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Refined beam theory for bending of thick beams subjected to various loading

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ABSTRACT
In this paper a new Trigonometric shear deformation theory is developed for the static flexure of thick isotropic beam, considering Trigonometric functions in terms of thickness co-ordinate associated with transverse shear deformation effect. The most important feature of the theory is that the transverse shear stress can be obtained directly from the constitutive relations satisfying the shear stress free surface conditions on the top and bottom of the beam. Hence the theory obviates the need of shear correction factor. Governing equations and boundary conditions of the theory are obtained using the principle of virtual work. Results obtained for static flexure of simply supported isotropic beam subjected to various loading cases are compared with those of other refined theories and exact solution.

Introduction

Classical Euler-Bernoulli theory of beam (ETB) bending is based on hypothesis that the plane section, which is perpendicular to the neutral axis before bending, remains plane and perpendicular to the neutral axis after bending. The theory should not apply to deep beams since it disregards the effect of shear deformation.

Timoshenko [1] has developed first order shear deformation theory (FSDT), which is based on hypothesis that the plane section, which is perpendicular to the neutral axis before bending, remains plane but not necessarily perpendicular to the neutral axis after bending. In this theory the transverse shear strain distribution over the cross-section of the beam is assumed to be constant through the thickness and thus require shear correction factor.

Ghugal and Sharma [2] have developed a variationally consistent refined hyperbolic shear deformation theory for flexure and free vibration of thick isotropic beam.

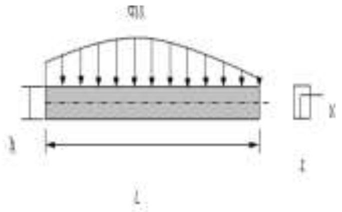
Ghugal and Nakhate [3] has developed trigonometric shear deformation theory for the static flexure of thick isotropic beam and obtained the general solution of thick isotropic beam with various support and loading conditions.

Sayyad and Ghugal [4] have developed new hyperbolic shear deformation theory for the flexure of thick beams, in which combined effect of shear and bending rotations is considered.

Sayyad and Ghugal [5] have carried out comparative study of refined beam theories for static flexure of deep beams. Ghugal [6] has developed trigonometric shear deformation theory for the flexure and vibration of thick beams. Ghugal and Waghe [IEI] have developed the trigonometric shear deformation theory for deep beams.

In this paper a variationally consistent new Trigonometric shear deformation theory for beam is developed. In this theory rotation of normal is taken as combined effect of shear slope and bending slope at the neutral axis. The theory is applied to simply supported isotropic beam of rectangular cross-section carrying various loading cases for static flexure analysis. A close form solution for simply supported beam subjected to single sine load is obtained. The results obtained are compared with those of elementary, refined and exact beam theories available in the literature.

Beam under Consideration



The beam under consideration occupies the region:

$$0 \le x \le L; \quad -\frac{b}{2} \le y \le \frac{b}{2}; \quad -\frac{h}{2} \le z \le \frac{h}{2} \qquad (1)$$

Where x, y, z are Cartesian coordinates, L is the length of beam, b is the width and h is the total depth of beam. The beam is subjected to transverse load of intensity $q(x)$ per unit length of the beam.

Assumptions Made in Theoretical Formulation

The axial displacement consists of two parts:
(a) Displacement given by elementary theory of beam bending.
(b) Displacement due to shear deformation, which is assumed to be Trigonometric in nature with respect to thickness coordinate, such that maximum shear stress occurs at neutral axis as predicted by the elementary theory of bending of beam
1.The axial displacement u is such that the resultant of axial stress σ_x, acting over the Cross- section should result in only bending moment and should not in force in x direction.
2.The transverse displacement w is assumed to be a function of longitudinal (length) co-ordinate 'x' direction
3.The displacements are small as compared to beam thickness.
4.The body forces are ignored in the analysis. (The body forces can be effectively taken into account by adding them to the external forces.)

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5.One dimensional constitutive laws are used.
6.The beam is subjected to lateral load only.

The Displacement Field

Based on the before mentioned assumptions, the displacement field of the present hyperbolic shear deformation theory is given as below:

$$u = -z\frac{dw}{dx} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x) \qquad (2)$$
$$w = w(x)$$

where u is axial displacement component x direction and w is transverse displacement in z direction. The trigonometric function in terms of thickness coordinate in the displacement field of u is associated with the transverse shear stress distribution through the thickness of beam and the function ϕ (x) is unknown function associated with shear slope/warping of the cross section of beam at neutral axis of beam

Strain-Displacement Relationships

Normal and shear strains are obtained within the framework of linear theory of elasticity using the displacement field given by equation (2) These relationships are given as follows: -
Normal Strain:

$$\varepsilon_x = \frac{du}{dx} = -z\frac{dw^2}{dx^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx} \qquad (3)$$

Shear Strain:

$$\gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx} = \cos\frac{\pi z}{h}\phi \qquad (4)$$

Stress-Strain Relationships:

The one-dimensional Hooke's law is applied. The axial stress σ_x is related to strain ε_x and the following constitutive relations relate shear stress to shear strain:

$$\sigma_x = E\varepsilon_x$$
$$\tau_{zx} = G\gamma_{zx} \qquad (5)$$

Where *E* and *G* are the elastic constants of the beam material.
Using the Eqns. (3) and (4) for strains, stresses and principle of virtual work, variationally consistent differential equations for the beam under consideration are obtained. The principle of virtual work when applied to the beam leads to:

$$\int_0^L\int_{-h/2}^{h/2}\left(\sigma_x\,\delta\varepsilon_x + \tau_{zx}\,\delta\gamma_{zx}\right)dx\,dz - \int_0^L q(x)\,\delta w\,dx = 0 \qquad (6)$$

substituting the value of $\sigma_x, \gamma_{zx,}\varepsilon_{x,}\delta\gamma_{zx}$ in above equation we get

$$\int_0^L\int_{-h/2}^{h/2} E\left(-z\frac{d^2w}{dx^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx}\right)\left(-z\frac{d\delta^2 w}{dx^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{d\delta\phi}{dx}\right)\left(G\cos\frac{\pi z}{h}\phi\right)\left(\cos\frac{\pi z}{h}\delta\phi\right) - \int_0^L q(x)\delta w\,dx = 0 \qquad (7)$$

where the symbol δ denotes the variational operator . Integrating Eqn. (6) by parts and collecting the coefficients of δw and $\delta\phi$ the governing equations obtained are as follows:

$$\frac{Eh^3}{12}\frac{d^4w}{dx^4} - \frac{2Eh^3}{\pi^3}\frac{d^3\phi}{dx^3} = q(x) \qquad (8)$$

$$\frac{2Eh^3}{\pi^3}\frac{d^3w}{dx^3} - \frac{Eh^3}{2\pi^2}\frac{d^2\phi}{dx^2} + \frac{Gh}{2}\phi = 0 \qquad (9)$$

The associated consistent natural boundary conditions obtained are of following form: At the ends x=0and x=L or w is prescribed

$$V_x = \frac{2Eh^3}{\pi^3}\frac{d^3\phi}{dx^3} - \frac{Eh^3}{12}\frac{d^3w}{dx^3} = 0 \quad \text{or w is prescribed} \qquad (10)$$

$$M_x = \frac{Eh^3}{12}\frac{d^2w}{dx^2} - \frac{2Eh^3}{\pi^3}\frac{d\phi}{dx} = 0 \quad \text{or } \frac{dw}{dx} \text{ is prescribe} \qquad (11)$$

$$\frac{Eh^3}{2\pi^2}\frac{d\phi}{dx} - \frac{2Eh^3}{\pi^3}\frac{d^2w}{d^2x} = 0 \quad \text{or } (\phi) \text{ is prescribed} \qquad (12)$$

The governing differential equations and associated boundary conditions for static flexure of beam under consideration can be obtained directly from Eqns. (8) through (12).

Thus, the variation ally consistent governing differential equations and boundary conditions are obtained. The static analysis of the beam is described by the solution of these equations and simultaneously satisfaction of the associated boundary conditions.

Illustrative Examples

A simply supported uniform beam of rectangular cross-section occupying the region given by expression (1) is considered for detailed numerical study.
In order to prove the efficacy of the present theory, the following numerical examples are considered. The following material properties for beam are used.

E = 210GPa, μ = 0.3 and $G = \frac{E}{2(1+\mu)}$

Where *E* is the Young's modulus, G is shear modulus and μ is the Poisson's ratio of beam material
1. Simply supported beam subjected to uniformly distributed load

The beam with origin on left hand side supported is simply supported at *x* = 0 and *L*. The beam is subjected to uniformly distributed load, *q*(*x*) at surface *z* = −*h*/2 acting in the downward *z*-direction (positive) as shown in Fig.2.

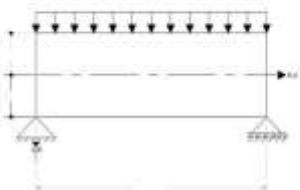


Fig. 2 simply supported beam subjected to uniformly distributed load *q* (*x*)

Simply Supported Beam with a concentrated load

The beam with origin at left hand side support, is simply supported at *x* = 0 and *L*. The beam is subjected to a concentrated load, *P* at mid span at surface *z =−h/2* acting in the *z* direction as shown in Fig.3

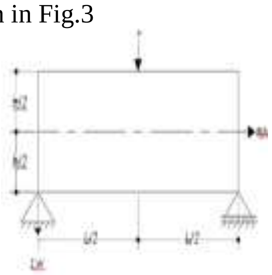


Fig.3 simply supported beam with a concentrated load *P*

Simply supported beam subjected to single sine load

The beam with origin on left hand side supported is simply supported at *x* = 0 and *L*. The beam is subjected to single sine load, *q* (*x*) at surface *z* = −*h*/2 acting in the downward *z*-direction (positive) as shown in Fig 4.

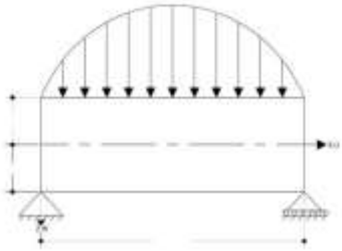


Fig.4 Simply Supported Beam subjected to single sine load

Simply Supported Beam subjected to linearly varying load

The beam with origin on left hand side supported is simply supported at $x = 0$ and L. The beam is subjected to linearly varying load, $q(x)$ at surface $z = -h/2$ acting in the downward z-direction (positive) as shown in Fig 5.

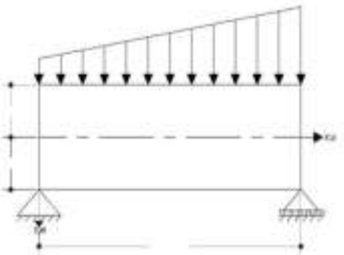


Fig.5 Simply Supported Beam subjected to linearly varying load

Simply Supported Beam subjected to parabolic load

The beam with origin on left hand side supported is simply supported at $x = 0$ and L. The beam is subjected to parabolic load, $q(x)$ at surface $z = -h/2$ acting in the downward z-direction (positive) as shown in Fig 3.6.

Fig. 6 Simply Supported Beam subjected to parabolic load

The Solution Scheme

Following solution scheme is assumed for the static flexure of simply supported thick isotropic beams

$$w = w_m \sin\left(\frac{m\pi x}{L}\right)$$
$$\phi = \phi_m \cos\left(\frac{m\pi x}{L}\right) \qquad (13)$$
$$q = q_m \sin\left(\frac{m\pi x}{L}\right)$$

Substituting equation (13) in General equation (8) and (9) we get

$$\left(\frac{Eh^3}{12}\frac{m^4\pi^4}{l^4}\right)w_m - \left(\frac{2Eh^3}{\pi^3}\frac{m^3\pi^3}{l^3}\right)\phi_m = q_m \qquad (14)$$

$$-\left(\frac{2Eh^3}{\pi^3}\frac{m^3\pi^3}{l^3}\right)w_m + \left[\frac{Eh^3}{2\pi^2}\left(\frac{m^2\pi^2}{l^2}\right) + \frac{Gh}{2}\right]\phi_m = 0 \; (15)$$

Equation (3.14) and (3.15) can be written in following matrix form

$$\begin{bmatrix} \frac{Eh^3}{12}\frac{m^4\pi^4}{l^4} & -\frac{2Eh^3}{\pi^3}\frac{m^3\pi^3}{l^3} \\ -\frac{2Eh^3}{\pi^3}\frac{m^3\pi^3}{l^3} & \frac{Eh^3}{2\pi^2}\left(\frac{m^2\pi^2}{l^2}\right)+\frac{Gh}{2} \end{bmatrix} \begin{bmatrix} w_m \\ \phi_m \end{bmatrix} = \begin{bmatrix} q_m \\ 0 \end{bmatrix} \qquad (16)$$

Crammer's rule is used to solve above equation

Where,

$$D = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = L_{11}L_{22} - L_{12}L_{21}$$
$$D_1 = \begin{bmatrix} q_m & L_{12} \\ 0 & L_{22} \end{bmatrix} = q_m L_{22}$$
$$D_2 = \begin{bmatrix} L_{11} & q_m \\ L_{\cdot} & x,u \end{bmatrix} = -q_m L_{2},$$
$$W_m = \frac{D_1}{D} q_m \text{ And } \phi_m = -\frac{D_2}{D} q_m$$
$$w_m = \overline{w} q_m \; \& \; \phi_m = \overline{\phi} q_m$$

Now substitute the value of w and ϕ in to displacement field to obtain displacement from equation ( ) and ( )

$$\therefore u = \left[\left(\frac{z}{h}\right) h \times \overline{w}\left(\frac{m\pi}{l}\right) + \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\overline{\phi}\right] q_m \cos\left(\frac{m\pi}{l}\right) \qquad (17)$$

$$\therefore w = \overline{w} q_m \sin\left(\frac{m\pi x}{L}\right) \qquad (18)$$

Stresses are obtain from equation (5) σ_x and τ_{zx}

$$\therefore \sigma_x = \left[\left(\frac{z}{h}\right) h \times \overline{w}\left(\frac{m^2\pi^2}{l^2}\right) - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\frac{m\pi}{l}\overline{\phi}\right] E q_m \sin\left(\frac{m\pi x}{l}\right) \qquad (19)$$

Determination of τ_{zx} via equation oequilibrium

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{zx}}{dz} = 0$$
$$\int \frac{d\tau_{zx}}{dz} dz = \int -\frac{d\sigma_x}{dx} dz$$
$$= -\left[\left(\frac{z}{h}\right) h \overline{w} \frac{m^3\pi^3}{L^3} - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\overline{\phi}\left(\frac{m^2\pi^2}{L^2}\right)\right] E q_m \cos\left(\frac{m\pi x}{L}\right) \qquad (20)$$
$$\tau_{zx} = -\left[\frac{z^2}{2}\overline{w}\frac{m^3\pi^3}{L^3} + \frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right)\frac{h}{\pi}\overline{\phi}\frac{m^2\pi^2}{L^2}\right] E q_m \cos\left(\frac{m\pi x}{L}\right) + c$$

Now to find the value of constant 'c'we have $\tau_{zx} = 0$ at $z = \pm h/2$

$$\therefore c = \left[\frac{h^2}{8}\overline{w}\frac{m^3\pi^3}{L^3}\right] E q_m \cos\left(\frac{m\pi x}{L}\right)$$ Substituting the value of 'c' in above equation we have

$$\therefore \tau_{zx}^{EE} = \left[\overline{w}\frac{m^3\pi^3}{L^3}\frac{h^2}{8}\left(1 - 4\left(\frac{z}{h}\right)^2\right) - \frac{h^2}{\pi^2}\cos\left(\frac{\pi z}{h}\right)\overline{\phi}\frac{m^2\pi^2}{L^2}\right] E q_m \cos\left(\frac{m\pi x}{L}\right) (21)$$

For various loading cases q_m is used as follows given by Navier

1. Simply supported beam subjected to Uniformly distributed load $q_m = \frac{4q_0}{m\pi}$ where

q_0 = intensity of loading

2. Simply supported beam subjected to a central concentrated load $q_m = \frac{2p}{L}\sin\frac{m\pi\xi}{L}$ where ξ distance of point load from one end is.
3. Simply supported beam subjected to single sine load $q_m = q_0$

4. Simply supported beam subjected to Linearly varying load $q_m = \dfrac{2q_0}{m\pi}$

5. Simply supported beam subjected to Linearly Parabolic load

$$q_m = \frac{2q_0}{m\pi}\left(\sin m\pi + \frac{\cos m\pi}{m\pi} - \cos m\pi - \frac{1}{m\pi}\right)$$

Numerical results

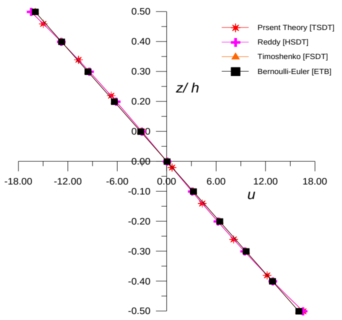
The results obtained for displacements and stresses are presented in the following non-dimensional form:

$$\bar{u} = \frac{Ebu}{q_0 h}; \bar{w} = \frac{Ew10h^3}{q_0 L^4}; \bar{\sigma}_x = \frac{b\sigma_x}{q_0}; \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0}; s = L/h$$

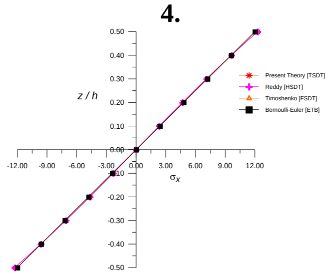
The percentage error in the results obtained by present and other theories with respect to the corresponding results obtained by the theory of elasticity is calculated as follows:

$$\%\text{error} = \frac{\text{value by a particular model} - \text{value by exact elasticity solution}}{\text{value by exact elasticity solution}} \times 100$$

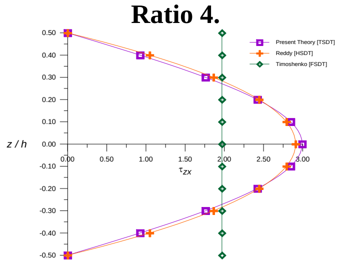
#Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]



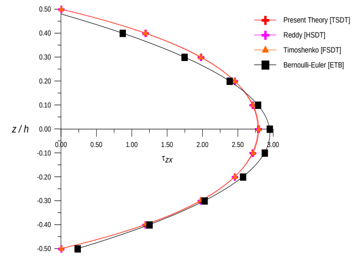
Graph 1. Variation Of Axial Displacement ($\bar{u}$) Through The Thickness Of Simply Supported Beam At(X = 0, Z) When Subjected To Uniformly Distributed Load For Aspect Ratio 4.



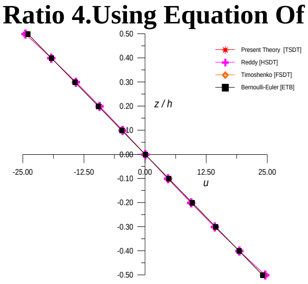
Graph.2: Variation Of Axial Bending Stress $(\bar{\sigma}_x)$ Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Uniformly Distributed Load For Aspect Ratio 4.



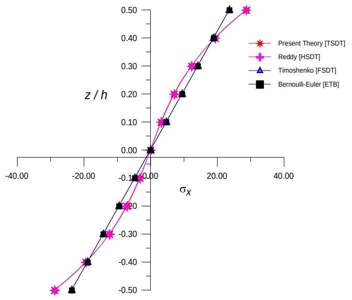
Graph 3: Variation of transverse shear stress $\bar{\tau}_{zx}^{CR}$ through the thickness of simply supported beam at (x=0,z) when subjected to uniformly distributed load for aspect ratio 4.using constitutive relation



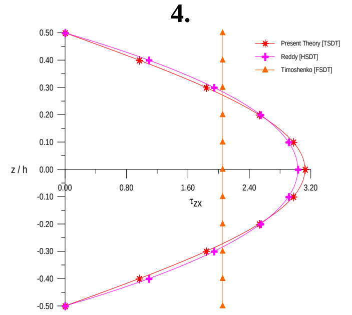
Graph .4: Variation Of Transverse Shear Stress $\bar{\tau}_{zx}^{EE}$ Through The Thickness Of Simply Supported Beam At (X=0,Z) When Subjected To Uniformly Distributed Load For Aspect Ratio 4.Using Equation Of Equilibrium.



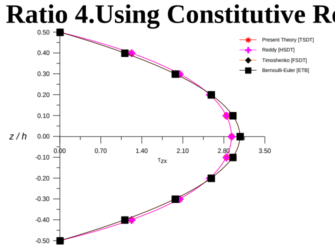
Graph .5: Variation Of Axial Displacement ($\bar{u}$) Through The Thickness Of Simply Supported Beam At (X=0, Z) When Subjected To Concentrated Point Load For Aspect Ratio 4.



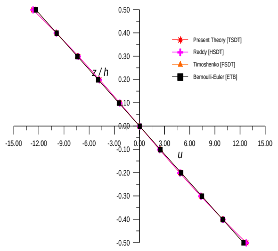
Graph .6: Variation Of Axial Bending Stress $\bar{\sigma}_x$ Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Concentrated Load For Aspect Ratio 4.



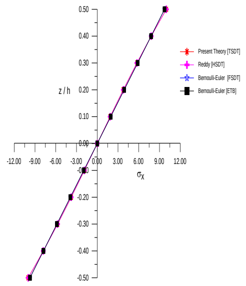
Graph .7: Variation Of Transverse Shear $\left(\bar{\tau}_{zx}^{CR}\right)$ Stress Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Concentrated Load For Aspect Ratio 4.Using Constitutive Relation



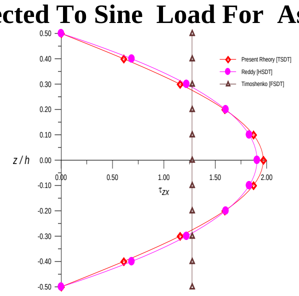
Graph .8: Variation Of Transverse Shear $\left(\bar{\tau}_{zx}^{EE}\right)$ Stress Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Concentrated Load For Aspect Ratio 4.Using Equilibrium Equation



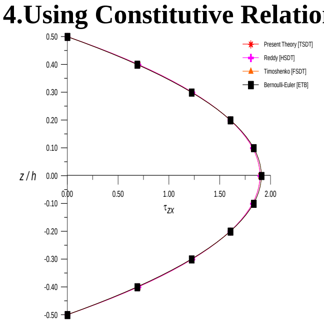
Graph .9: Variation Of Axial Displacement ($\bar{u}$) Stress Through The Thickness Of Simply Supported Beam At (X=0,Z= $\pm$H/2 When Subjected To Sine Load For Aspect Ratio 4.



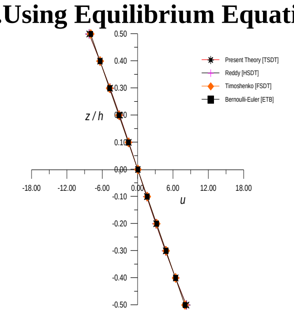
Graph .10: Variation Of Axial Bending Stress ($\bar{\sigma}_x$)Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Sine Load For Aspect Ratio 4.



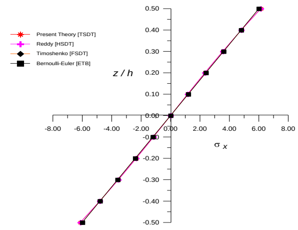
Graph .11: Variation Of Transverse Shear Stress $\bar{\tau}_{zx}^{CR}$ Through The Thickness Of Simply Supported Beam At (X=0,Z) When Subjected To Sine Load For Aspect Ratio 4.Using Constitutive Relation



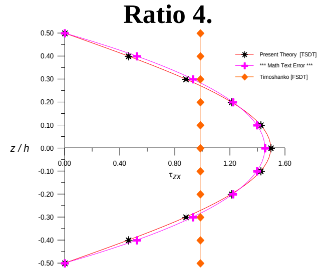
Graph 12: Variation Of Transverse Shear Stress $\bar{\tau}_{zx}^{EE}$ Through The Thickness Of Simply Supported Beam At (X=0,Z) When Subjected To Sine Load For Aspect Ratio 4.Using Equilibrium Equation



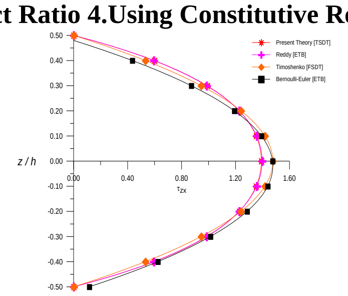
Graph 13: Variation Of Axial Displacement U Through The Thickness Of Simply Supported Beam At (X=L,Z) When Subjected To Linearly Varying Load For Aspect Ratio 4.



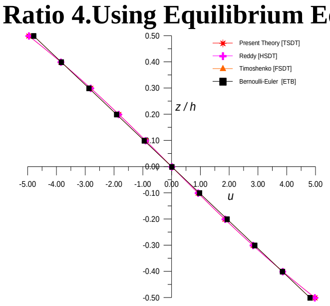
Graph 14: Variation Of Axial Bending Stress $\bar{\sigma}_x$ Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Linearly Varying Load For Aspect Ratio 4.



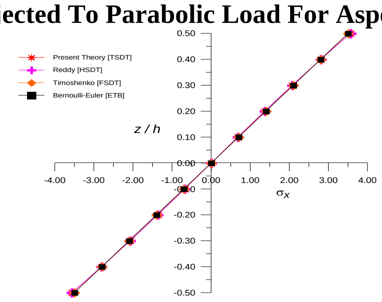
Graph 15: Variation Of Transverse Shear Stress $\bar{\tau}_{zx}^{CR}$ Through The Thickness Of Simply Supported Beam At (X=0,Z) When Subjected To Linearly Varying Load For Aspect Ratio 4.Using Constitutive Relation



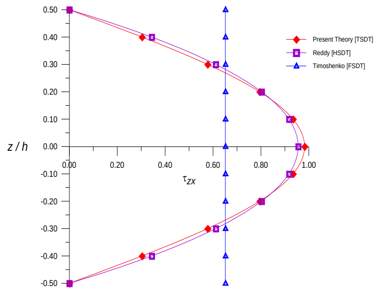
Graph 16: Variation Of Transverse Shear Stress $\bar{\tau}_{zx}^{EE}$ Through The Thickness Of Simply Supported Beam At (X=0, Z) When Subjected To Linearly Varying Load For Aspect Ratio 4.Using Equilibrium Equation.



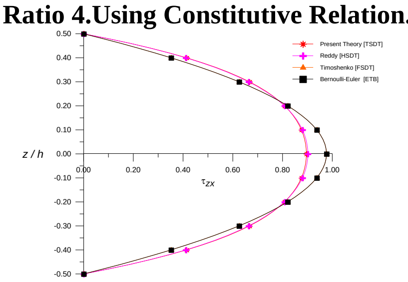
Graph 17: Variation Of Axial Displacement 'U' Through The Thickness Of Simply Supported Beam At (X=L,Z) When Subjected To Parabolic Load For Aspect Ratio 4.



Graph 18: Variation Of Axial Bending Stress $\bar{\sigma}_x$ Through The Thickness Of Simply Supported Beam At (X=L/2,Z) When Subjected To Parabolic Load For



Graph 19: Variation Of Transverse Shear Stress $\bar{\tau}_{zx}^{CR}$ Through The Thickness Of Simply Supported Beam At (X=0, Z) When Subjected To Parabolic Load For Aspect Ratio 4.Using Constitutive Relation.



Graph 20: Variation Of Transverse Shear Stress Through The Thickness Of Simply Supported Beam At (X=0, Z) When Subjected To Parabolic Load For Aspect Ratio 4.Using Equilibrium Equation.

Discussion of results:

The results obtained from the present theory are compared with those of the elementary beam theory (ETB), first order shear deformation theory of Timoshenko, higher order theories of Ghugal , Reddy, refined theory of beam by Ghugal and exact elasticity solutions given by Timoshenko and Goodier,

Example 1: Simply supported isotropic beam subjected to uniformly distributed load

Comparison of axial displacement transverse displacement, axial bending stress and transverse shear stress for simply supported isotropic beam subjected to uniformly distributed load is presented in Tables 4.1 through 4.5

a) The axial displacement predicted by present theory for simply supported isotropic beam subjected to uniformly distribute load is in error by 4.652% and 0.745% for aspect ratio 4 and 10 respectively

b) The axial displacement predicted by HSDT for simply supported isotropic beam subjected to uniformly distribute load is in error by 4.456 and 0.709% for aspect ratio 4 and 10 respectively

c) Transverse displacement predicted by present theory overestimate the Transverse displacement by 1.12and0.112% for aspect ratio 4 and 10 respectively whereas ETB underestimate the same.

d) Present theory and HSDT of Reddy is in excellent agreement with exact solution for all aspect ratios (See Table 4.2).

e) The deflection predicted by ETB is lower than Present TSDT and Exact elasticity solution due to neglect of effect of shear deformation in ETB (See Table 4.2).

f) Axial bending stress predicted by present theory and HSDT of Reddy good agreement with exact solution.

g) ETB and FSDT under estimate the axial bending stress by 1.693 and 0.264% for aspect ratio 4 and 10 respectively (See Table 4.4).

h) Transverse shear stress $\bar{\tau}_{zx}^{CR}$ predicted by present theory close agreement with exact solution while HSDT, FSDT and ETB underestimate the same (See Table 4.4).

i) The Transverse shear stress $\bar{\tau}_{zx}^{EE}$ predicted by present theory for simply supported isotropic beam subjected to uniformly distribute load is in error by –7.233% and –2.733% for aspect ratio 4 and 10 respectively (See Table 4.5)

Example 2: Simply Supported Beam with a concentrated load

Comparison of axial displacement, transverse displacement, axial bending stress and transverse shear stress for simply supported isotropic beam subjected to concentrated load is presented in Table 6 through 10

a) For simply supported isotropic beam with concentrated load for axial displacement, percentage error is not quoted due to non-availability/non existence of exact solution.

b) Present theory over estimate Transverse displacement by 2.9706 and 0.288 % for aspect ratio 4 and 10 respectively, also HSDT, FSDT, over estimate the same. While ETB underestimate the same.

c) Present theory overestimates the axial bending stress 12.838 and 4.468 for aspect ratio 4 and 10 respectively. While FSDT and ETB underestimate the same.

d) The examination of table 4.9 revels that present theory and HSDT theory overestimate transverse shear stress, while FSDT underestimate the same.

e) For simply supported isotropic beam with concentrated load for Transverse shear stress $\bar{\tau}_{zx}^{EE}$ percentage error is not quoted due to non-availability/non existence of exact solution.

Example 3: Simply Supported Beam subjected to single sine load

Comparison of axial displacement, transverse displacement, axial bending stress and transverse shear stress for simply supported isotropic beam subjected to single sine load is presented in Table 11 through 15

a) The examination of table 4.11 reveals that maximum axial deflection obtained by present theory overestimate the value by 3.569and 0.746 % for aspect ratio 4 and 6 respectively as compared to exact solution is given by Ghugal and also over estimate the value for HSDT, FSDT and ETB

b) Transverse displacement predicted by present theory, HSDT and FSDT close agreement with exact solution. ETB under estimate the transverse displacement by 12.693 and 2.308% for aspect ratio 4 and 6 respectively.

c) Present theory over estimate the axial bending stress by 0.4495 and0.250% for aspect ratio 4 and 6 respectively where FSDT and ETB Underestimate the same.

d) HSDT of Reddy show excellent results of transverse shear stress for all aspect ratio.

e) The examination of Table 4.14 reveals that transverse shear stress obtained by present theory overestimate the value by 3.495% and 3.224% for aspect ratio 4 and 10 respectively. While HSDT of Reddy shows close agreement with exact solution

f) For simply supported isotropic beam subjected to single sine load percentage error for Transverse shear stress $\bar{\tau}_{zx}^{EE}$ is not quoted due to non-availability/non existence of exact solution (See Table 15)

Example 4: Simply Supported Beam subjected to linearly varying load

Comparison of axial displacement, transverse displacement axial bending stress and transverse shear stress for simply supported isotropic beam subjected to linearly varying load. is presented in Table 16 through 20

a) The examination of Table 16 reveals that axial displacement obtained by present Theory, HSDT of Reddy, FSDT and ETB over estimate the value for aspect ratio 4 and 10 as compared to Theory of elasticity given by Timoshenko and Goodier

b) Transverse displacement predicted by present theory and HSDT of Reddy is in good agreement with exact solution. FSDT over estimate the value by 1.2265% and 0.222% for aspect ratio 4 and 10 respectively for the same, while ETB underestimate the same.

c) Axial bending stress predicted by present theory and HSDT of Reddy is in excellent agreement with exact solution. ETB and FSDT under estimate the same.

d) The examination of Table.19 reveals that present theory, HSDT, FSDT, underestimate Transverse shear stress for all aspect ratio as compared to exact solution.

e) For simply supported isotropic beam subjected to linearly varying load for Transverse shear stress $\bar{\tau}_{zx}^{EE}$ percentage error is not quoted due to non-availability/non existence of exact solution

Example 5: Simply Supported Beam subjected to parabolic load

Comparison of axial displacement, transverse displacement axial bending stress $\bar{\sigma}_x$ at and transverse shear stress for Simply supported isotropic beam subjected to parabolic load is presented in Table no 21through 25.

a) For simply supported isotropic beam with parabolic load axial displacement $\bar{u}$, transverse displacement $\bar{w}$, axial bending stress $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{zx}$ percentage error is not quoted due to non availability/non existence of exact solution

Conclusions

A refined shear deformation theory for bending of thick isotropic beam is presented and results obtained are discussed with those of other theories. The present theory has several features as given below:

a) It is a displacement based, refined shear deformation theory which includes the transverse shear effects.

b) The number of unknown variables is same as that in FSDT.

c) The shear deformation in the beam is properly accounted for.

d) Constitutive relations are satisfied in respect of axial stress and transverse shear stress.

e) Transverse shear stress satisfies zero shear stress boundary conditions on top and bottom surfaces of the beam perfectly.

f) The theory obviates the need of shear correction factor.

The Theories having above features is used for static flexural and free flexural vibration analysis of thick isotropic beam. From this analysis, following conclusions are drawn.

1) Present theory gives good result in respect of axial displacements.

2) The use of present theory gives good result in respect of transverse displacements.

3) The results of axial stress obtained by present theory are matching with results of available higher-order and refined shear deformation theories.

4) The governing differential equations and the associated boundary conditions are variation ally consistent

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Table 1 Comparison of axial displacement $\bar{u}$ at ($x = L$, $z = \pm h/2$) for isotropic beam subjected to uniformly distributed load.

Theory	Model	S = 2	% error#	S = 4	%error#	S = 10	% error#
Present	TSDT	2.259	2.682	16.535	4.652	251.35	0.745
Reddy	HSDT	2.245	2.045	16.504	4.456	251.27	0.709
Timoshenko	FSDT	2.000	-9.091	16.000	1.265	250.00	0.200
Bernoulli-Euler	ETB	2.000	-9.091	16.000	1.265	250.00	0.200
Timoshenko and Goodier	Elasticity	2.200	0.0	15.800	0.0	249.50	0.0

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table 2 Comparison of transverse displacement $\bar{w}$ at ($x = L/2$, z = 0) for isotropic beam subjected to uniformly distributed load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	2.529	3.098	1.805	1.120	1.601	1.12
Reddy	HSDT	2.532	3.221	1.806	1.176	1.602	0.250
Timoshenko	FSDT	2.538	3.465	1.806	1.176	1.602	0.250
Bernoulli-Euler	ETB	1.563	-36.282	1.563	-12.437	1.563	-2.190
Timoshenko and Goodier	Elasticity	2.453	0.0	1.785	0.0	1.598	0.0

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table 3 Comparison of axial bending stress $\bar{\sigma}_x$ at ($x = L/2$, $z = \pm h/2$) for isotropic beam subjected to uniformly distributed load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	3.278	2.438	12.280	0.656	75.284	0.112
Reddy	HSDT	3.261	1.960	12.263	0.516	75.268	0.090
Timoshenko	FSDT	3.000	-6.25	12.000	-1.693	75.000	-0.264
Bernoulli-Euler	ETB	3.000	-6.25	12.000	-1.693	75.000	-0.264
Timoshenko and Goodier	Elasticity	3.200	0.0	12.200	0.0	75.200	0.0

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table .4 Comparison of transverse shear stress $\bar{\tau}_{zx}^{CR}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to uniformly distributed load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.451	-3.267	2.993	-0.233	7.591	1.2133
Reddy	HSDT	1.415	-5.667	2.908	-3.067	7.361	-1.853
Timoshenko	FSDT	0.984	-34.4	1.969	-34.367	4.922	-34.373
Bernoulli-Euler	ETB	---		---		---	
Timoshenko and Goodier	Elasticity	1.500	0.0	3.000	0.0	7.500	0.0

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table .5 Comparison of transverse shear stress $\bar{\tau}_{zx}^{EE}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to uniformly distributed load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.250	-16.667	2.783	-7.233	7.295	-2.733
Reddy	HSDT	1.262	-15.867	2.795	-6.833	7.304	-2.61
Timoshenko	FSDT	1.477	-1.533	2.953	-1.567	7.383	-1.56
Bernoulli-Euler	ETB	1.477	-1.533	2.953	-1.567	7.383	-1.56
Timoshenko and Goodier	Elasticity	1.500	0.0	3.000	0.0	7.500	0.0

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table .6 Comparison of axial displacement $\bar{u}$ at ($x = L$, $z = \pm h/2$) for isotropic beam subjected to concentrated load.

Theory	Model	S = 2	S = 4	S = 10
Present	TSDT	3.2776	24.5591	376.4214
Reddy	HSDT	3.2611	24.5263	376.3385
Timoshenko	FSDT	3.0001	24.0007	375.0122
Bernoulli-Euler	ETB	3.0001	24.0007	375.0109
Timoshenko and Goodier	Elasticity	---	---	---

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table.7 Comparison of transverse displacement $\bar{w}$ at ($x = L/2$, $z = 0$) for isotropic beam subjected to concentrated load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	4.3257	7.63	2.9706	1.995	2.5764	0.288
Reddy	HSDT	4.3399	7.899	2.9726	20.635	2.5765	0.292
Timoshenko	FSDT	4.4198	9.978	2.9799	23.142	2.5768	0.304
Bernoulli-Euler	ETB	2.5000	-37.792	2.5000	-14.1631	2.5000	-2.686
Timoshenko and Goodier	Elasticity	4.0188	0.00	2.9125	0.00	2.5690	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table .8 Comparison of axial bending stress $\bar{\sigma}_x$ at ($x = L/2, z=\pm h/2$) for isotropic beam subjected to concentrated load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	9.3101	67.907	28.7619	12.838	154.3242	4.468
Reddy	HSDT	9.3469	68.571	28.6790	12.513	154.0091	4.255
Timoshenko	FSDT	5.9065	6.523	23.6261	-7.311	147.6634	-0.041
Bernoulli-Euler	ETB	5.9065	6.523	23.6261	-7.311	147.6630	-0.0412
Timoshenko and Goodier	Elasticity	5.5448	0.0	25.4896	0.0	147.7239	0.0

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table .9 Comparison of transverse shear stress $\bar{\tau}_{zx}$ at (x=0, z=0) for isotropic beam subjected to concentrated load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.5532	--	3.1253	4.177	7.8912	5.216
Reddy	HSDT	1.5059	--	3.0319	1.063	7.6519	2.025
Timoshenko	FSDT	1.0244	--	2.0489	-31.703	5.1223	-31.702
Bernoulli-Euler	ETB	---	--	---	--	---	--
Timoshenko and Goodier	Elasticity	---	--	3.000	0.00	7.500	0.00

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table.10 Comparison of transverse shear stress $\bar{\tau}_{zx}^{EE}$ at (x = 0, z= 0) for isotropic beam subjected to concentrated load.

Theory	Model	S = 2	% Erro	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.4347	--	2.9283	--	7.5636	--
Reddy	HSDT	1.4290	--	2.9284	--	7.5733	--
Timoshenko	FSDT	1.5367	--	3.0733	--	7.6834	--
Bernoulli-Euler	ETB	1.5367	--	3.0733	--	7.6834	--
Timoshenko and Goodier	Elasticity	---	--	---	--	---	--

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table .11 Comparison of axial displacement $\bar{u}$ at (x = 0, $z = \pm h / 2$) for isotropic beam subjected to Sine load.

Theory	Model	S = 2	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.7225	12.7359	3.569	194.3895	0.746
Reddy	HSDT	1.7124	12.7150	3.399	194.3370	0.719
Timoshenko	FSDT	1.5481	12.3846	0.712	193.5098	0.290
Bernoulli-Euler	ETB	1.5481	12.3846	0.712	193.5092	0.289
Ghugal	Exact	---	12.2970	0.00	192.9500	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [65]

Table .12 Comparison of transverse displacement $\bar{w}$ at ($x = L/2$, z = 0) for isotropic beam subjected to Sine load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	2.0138	--	1.4288	1.262	1.2635	0.198
Reddy	HSDT	2.0163	--	1.4291	1.283	1.2635	0.198
Timoshenko	FSDT	2.0223	--	1.4295	1.311	1.2635	0.198
Bernoulli-Euler	ETB	1.2319	--	1.2319	-12.693	1.2319	-2.308
Ghugal	Exact	---	--	1.4110	0.00	1.2610	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [65]

Table .1 3 Comparison of axial bending stress $\bar{\sigma}_x$ at ($x = L/2, z = \pm$ h/2) for isotropic beam subjected to Sine load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	2.7057	--	10.0028	0.449	61.0693	0.250
Reddy	HSDT	2.6898	--	9.9864	0.285	61.0528	0.223
Timoshenko	FSDT	2.4317	--	9.7268	-2.322	60.7929	-0.204
Bernoulli-Euler	ETB	2.4317	--	9.7268	-2.322	60.7927	-0.204
Ghugal	Exact	---	--	9.9580	0.00	60.9170	0.00

 # Percentage error quoted is with respect to the corresponding value of exact Ghugal solution [65]

Table .1 4 Comparison of transverse shear stress $\bar{\tau}_{zx}$ at (*x* = 0, *z* = 0) for isotropic beam subjected to Sine load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.9764	--	1.9664	3.495	4.9258	3.244
Reddy	HSDT	0.9477	--	1.9062	0.326	4.7732	0.0461
Timoshenko	FSDT	0.6366	--	1.2732	-32.989	3.1831	-33.282
Bernoulli-Euler	ETB	---	--	---	--	---	--
Ghugal	Exact	---	--	1.900	0.00	4.7710	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticitySolution [65]

Table.1 5 Comparison of transverse shear stress $\bar{\tau}_{zx}^{EE}$ at (*x* = 0, *z* = 0) for isotropic beam subjected to Sine load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.9263	--	1.8955	---	4.7689	--
Reddy	HSDT	0.9296	--	1.8971	---	4.7696	--
Timoshenko	FSDT	0.9549	--	1.9099	--	4.7747	--
Bernoulli-Euler	ETB	0.9549	--	1.9099	--	4.7746	--
Ghugal	Exact	---	--	---	--	---	--

 # Percentage error quoted is with respect to the corresponding value of exact elasticitySolution [65]

Table 16 Comparison of axial displacement $\bar{u}$ at (*x* = *L*, *z* = ± *h/2*) for isotropic beam subjected to linearly varying load.

Theory	Model	S = 2	% error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.1295	2.682	8.2675	4.652	125.675	0.7415
Reddy	HSDT	1.1225	2.045	8.2520	4.456	125.635	0.709
Timoshenko	FSDT	1.0000	-9.091	8.0000	1.266	125.000	0.200
Bernoulli-Euler	ETB	1.0000	-9.091	8.0000	1.266	125.000	0.200
Timoshenko and Goodier	Elasticity	1.1000	0.00	7.9000	0.00	124.750	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table 17 Comparison of axial displacement transverse displacement $\bar{w}$ at (*x* = *L*/2, z = 0) for isotropic beam subjected to linearly varying load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.2645	3.098	0.9025	1.1204	0.8005	0.1877
Reddy	HSDT	1.2660	3.2205	0.9030	1.1765	0.8010	0.250
Timoshenko	FSDT	1.2690	3.465	0.9030	1.1765	0.8010	0.250
Bernoulli-Euler	ETB	0.7815	-36.282	0.7815	-12.437	0.7815	-2.190
Timoshenko and Goodier	Elasticity	1.2265	0.00	0.8925	0.00	0.7990	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table 18 Comparison of axial bending stress $\bar{\sigma}_x$ at (*x* = *L*/2, z = ± *h/2*) for isotropic beam subjected to linearly varying load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	1.6390	2.4375	6.1400	0.6557	37.642	0.112
Reddy	HSDT	1.6310	1.938	6.1315	0.5164	37.634	0.090
Timoshenko	FSDT	1.5000	-6.250	6.0000	-1.6393	37.500	-0.266
Bernoulli-Euler	ETB	1.5000	-6.250	6.0000	-1.6393	37.500	-0.266
Timoshenko and Goodier	Elasticity	1.6000	0.00	6.1000	0.00	37.600	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table 19 Comparison of for transverse shear stress $\bar{\tau}_{zx}$ at (x = 0, z = 0) isotropic beam subjected to linearly varying load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.7255	-3.267	1.4965	-0.233	3.7955	1.2133
Reddy	HSDT	0.7075	-5.667	1.4540	-3.067	3.6805	-1.853
Timoshenko	FSDT	0.6000	-20.00	1.2000	-20.00	3.0000	-20.00
Bernoulli-Euler	ETB	---	--	---	--	---	--
Timoshenko and Goodier	Elasticity	0.7500	0.00	1.5000	0.00	3.7500	0.00

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table .20 Comparison of for transverse shear stress $\bar{\tau}_{zx}^{EE}$ at (x = 0, z = 0) isotropic beam subjected to linearly varying load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.6250	--	1.3195	--	3.6475	--
Reddy	HSDT	0.6310	--	1.3975	--	3.6520	--
Timoshenko	FSDT	0.7385	--	1.4765	--	3.6915	--
Bernoulli-Euler	ETB	0.7500	--	1.4765	--	3.6915	--
Timoshenko and Goodier	Elasticity	--	--	--	--	--	--

 # Percentage error quoted is with respect to the corresponding value of exact elasticity solution [69]

Table 21 Comparison of axial displacement $\bar{u}$ at (x = L, z= ± h/2) for isotropic beam subjected to parabolic load.

Theory	Model	S = 2	% Error#	S = 4	z% Error#	S = 10	% Error#
Present	TSDT	0.6840	--	4.9754	--	75.4467	--
Reddy	HSDT	0.6797	--	4.9655	--	75.4204	--
Timoshenko	FSDT	0.6000	--	4.7999	--	74.9991	--
Bernoulli-Euler	ETB	0.6000	--	4.7999	--	74.9988	--
Timoshenko and Goodier	Elasticity	--	--	--	--	--	--

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table.22 Comparison of transverse displacement $\bar{w}$ at (x = L/2, z =0), for isotropic beam subjected to parabolic load.

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.7457	--	0.5344	--	0.4749	--
Reddy	HSDT	0.7464	--	0.5346	--	0.4749	--
Timoshenko	FSDT	0.7480	--	0.5346	--	0.4749	--
Bernoulli-Euler	ETB	0.4635	--	0.4635	--	0.4635	--
Timoshenko and Goodier	Elasticity	--	--	--	--	--	--

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table 23 Comparison of axial bending stress $\bar{\sigma}_x$ at (x = L/2, z = ± h/2) for isotropic beam subjected to parabolic load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.9452	--	3.5622	--	21.9480	--
Reddy	HSDT	0.9410	--	3.5667	--	21.9438	--
Timoshenko	FSDT	0.8750	--	3.5002	--	21.8762	--
Bernoulli-Euler	ETB	0.8750	--	3.5002	--	21.8762	--
Timoshenko and Goodier	Elasticity	--	--	--	--	--	--

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table 24 Comparison of transverse shear stress $\bar{\tau}_{zx}^{CR}$ at (x = 0, z = 0) for isotropic beam subjected to parabolic load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.4704	--	0.9820	--	2.5062	--
Reddy	HSDT	0.4598	--	0.9552	--	2.4307	--
Timoshenko	FSDT	0.3255	--	0.6511	--	1.6277	--
Bernoulli-Euler	ETB	--	--	--	--	--	--
Timoshenko and Goodier	Elasticity	--	--	--	--	--	--

 # Percentage error is not quoted due to non availability/non-existence of exact solution

Table 25 Comparison of transverse shear stress $\bar{\tau}_{zx}^{EE}$ at (x = 0, z = 0) for isotropic beam subjected to parabolic load

Theory	Model	S = 2	% Error#	S = 4	% Error#	S = 10	% Error#
Present	TSDT	0.3838	--	0.8957	--	2.3991	--
Reddy	HSDT	0.3885	--	0.9016	--	2.4035	--
Timoshenko	FSDT	0.4883	--	0.9766	--	2.4416	--
Bernoulli-Euler	ETB	0.4883	--	0.9766	--	2.4416	--
Timoshenko and Goodier	Elasticity	--	--	--	--	--	--

 # Percentage error is not quoted due to non availability/non-existence of exact solution