left(\frac{\tilde{k}_p-1}{\tilde{k}_p+2}\right) \tag{49}$$

We observe that when there is no conductivity variation (that is $\eta = 1,$ as when $\tilde{k}_s = 1$ and $\tilde{k}_p = 1$) the effect of viscosity variation is to increase the critical Rayleigh number by a factor ν. The additional effect of conductivity variation η is expressed by Eq. (49). When $\tilde{k}_s = 1$, the maximum value of η is (2.5$\left(\phi_1^*+\phi_0^*\right)$) attained when $\varepsilon = 1$ and $\tilde{k}_p \to \infty$. It is worth noting that the factor ν comes from the mean value of $\tilde{\mu}(z)$ over the range [0,1] and the factor η is the mean value of $\tilde{k}(z)$ over the same range. When evaluating the critical Rayleigh number, it is a good approximation to base that number on the mean values of the viscosity and conductivity based in turn on the basic solution for the nanofluid fraction.

Linear Stability Analysis

Stationary Mode

For the validity of principle of exchange of stabilities (i.e., steady case), we have s = 0 $\left(i.e., s = s_r + is_i = s_r = s_i = 0\right)$ at the margin of stability. The most unstable mode corresponds to $N = 1$ (the fundamental mode). Then the Rayleigh number for $N = 1$ at which marginally stable steady mode exists becomes,

$$Ra_T^{St} = \frac{1}{\alpha^2}\left(\frac{JN_{TC}}{\varepsilon}\left(\frac{J}{Ln}\right)-\frac{1}{Le}(J)\left(\frac{J}{Ln}\right)\right)^{-1}\left[\left((-J)\left(\frac{J\eta}{Le}\right)\left(\frac{J}{Ln}\right)(J)+J^3 N_{TC}N_{CT}\left(\frac{J}{Ln}\right)\right)\nu\right. \tag{50}$$

$$+\frac{Rs\,\alpha^2}{Le}\left(-JN_{CT}\left(\frac{J}{Ln}\right)+\left(\frac{J\eta}{\varepsilon}\right)\left(\frac{J}{Ln}\right)\right)+Rn\,\alpha^2\left(\frac{N_A J}{Le\,Ln}(J)+\left(\frac{J\eta}{Le\,\varepsilon}\right)(J)\right.$$

$$\left.\left.-JN_{TC}\left(\frac{N_A J}{\varepsilon\,Ln}+\frac{JN_{CT}}{\varepsilon}\right)\right)\right].$$

In the case of double diffusion in a regular fluid, when $\varepsilon = 1$, N_{TC}, N_{CT} and N_A are all zero, Eq. (50) reduces to $Ra_T^{St} + Rs = R_0$, where ($R_0 = 4\pi^2 = 39.48$ with $\alpha_c = \pi = 3.14$) which is the same as obtained by Nield [3]. The stationary boundary does not depend on the value of Pr.

Oscillatory Mode

We now set $s = i\omega_i$, where ω_i is real, in Eq. (48) and clear the complex quantities from the denominator to obtain

$$Ra_T = \Delta_1 + i\,\omega\Delta_2 \tag{51}$$

For oscillatory onset $\Delta_2 = 0$ $\left(\omega_i \neq 0\right)$ and this gives a dispersion relation of the form (on dropping the subscript *i*)

$$b_1\left(\omega^2\right)^2 + b_2\left(\omega^2\right) + b_3 = 0 \tag{52}$$

Now Eq. (48) with $\Delta_2 = 0$ gives

$$Ra_T^{Osc} = a_0\left(a_1 + \omega^2 a_2\right) \tag{53}$$

where $b_1,\ b_2,\ b_3$ and $a_0,\ a_1,\ a_2$, and Δ_1 and Δ_2 are not presented here for brevity.

We find the oscillatory neutral solutions from Eq. (53). It proceeds as follows: First determine the number of positive solutions of Eq. (52). If there are none, then no oscillatory instability is possible. If there are two, then the minimum (over a^2) of Eq. (53) with

ω^2 given by Eq. (52) gives the oscillatory neutral Rayleigh number. Since Eq. (52) is quadratic in ω^2, it can give rise to more than one positive value of ω^2 for fixed values of the parameters *Rn, Ln,* N_A, σ, γ_a, ν, η, and λ. However, our numerical solution of Eq. (52) for the range of parameters considered here gives only one positive value of ω^2 indicating that there exists only one oscillatory neutral solution. The analytical expression for oscillatory Rayleigh number given by Eq. (53) is minimized with respect to the wave number numerically, after substituting for ω^2 (> 0) from Eq. (52), for various values of physical parameters in order to know their effects on the onset of oscillatory convection.

4. Non-Linear Stability Analysis:

For simplicity, we consider the case of two dimensional rolls, assuming all physical quantities to be independent of *y*. On eliminating the pressure and introducing the stream function, we obtain:

$$\left(\nu+s\gamma_a\left(1+\lambda s\right)\right)\nabla^2\Psi+\left(1+\lambda s\right)Ra_T\frac{\partial T}{\partial x}+\left(1+\lambda s\right)\left(\frac{Rs}{Le}\right)\frac{\partial C}{\partial x}-\left(1+\lambda s\right)Rn\frac{\partial S}{\partial x}=0 \tag{54}$$

$$\frac{\partial T}{\partial t}+\frac{\partial\Psi}{\partial x}=\eta\nabla^2T+\frac{\partial\left(\Psi,T\right)}{\partial\left(x,z\right)}+N_{TC}\nabla^2C \tag{55}$$

$$\frac{1}{\sigma}\frac{\partial S}{\partial T}+\frac{1}{\varepsilon}\frac{\partial\Psi}{\partial x}=\frac{1}{Ln}\nabla^2S+\frac{N_A}{Ln}\nabla^2T+\frac{1}{\varepsilon}\frac{\partial(\Psi,S)}{\partial(x,z)} \tag{56}$$

$$\frac{1}{\sigma}\frac{\partial C}{\partial T}+\frac{1}{\varepsilon}\frac{\partial\Psi}{\partial x}=\frac{1}{Le}\nabla^2C+N_{CT}\nabla^2T+\frac{1}{\varepsilon}\frac{\partial(\Psi,C)}{\partial(x,z)} \tag{57}$$

We solve Eqs. (54)–(57) subjecting them to stress-free, isothermal, iso-volume fraction iso-concentration boundary conditions:

$$\psi=\frac{\partial^2\psi}{\partial z^2}=T=S=C=0 \quad \text{at } z = 0, 1.$$

To perform a local non-linear stability analysis, we take the following Fourier expressions:

$$\psi=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\left(t\right)sin\left(m\alpha x\right)sin\left(n\pi z\right),$$

$$T=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}(t)\cos\left(m\alpha x\right)sin\left(n\pi z\right),$$

$$S=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{mn}(t)\cos(m\alpha x)\,sin(n\pi z),$$

$$C=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{mn}(t)\cos(m\alpha x)\,sin(n\pi z), \tag{58}$$

Further, we take the modes (1, 1) for stream function, and (0, 2) and (1, 1) for temperature, nanoparticle volume fraction and concentration, to get

$$\psi=A_{11}(t)\,sin(\alpha x)\,sin(\pi z),$$

$$T=B_{11}(t)\cos(\alpha x)\,sin(\pi z)+B_{02}(t)\,sin(2\pi z),$$

$$S=C_{11}(t)\cos(\alpha x)\,sin(\pi z)+C_{02}(t)\,sin(2\pi z),$$

$$C=D_{11}(t)\cos(\alpha x)\,sin(\pi z)+D_{02}(t)\,sin(2\pi z), \tag{59}$$

where the amplitudes $A_{11}(t)$, $B_{11}(t)$, $B_{02}(t)$, $C_{11}(t)$, $C_{02}(t)$, $D_{11}(t)$ and $D_{02}(t)$ are functions of time and are to be determined.

Taking the orthogonality condition with the eigenfunctions associated with the considered minimal model, we get

$$\frac{dA_{11}(t)}{dt}=\frac{1}{\gamma_a\delta^2}\left[\begin{aligned}&\alpha Rn\,C_{11}(t)-\alpha Ra_T\,B_{11}(t)-\frac{\alpha Rs\,D_{11}(t)}{Le}-\nu\delta^2A_{11}(t)-\lambda\gamma_a\,\delta^2B_{02}(t)\\&-\lambda\,\alpha\,Ra\,B_{02}(t)+\lambda\,\alpha\,Rn\,C_{02}(t)-\lambda\,\alpha\frac{Rs}{Le}C_{02}(t)\end{aligned}\right],$$

$$\frac{dB_{11}}{dt}=-\left[\alpha A_{11}(t)+\eta\,\delta^2B_{11}(t)+\alpha\,\pi\,A_{11}(t)\,B_{02}(t)+N_{TC}\,\delta^2D_{11}(t)\right],$$

$$\frac{dB_{02}}{dt}=-\eta\,4\pi^2B_{02}(t)+\frac{\alpha\pi}{2}A_{11}(t)B_{11}(t)-N_{TC}\,4\pi^2D_{02}(t),$$

$$\frac{dC_{11}}{dt}=-\sigma\left[\frac{1}{\varepsilon}\alpha A_{11}(t)+\delta^2\left(\frac{C_{11}(t)}{Ln}+\frac{N_A}{Ln}B_{11}(t)\right)+\frac{1}{\varepsilon}\alpha\pi A_{11}(t)C_{02}(t)\right],$$

$$\frac{dC_{02}}{dt}=-\sigma\left[\frac{1}{Ln}4\pi^2C_{02}(t)+4\pi^2B_{02}(t)\frac{N_A}{Ln}-\frac{a\pi}{2\varepsilon}A_{11}(t)C_{11}(t)\right],$$

$$\frac{dD_{11}}{dt}=-\sigma\left[\frac{1}{\varepsilon}\alpha A_{11}(t)+\delta^2\left(\frac{D_{11}(t)}{Le}+N_{CT}B_{11}(t)\right)+\frac{1}{\varepsilon}\alpha\pi A_{11}(t)D_{02}(t)\right],$$

$$\frac{dD_{02}}{dt}=-\sigma\left[\frac{1}{Le}4\pi^2D_{02}(t)+4\pi^2B_{02}(t)N_{CT}-\frac{a\pi}{2\varepsilon}A_1(t)D_{11}(t)\right], \tag{60}$$

In case of steady motion $\frac{d(\ )}{dt}=D_i=0$, (i = 1, 2, .., 7), we may write all D_i's in terms of A_{11}.

Retaining the original form, we get:

$$D_1=\frac{1}{\gamma_a\delta^2}\left[\begin{array}{r}\alpha Rn\,C_{11}(t)-\alpha Ra_T\,B_{11}(t)-\frac{\alpha Rs\,D_{11}(t)}{Le}-\nu\,\delta^2A_{11}(t)-\lambda\,\gamma_a\,\delta^2B_{02}(t)\\-\lambda\,\alpha\,Ra\,B_{02}(t)+\lambda\,\alpha\,Rn\,C_{02}(t)-\lambda\,\alpha\frac{Rs}{Le}C_{02}(t)\end{array}\right],$$

$$D_2=-\left[\alpha A_{11}(t)+\eta\,\delta^2B_{11}(t)+\alpha\,\pi\,A_{11}(t)\,B_{02}(t)+N_{TC}\,\delta^2D_{11}(t)\right],$$

$$D_3=-\eta\,4\pi^2B_{02}(t)+\frac{\alpha\pi}{2}A_{11}(t)B_{11}(t)-N_{TC}\,4\pi^2D_{02}(t),$$

$$D_4=-\sigma\left[\frac{1}{\varepsilon}\alpha A_{11}(t)+\delta^2\left(\frac{C_{11}(t)}{Ln}+\frac{N_A}{Ln}B_{11}(t)\right)+\frac{1}{\varepsilon}\alpha\pi A_{11}(t)C_{02}(t)\right],$$

$$D_5=-\sigma\left[\frac{1}{Ln}4\pi^2C_{02}(t)+4\pi^2B_{02}(t)\frac{N_A}{Ln}-\frac{a\pi}{2\varepsilon}A_{11}(t)C_{11}(t)\right],$$

$$D_6=-\sigma\left[\frac{1}{\varepsilon}\alpha A_{11}(t)+\delta^2\left(\frac{D_{11}(t)}{Le}+N_{CT}B_{11}(t)\right)+\frac{1}{\varepsilon}\alpha\pi A_{11}(t)D_{02}(t)\right],$$

$$D_7=-\sigma\left[\frac{1}{Le}4\pi^2D_{02}(t)+4\pi^2B_{02}(t)N_{CT}-\frac{a\pi}{2\varepsilon}A_1(t)D_{11}(t)\right],$$

and for steady nonlinear mode we have:

$$D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = D_7 = 0. \tag{61}$$

The above system of simultaneous autonomous ordinary differential equations is solved numerically using Runge–Kutta–Gill method. One may also conclude that the trajectories of the above equations will be confined to the finiteness of the ellipsoid. Thus, the effect of the parameters *Rn, Ln,* N_A on the trajectories is to attract them to a set of measure zero, or to a fixed point to say.

Heat and Nanoparticle Concentration Transport

The thermal Nusselt number *NuT* is defined as

$$\mathrm{NuT}=\frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by conduction}} \tag{62}$$

$$=1+\left[\frac{\int_0^{2\pi/\alpha}\frac{\partial \mathrm{T}}{\partial \mathrm{z}}\,\mathrm{dx}}{\int_0^{2\pi/\alpha}\frac{\partial \mathrm{T_B}}{\partial \mathrm{z}}\,\mathrm{dx}}\right]_{\mathrm{z}=0}.$$

Substituting dimensional form of Eq. (26) and (59) in above equation we get

$$NuT=1-2\pi B_{02}(t)$$

The nanoparticle concentration Nusselt number *NuF* is defined similar to the thermal Nusselt number. Following the procedure adopted for arriving at *NuT,* one can obtain the expression for *NuF* in the form:

$$NuF=\left(1-2\pi C_{02}(t)\right)+N_A\left(1-2\pi B_{02}(t)\right) \tag{63}$$

The solute concentration Nusselt number *NuC* is defined similar to the thermal Nusselt number. Likewise one can obtain the expression for *NuC* in the form:

$$NuC = \left(1-2\pi D_{02}(t)\right) + N_{CT}\left(1-2\pi B_{02}(t)\right) \tag{64}$$

Results and discussions

Figure 1(a)-(e) shows the effect of various parameters on the neutral stability curves for stationary convection for Rn = -0.1, Ln = 50, N_A = 4, ε = 0.9, ν = 1, η = 1, N_{TC} = 0.001, Rs = 5, Le = 0.75, N_{CT} = 1 with variation in one of these parameters. We draw the effect of Soret parameter N_{CT} and Dufour parameter N_{TC} on the thermal Rayleigh number in Figs. 1(a) and 1(b) respectively. It can be seen that as N_{CT} and N_{TC} increase, Ra_T increases and hence N_{CT} and N_{TC} have a stabilizing effect on the stationary convection system. From Fig. 1(c), one can observe that as solutal Rayleigh number Rs increases, thermal Rayleigh number decreases which means that the solutal Rayleigh number Rs advances the onset of convection. The effect of viscosity ratio ν and conductivity ratio η on the thermal Rayleigh number is depicted in Figs. 1(d) and 1(e) respectively, these figures show that as ν and η increases, Ra_T increases which indicates that ν and η will stabilize the system. The effect of Soret parameter N_{CT}, Dufour parameter N_{TC} and solutal Rayleigh number Rs on thermal Rayleigh number Ra_T for stationary convection show the similar results obtained by Agarwal et al. [12].

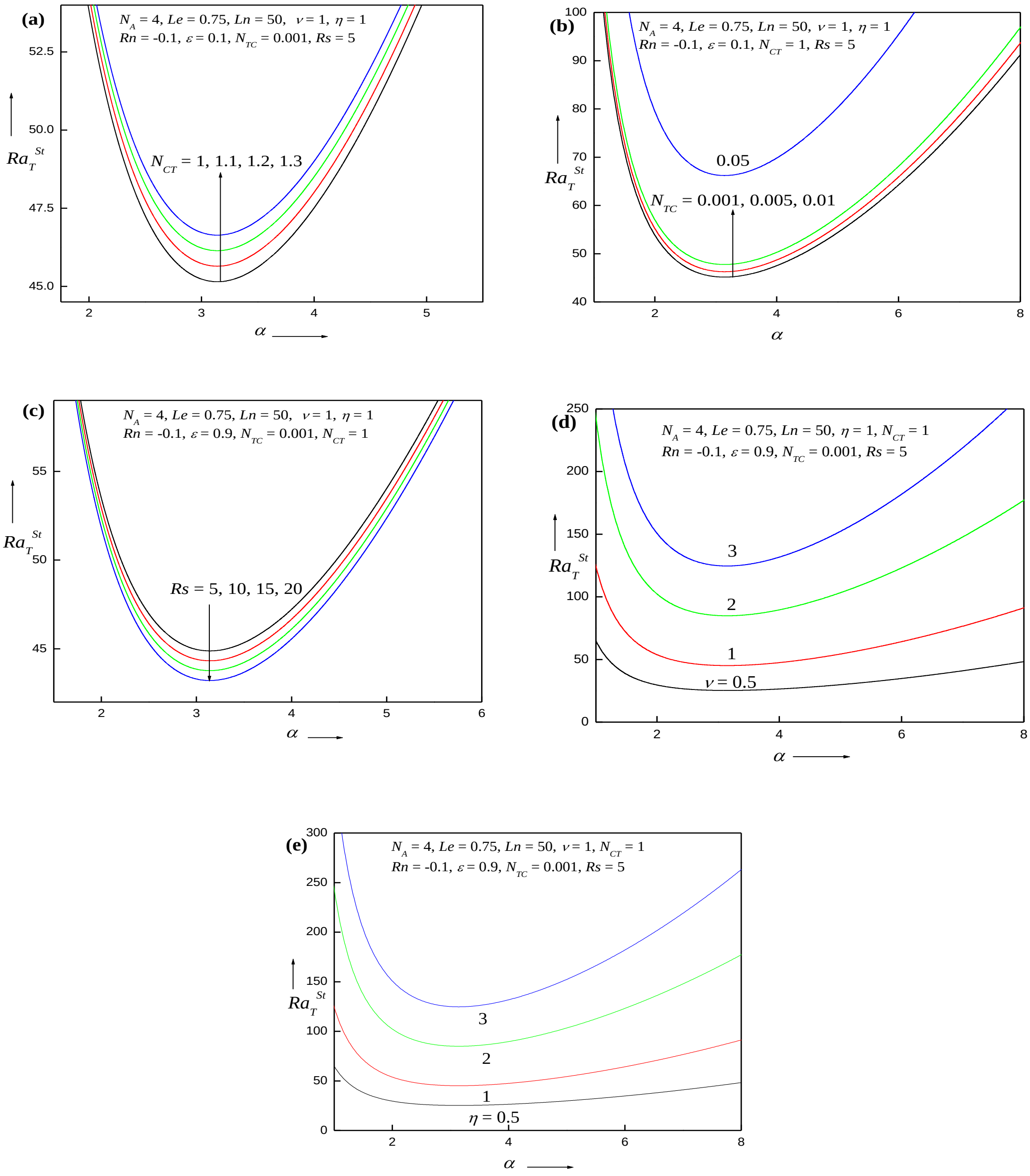

Fig. 1. Neutral curves on stationary convection for different values of (a) Soret parameter N_{CT}, (b) Dufour parameter N_{TC}, (c) solutal Rayleigh number *Rs*, (d) viscosity ratio ν, and (e) conductivity ratio η.

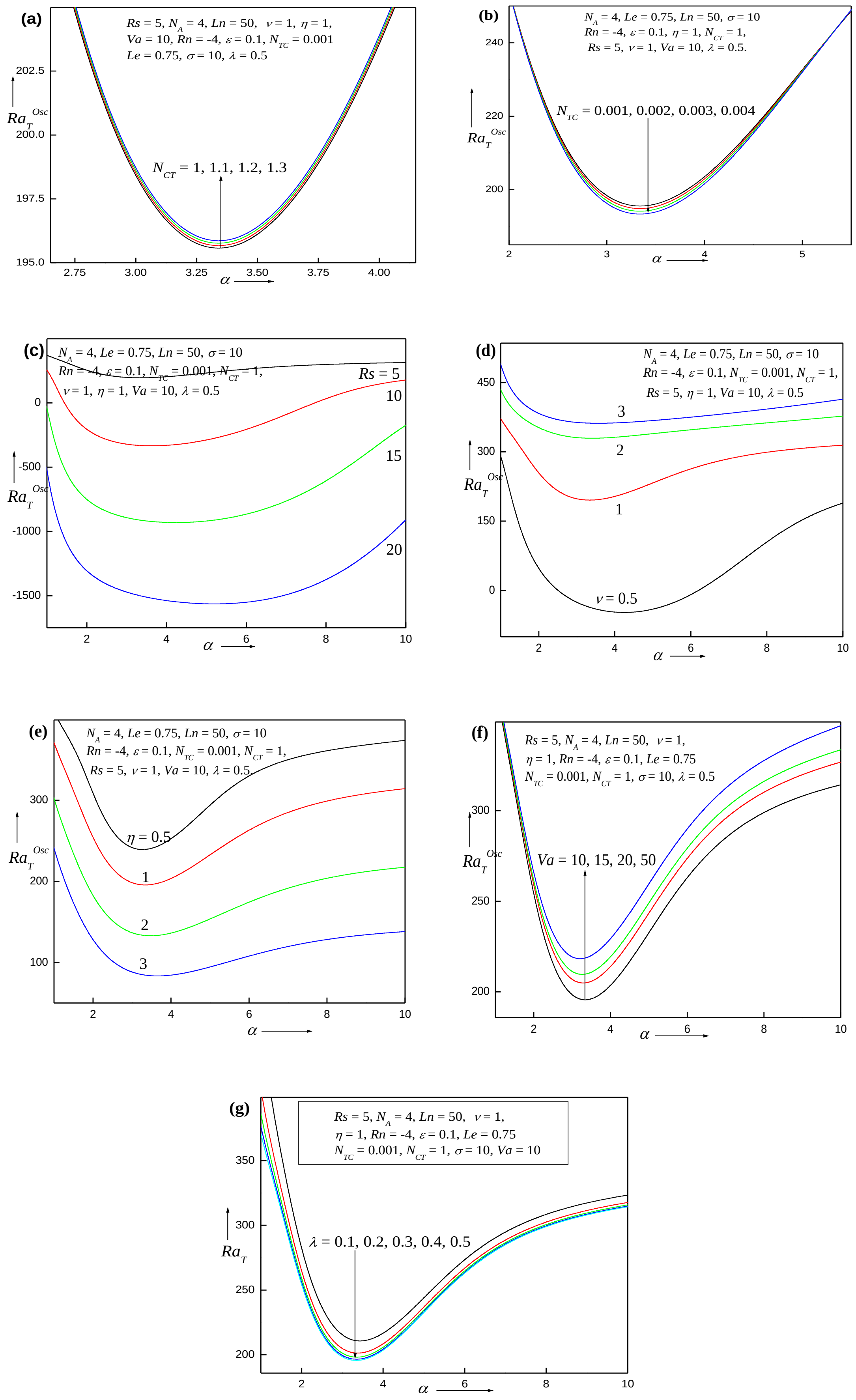

Fig. 2. Neutral curves on oscillatory convection for different values of (a) Soret parameter N_{CT}, (b) Dufour parameter N_{TC}, (c) solutal Rayleigh number *Rs*, (d) viscosity ratio ν, (e) conductivity ratio η, (f) Vadász number *Va*, and (g) relaxation parameter λ.

Figure 2(a)-(g) displays the variation of thermal Rayleigh number for oscillatory convection with respect to various parameters. The effect of Soret parameter N_{CT} and Dufour parameter N_{TC} on the thermal Rayleigh number is shown in Figs. 2(a) and 2(b) respectively, and we find that as N_{CT} and N_{TC} increase Ra_T increases, and hence N_{CT} and N_{TC} have a stabilizing effect on the oscillatory convection system. From Fig. 2(c), one can observe that as solutal Rayleigh number *Rs* increases, thermal Rayleigh number decreases which implies that the solutal Rayleigh number *Rs* advances the onset of convection in oscillatory form. From the pictures 2(d) and 2(f), one can reveal that both the viscosity ratio ν and Vadász number *Va* stabilize the system for oscillatory convection; that is, an increase in ν and *Va* increases the thermal Rayleigh number thus delaying the onset of convection. In Fig. 2(e), we see that the thermal Rayleigh number decreases as conductivity ratio η increases which will advance the onset of convection. The effect of relaxation parameter λ on the thermal Rayleigh number is shown in Fig. 2(g). As λ increases Ra_T decreases and hence λ has a destabilizing effect on the system.

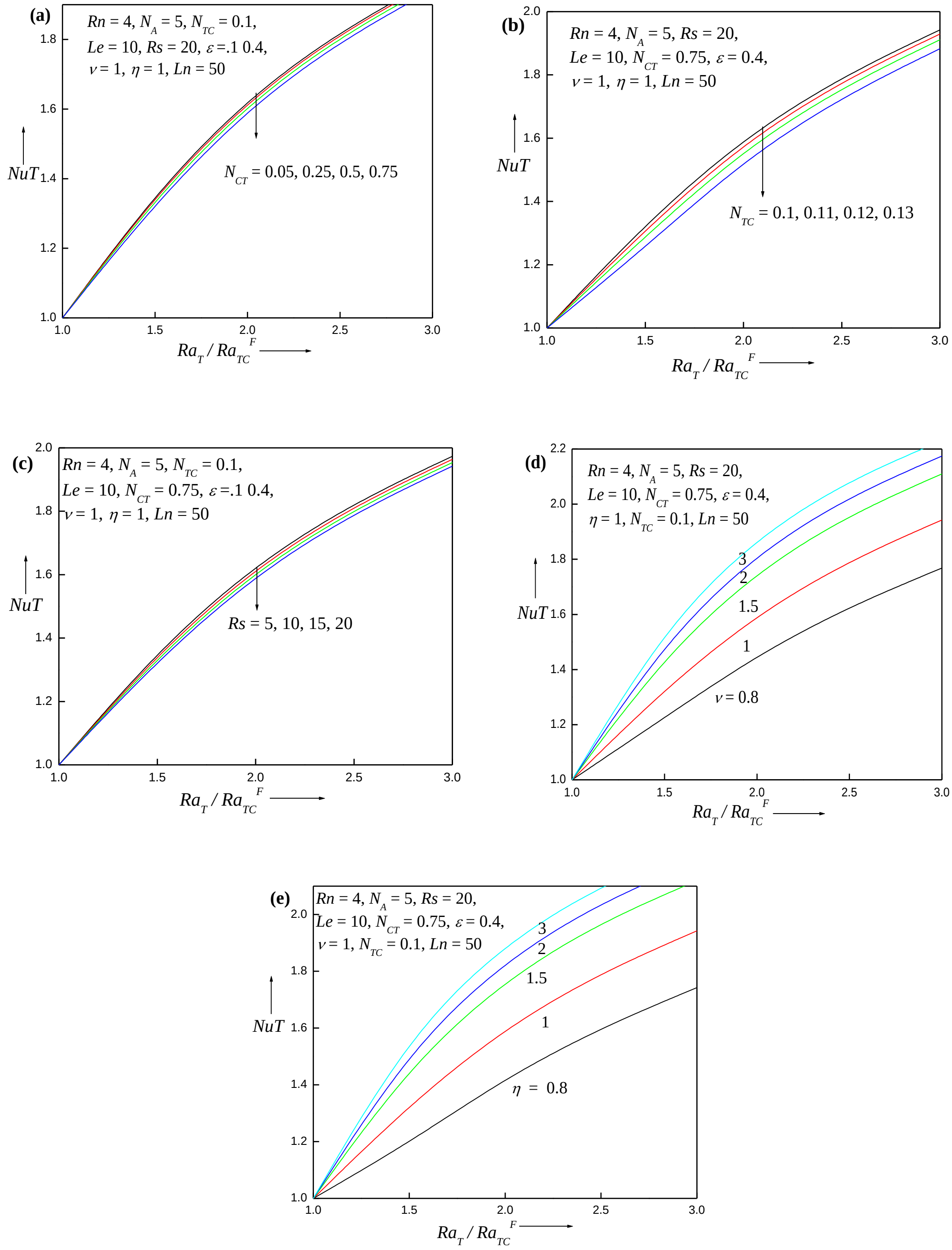

Fig. 3. Variation of thermal Nusselt number *NuT* with critical Rayleigh Number for different values of (a) Soret parameter N_{CT}, (b) Dufour parameter N_{TC}, (c) solutal Rayleigh number *Rs*, (d) viscosity ratio ν, and (e) conductivity ratio η.

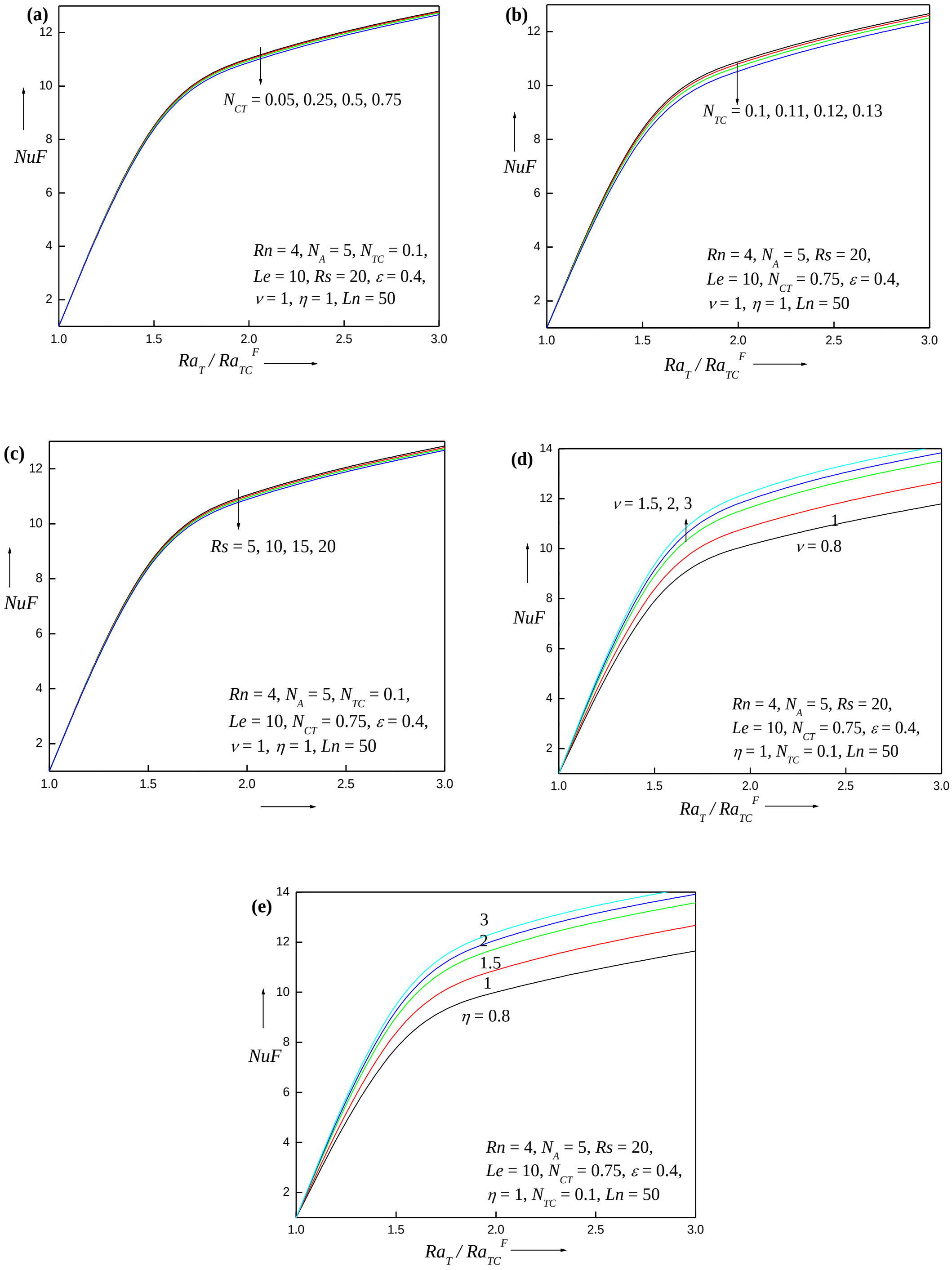

Fig. 4. Variation of nanoparticle concentration Nusselt number *NuF* with critical Rayleigh Number for different values of (a) Soret parameter N_{CT}, (b) Dufour parameter N_{TC}, (c) solutal Rayleigh number *Rs*, (d) viscosity ratio ν, and (e) conductivity ratio η.

The nonlinear analysis provides not only the onset threshold of finite amplitude motion but also the information of heat and mass transports which are given in terms of thermal Nusselt number *NuT*, nanoparticle concentration Nusselt number *NuF* and solute concentration Nusselt number *NuC*. The Nusselt numbers are computed as the functions of Ra_T, and the variations of these non-dimensional numbers with Ra_T for different parameter values are depicted in Figs. 3(a)-(e), 4(a)-(e) and 5(a)-(e) respectively. In the three figures, it is observed that in each case, nanoparticle concentration Nusselt number *NuF* is always greater than both thermal Nusselt number *NuT* and solute concentration Nusselt number *NuC*. Before the value Ra_T / Ra_{Tc} reaching unity (not shown), the all three Nusselt numbers remain the value of 1; that is, all Nusselt numbers start with the conduction state value at the point of onset of steady finite amplitude convection. When Ra_T is increased beyond Ra_{Tc}, there is a sharp increase in the values of Nusselt numbers. However further increase in Ra_T will not change *NuF* and *NuC* significantly. It is to be noted that the upper bound of *NuT* is 3 (similar results were obtained by Malashetty et al. [9]). It should also be noted that the upper bounds of *NuF* and *NuC* are not 3 (similar results were obtained by Bhadauria and Agarwal [23]). The upper bound of *NuT* remains 3 only for both clear and nanofluid. However the upper bound for *NuF* and *NuC* for clear fluid is 3, whereas for nanofluid it is not fixed.

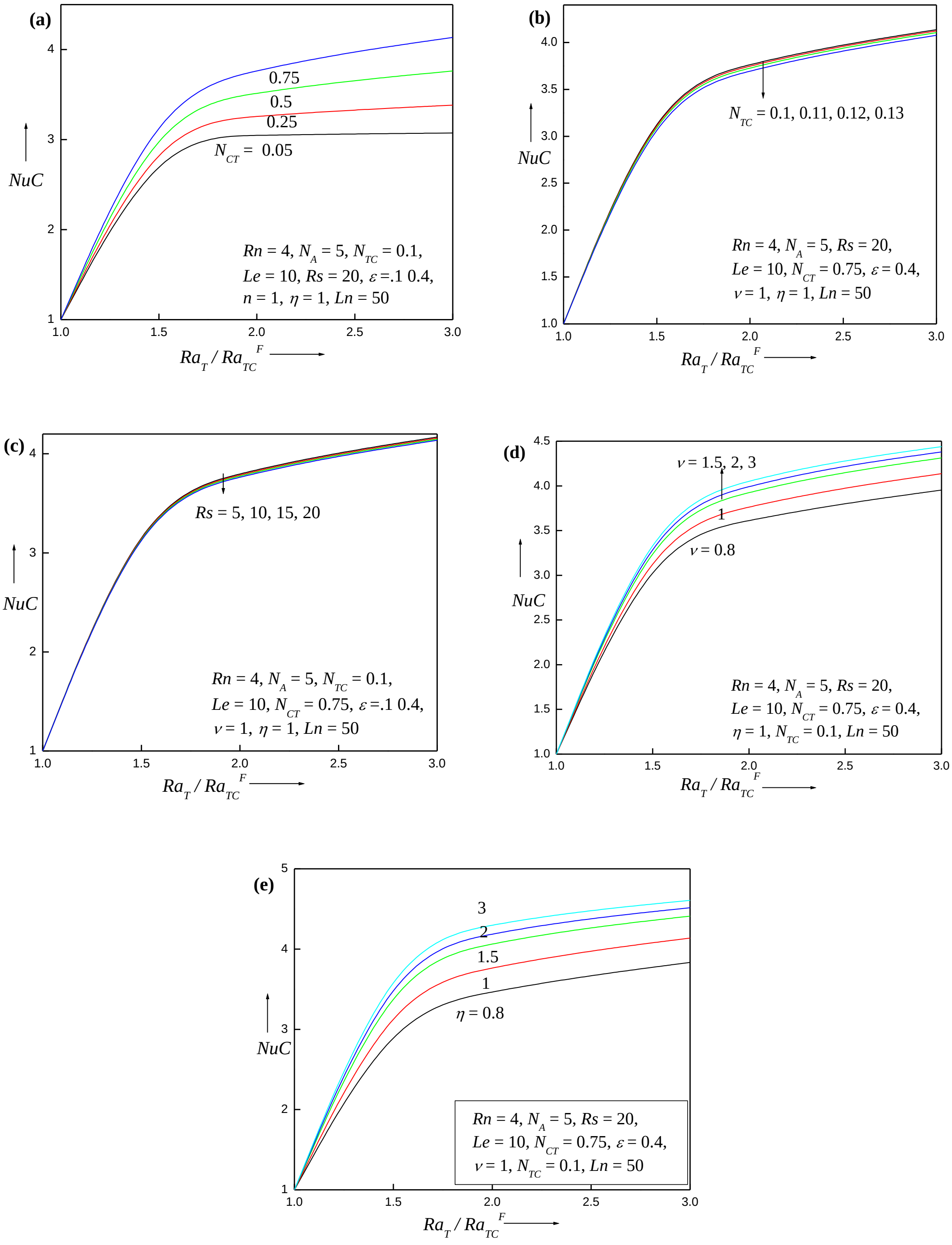

Fig. 5. Variation of solute concentration Nusselt number *NuC* with critical Rayleigh Number for different values of (a) Soret parameter N_{CT}, (b) Dufour parameter N_{TC}, (c) solutal Rayleigh number *Rs*, (d) viscosity ratio ν, and (e) conductivity ratio η.

From Figs. 3(a) and 4(a) we observe that as the Soret parameter N_{CT} increases, the value of *NuT* and *NuF* decreases, thus showing a decrease in the rate of heat and mass transport, while the solute concentration Nusselt number *NuC* (Fig. 5(a)) increases with increase in Soret parameter N_{CT} implying that Soret parameter N_{CT} enhances the solute concentration Nusselt number. We also find that as the Dufour parameter N_{TC} (Figs. 3(b), 4(b) and 5(b)) and solutal Rayleigh number *Rs* (Figs. 3(c), 4(c) and 5(c)) increases, the values of *NuT*, *NuF* and *NuC* decrease, thus showing a decrease in the rate of heat and mass transport. As the viscosity ratio ν (Figs. 3(d), 4(d) and 5(d)) and conductivity ratio η (Figs. 3(e), 4(e) and 5(e)) increase all the Nusselt numbers increases implying that ν and η enhances the heat and mass transports.

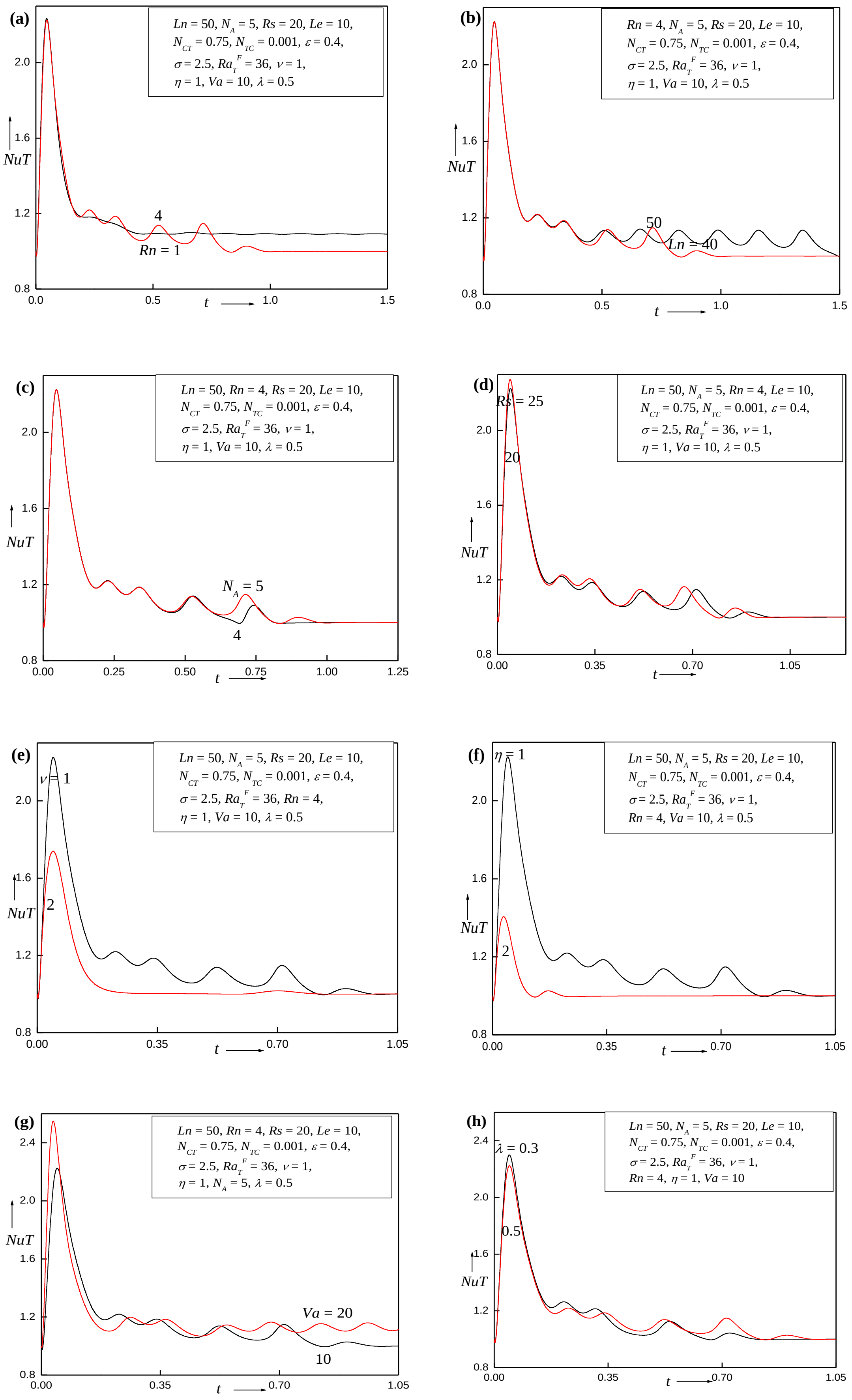

Fig. 6. Transient thermal Nusselt number *NuT* with time for different values of (a) nanoparticle concentration Rayleigh number *Rn*, (b) thermo-nanofluid Lewis number *Ln*, (c) modified diffusivity ratio N_A, (d) solutal Rayleigh number *Rs*, (e) viscosity ratio ν, (f) conductivity ratio η, (g) Vadász number *Va*, and (h) relaxation parameter λ.

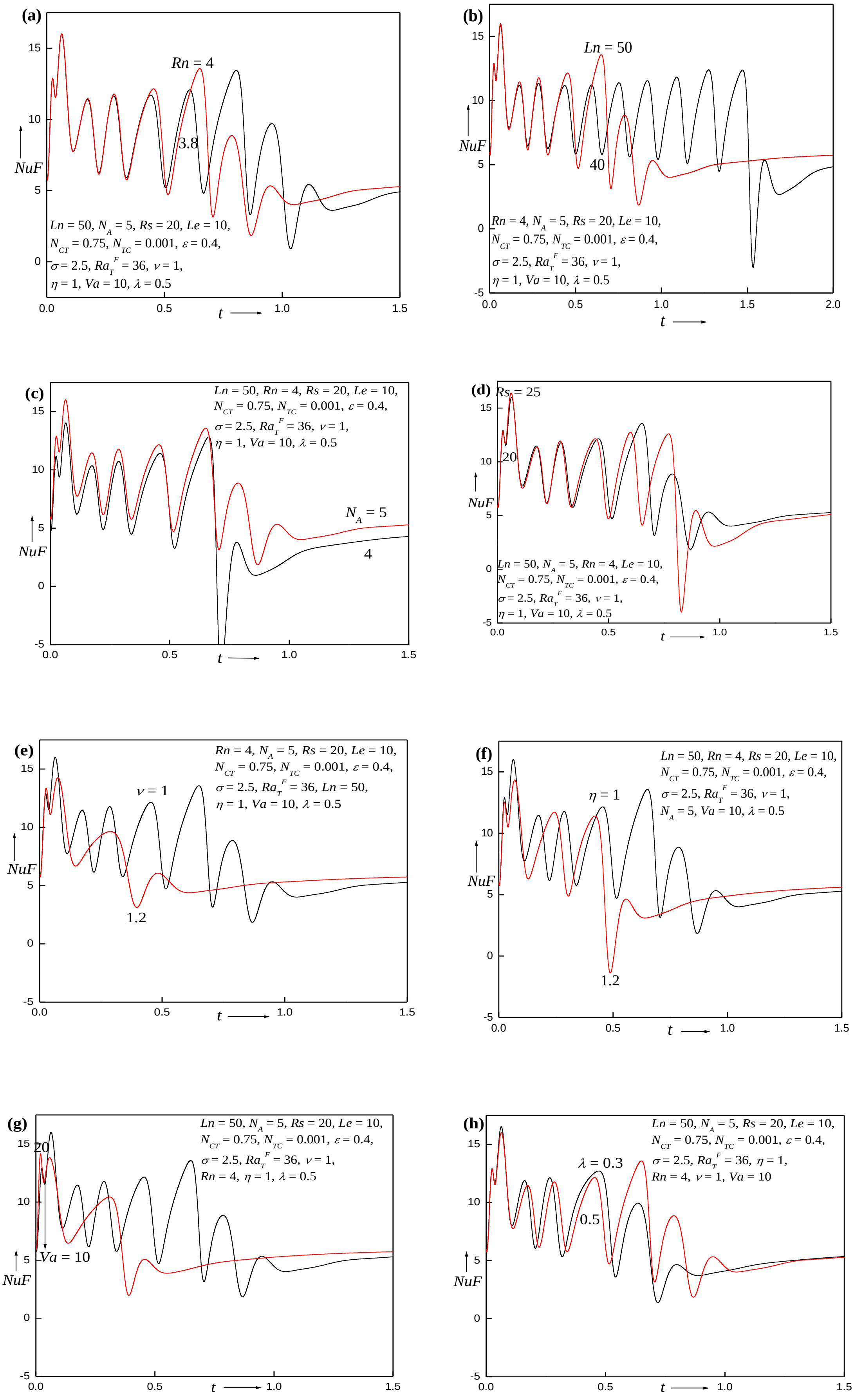

Fig. 7. Transient nanoparticle concentration Nusselt number *NuF* with time for different values of (a) nanoparticle concentration Rayleigh number *Rn*, (b) thermo-nanofluid Lewis number *Ln*, (c) modified diffusivity ratio N_A, (d) solutal Rayleigh number *Rs*, (e) viscosity ratio ν, (f) conductivity ratio η, (g) Vadász number *Va*, and (h) relaxation parameter λ.

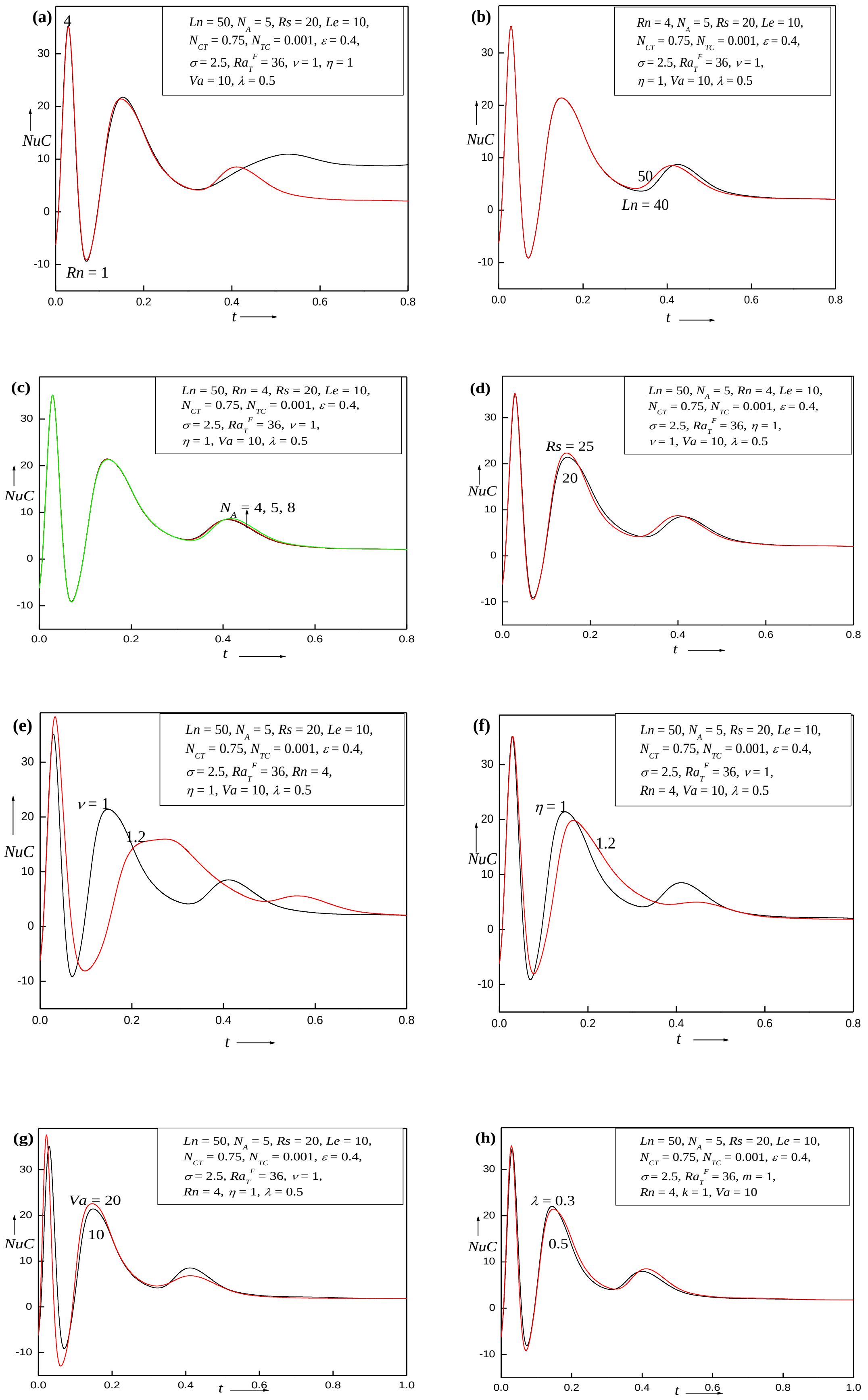

Fig. 8. Transient solute concentration Nusselt number *NuC* with time for different values of (a) nanoparticle concentration Rayleigh number *Rn*, (b) thermo-nanofluid Lewis number *Ln*, (c) modified diffusivity ratio N_A, (d) solutal Rayleigh number *Rs*, (e) viscosity ratio ν, (f) conductivity ratio η, (g) Vadász number *Va*, and (h) relaxation parameter λ.

The linear solution exhibits a considerable variety of behavior of the thermal convection system, and the transition from linear to non-linear convection can be quite complicated, but interesting to deal with. It is needed to study a time dependent results to analyze the same. The transition can be well understood by the analysis of Eq. (60) whose solution gives a detailed description of the two dimensional non-linear evolution problems. The autonomous system of unsteady finite amplitude equations is solved numerically using the Runge-Kutta method. The Nusselt numbers are evaluated as the functions of time *t*, the unsteady transient behavior of *NuT*, *NuF* and *NuC* is shown graphically in Figs. 6(a)-(h), 7(a)-(h) and 8(a)-(h), respectively.

These figures indicate that initially when time is small, there occur large scale oscillations in the values of Nusselt numbers indicating an unsteady rate of heat and mass transport in the thermal convection system. As time passes by, these values approach to steady state corresponding to a near convection stage.

Figures (6a, 7a, 8a), (6b, 7b, 8b), (6c, 7c, 8c) and (6d, 7d, 8d) depict the transient nature of thermal Nusselt number *NuT*, concentration Nusselt number *NuF* number and solute Nusselt number *NuC* for varying values of nanoparticle concentration Rayleigh number *Rn*, nanofluid Lewis number *Ln*, modified diffusivity ratio N_A and solutal Rayleigh number *Rs*. It is observed that as *Rn, Ln*, N_A and *Rs* increase, the values of *NuT*, *NuF* and *NuC* show slightly incremented, but the difference can be observed with further increase in *Rn, Ln,* N_A and *Rs,* especially for *NuF*, thus showing an increase in the heat and mass transport, which are the similar results observed by Agarwal et al. [12]. From Figs. (6e, 7e, 8e) we observe that viscosity ratio ν increases the heat and mass transports and in Figs. (6f, 7f, 8f) we observe that as conductivity ratio η increases the *NuT*, *NuF* and *NuC* decrease, indicating that there is retardation on heat and mass transports. Figures (6g, 7g, 8g) depict the transient nature of Vadász number *Va* on Nusselt numbers. It is observed that as *Va* increases, the evolution of *NuF* presents a significant variation in comparison with those for *NuT* and *NuC.* Figures (6h, 7h and 8h) depict the transient nature of Nusselt number for two values of relaxation parameter λ. It is observed that as λ increases the evolution of *NuT*, *NuF* and *NuC*, respectively, shows nearly unchanged variation except for medium time; say during $0.2<t<1.0$ for *NuF*.

From these figures with the parameters specified, we can observe that the value of thermal Nusselt number *NuT* starts from 1, the value of nanoparticle concentration Nusselt number *NuF* starts from 6 and the value of solute concentration Nusselt number *NuC* starts form 1.75.

Conclusions

We performed a linear and weakly nonlinear stability analysis in a horizontal porous medium saturated by a nanofluid, heated from below and cooled from above, using Darcy model which incorporates the effect of Brownian motion along with thermophoresis. Further the viscosity- and conductivity-dependence on nanoparticle volume fraction model was also adopted following Tiwari and Das [18]. Linear analysis has been made using normal mode technique, whereas for weakly nonlinear analysis a truncated Fourier series representation having only two terms is considered. We draw the following conclusions:

1. For stationary mode Soret parameter N_{CT}, Dufour parameter N_{TC}, viscosity ratio ν and conductivity ratio η have a stabilizing effect while solutal Rayleigh number *Rs* destabilizes the system.
2. For oscillatory mode Soret parameter N_{CT}, Dufour parameter N_{TC}, viscosity ratio ν and Vadász number *Va* have a stabilizing effect while solutal Rayleigh number *Rs* , conductivity ratio η and relaxation parameter λ destabilize the system.
3. With the assigned parameters the value of transient Nusselt numbers *NuT* starts from 1, *NuF* starts from 6, and *NuC* starts form 1.75.
4. The time evolution of thermal Nusselt number, nanoparticle concentration Nusselt number and solute concentration Nusselt number is found to be oscillatory when *t* is small. However, when *t* becomes very large all Nusselt numbers approach to their steady values.

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