# Existence results for fractional semilinear integrodifferential systems with infinite delay in banach spaces 

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#### Abstract

In this paper, we prove the existence of mild solutions for fractional semilinear integrodifferential systems with infinite delay in $\alpha$-norm in Banach spaces. The results are obtained by using Banach contraction principle and Schauder's fixed point theorem. In the end, we give an example to illustrate the applications of the abstract results.


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## Introduction

The study of fractional differential equations has emerged as a new branch of applied mathematics, which has been used for construction and various fields of engineering and sciences. We can find numerous applications in viscoelascity, electrochemistry, control, porous media, electromagnetic, etc. (see for instance [9, 13, 19]). In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al. [16], Miller et al. [20], Podlubny [23], Lakshmikantham et al. [18] and the papers [1-4, $7,8,12,15,17,26-30]$, where numerous properties of their solutions are studied and detailed bibliographies are given.

The starting point of this paper is the works in papers [5, 14, 21, 24]. Especially, the authors of [5] investigated the existence results for semilinear fractional order integrodifferential equations with nonlocal conditions in Banach Space $X$ :

$$
\begin{gathered}
D^{q} x(t)+A x(t)=f\left(t, x(t), \int_{0}^{t} e(t, s, x(s)) d s\right), \quad t \in J:=[0, T] \\
x(0)=g(x)+x_{0}
\end{gathered}
$$

by using Banach contraction principle and Sadovskii's fixed point theorem. And in [14], the authors studied the following fractional order semilinear integrodifferential equations with infinite delay in $\alpha$-norm:

$$
\begin{gathered}
D^{q} x(t)=A x(t)+f\left(t, x_{t} \int_{0}^{t} a\left(t, s, x_{s}\right) \mathrm{d} s\right), \quad t \in J:=[0, T], \\
x(t)=\phi(t) \in \mathcal{B}, t \in(-\infty, 0]
\end{gathered}
$$

by using the standard fixed point theorems and $\mathcal{B}$ is a phase space axioms which is introduced by Hale and Kato [11].

Motivated by the above mentioned works [5, 24, 25], the main purpose of this paper is to discuss the following fractional semilinear integrodifferential systems with infinite delay: $D^{q} x(t)=A x(t)+f\left(t, x_{t}, \int_{0}^{t} k\left(t, s, x_{s}\right) d s\right), \quad t \in J:=[0, b]_{,}(1.1)$
$x(t)=\phi(t) \in \mathcal{B}_{h}, t \in(-\infty, 0]$,
where $b>0,0<q<1, A$ is infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t), t \geq 0\}$ on a Banach space $X$, the derivative $D^{q}$ is understood here in the Caputo derivative sense and $k, f$ are functions specified later. The histories $x_{t}:(-\infty, 0] \rightarrow X$, defined by $x_{t}(\theta)=x(t+\theta), \theta \leq 0$, belongs to some abstract phase space $\mathcal{B}_{h}$.

The rest of this paper is organized as follows. In section 2, we give some preliminaries. In section 3, we study the existence and uniqueness results of mild solutions for the problem (1.1) - (1.2). At last, an example is given to demonstrate the applicability of our results in section 4.

## Preliminaries

At first, we present the abstract phase space $\mathcal{B}_{h}$, which has been used in [6]. Assume that $h:(-\infty, 0] \rightarrow(0,+\infty)$ is a continuous function with $l=\int_{-\infty}^{0} h(t) d t<+\infty$. For any $a>0$, we define
$\mathcal{B}=\{\psi:[-a, 0] \rightarrow X$ suchthat $\psi(t)$ is bounded and measurable $\}$, and equip the space $\mathcal{B}$ with the norm

$$
\|\psi\|_{[-a, 0]}=\sup _{s \in[-a, 0]}|\psi(s)|, \quad \forall \psi \in \mathcal{B}
$$

Let us define

$$
\mathcal{B}_{h}=\left\{\psi:(-\infty, 0] \rightarrow X \text { suchthat for any } c>0,\left.\psi\right|_{[-c, 0]} \in \mathcal{B}\right.
$$ and $\left.\int_{-\infty}^{0} h(s)\|\psi\|_{[s, 0]} d s<+\infty\right\}$.

If $\mathcal{B}_{h}$ is endowed with the norm

$$
\|\psi\|_{\mathcal{B}_{h 2}}=\int_{-\infty}^{0} h(s)\|\psi\|_{[s, 0]} d s, \forall \psi \in \mathcal{B}_{h}
$$

then it is clear that $\left(\mathcal{B}_{h},\|\cdot\|_{\mathcal{B}_{h}}\right)$ is a Banach space.
Now we consider the space

[^0]$$
\mathcal{B}_{h}^{\prime}=\left\{x:(-\infty, b] \rightarrow X \text { suchthat }\left.x\right|_{J} \in C(J, X), x(t)=\phi(t) \in B_{h}\right\} .
$$

Set $\|\cdot\|_{b}$ be a seminorm in $\mathcal{B}_{h}^{\prime}$ defined by
$\|x\|_{b}=\|\phi\|_{\mathcal{B}_{h}}+\sup \{|x(s)|: s \in[0, b]\}, x \in \mathcal{B}_{h}^{\prime}$.
To set the frame work for our main results, we will make use of the following definitions, notations and preliminary facts which are used throughout this paper.

Let $X$ be a Banach space provided with norm $\|\cdot\|$. Let A: $D(A) \rightarrow X$ is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ of uniformly bounded linear operators on $X$, that is to say, there exists some constant $M \geq 1$ such that $\|T(t)\| \leq M$ for every $t \in J$. Let $0 \in \rho(A)$, then it is possible to define the fractional power $A^{\alpha}$, for $0<\alpha \leq 1$, as a closed linear operator on its domain $D\left(A^{\alpha}\right)$. Further more, the subspace $D\left(A^{\alpha}\right)$ is dense in $X$ and the expression

$$
\|x\|_{\alpha}=\left\|A^{\alpha} x\right\|, \quad x \in D\left(A^{\alpha}\right)
$$

defines a norm on $D\left(A^{\alpha}\right)$. Hereafter, let $X_{\alpha}$ be the Banach space $D\left(A^{\alpha}\right)$ endowed with the norm $\|x\|_{\alpha}$. For $0<\beta \leq \alpha \leq 1, X_{\alpha} \rightarrow X_{\beta}$ and the imbedding is compact whenever the resolvent operator of $A$ is compact. Also for every $0<\alpha \leq 1$, there exists a positive constant $M_{\alpha \alpha}$ such that

$$
\left\|A^{\alpha} T(t)\right\| \leq \frac{M_{\alpha}}{t^{\alpha}}, \quad 0<t \leq b
$$

Let $C\left(J, X_{\alpha}\right)$ be the Banach space of all continuous functions from $J$ into $X_{\alpha}$ with the norm
$\|x\|_{J}=\sup _{t \in J}\|\cdot\|_{\alpha^{*}}$
Let us recall the following definitions [16, 23].
Definition 2.1.The fractional integral of order $\alpha$ with the lower limit zero for a function $f$ is defined as

$$
I^{\alpha}=\frac{1}{\Gamma(p)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} d s, \quad t>0, \alpha>0
$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.
Definition 2.2.The Caputo derivative of order $\boldsymbol{\alpha}$ for a function $f:[0, \infty) \rightarrow R$ can be written as

$$
D^{\alpha} f(t)=D^{\alpha}\left(f(t)-\sum_{k=0}^{n-1} \frac{t^{k}}{k!} f^{(k)}(0)\right), \quad t>0, n-1<\alpha<n
$$

Definition2.3 ([25]). A continuous function $x:(-\infty, b] \rightarrow X_{\alpha}$ is said to be a mild solution of the system (1.1) - (1.2) if $x(t)=\phi(t) \in \mathcal{B}_{h}$ on $(-\infty, 0]$ and the following integral equation
$x(t)=S_{q}(t) \phi(t)+\int_{0}^{s}(t-s)^{q-1} T_{q}(t-s) f\left(s_{1} x_{v} \int_{0}^{s} k\left(s_{j}, r_{j} x_{n}\right) d r\right) d s, \quad t \in J$,
is satisfied, where

$$
\begin{gathered}
S_{q}(t)=\int_{0}^{\infty} \xi_{q}(\theta) T\left(t^{q} \theta\right) d \theta, \\
T_{q}(t)=q \int_{0}^{\infty} \theta \xi_{q}(\theta) T\left(t^{q} \theta\right) d \theta, \\
\xi_{q}(\theta)=\frac{1}{q} \theta^{-1-\frac{1}{q}} \bar{w}_{q}\left(\theta^{-\frac{1}{q}}\right) \geq 0, \\
\bar{w}_{q}(\theta)=\frac{1}{\pi} \sum_{n=1}^{\infty}(-1)^{n-1} \theta^{-n q-1} \frac{\Gamma(n q+1)}{n!} \sin (n \pi q), \theta \in(0, \infty), \\
\xi_{q} \text { is a probability density function defined on }(0, \infty), \text { that is } \\
\xi_{q}(\theta) \geq 0, \quad \theta \in(0, \infty) \text { and } \int_{0}^{\infty} \xi_{q}(\theta) d \theta=1 .
\end{gathered}
$$

Lemma 2.1 ([25]). The above defined operators $S_{q}$ and $T_{q}$ have the following properties:
(i).For any fixed $t \geq 0, S_{q}$ and $T_{q}$ are linear and bounded operators, i.e., for any $x \in X$,
$\left\|S_{q}(t) x\right\| \leq M\|x\|, \quad\left\|T_{q}(t) x\right\| \leq \frac{q M}{r(q+1)}\|x\|$.
(ii). $\left\{S_{q}(t), t \geq 0\right\}$ and $\left\{T_{q}(t), t \geq 0\right\}$ are strongly continuous.
(iii). For every $t>0, S_{q}(t)$ and $T_{q}(t)$ are also compact operators.
(iv). For any $x \in X, \alpha, \beta \in(0,1)$, we have

$$
\begin{array}{r}
A T_{q}(t) x=A^{1-\beta_{2}} T_{q}(t) A^{\beta} x, \quad t \in J, \\
\left\|A^{\alpha} T_{q}(t)\right\| \leq \frac{q M_{\alpha} \Gamma(2-\alpha)}{r(1+q(1-\alpha))} t^{-\alpha q}, \quad 0<t \leq b .
\end{array}
$$

Remark 2.1 ([25]). It is not difficult to verify that for $v \in[0,1]$.

$$
\int_{0}^{\infty} \theta^{v} \xi_{q}(\theta) d \theta=\int_{0}^{\infty} \theta^{-q v} \bar{w}_{q}(\theta) d \theta=\frac{\Gamma(1+v)}{\Gamma(1+q v)^{*}}
$$

By Definition 2.3, Lemma 2.1(i) and the above results. For fixed $t \geq 0$ and any $x \in X_{\alpha}$, we have
$\left\|S_{q}(t) x\right\|_{\alpha} \leq M\|x\|_{\alpha^{\prime}} \quad\left\|T_{q}(t) x\right\|_{\alpha} \leq \frac{q M}{r(q+1)}\|x\|_{\alpha^{*}}$.
Lemma 2.2([6]). Assume $x \in \mathcal{B}_{h}^{\prime}$, then for $t \in J, x_{t} \in \mathcal{B}_{h}$. Moreover,

$$
l|x(t)| \leq\left\|x_{t}\right\|_{\mathcal{B}_{h}} \leq\|\phi\|_{\mathcal{B}_{h}}+l \sup _{s \in[0, t]}|x(s)|
$$

where $l=\int_{-\infty}^{0} h(t) d t<+\infty$.
Lemma 2.3 ([10], Schauder's Fixed Point Theorem). If $K$ is a closed bounded and convex subset of a Banach space $X$ and $F: K \rightarrow K$ is completely continuous, then $F$ has a fixed point in $K$.

## Existence Results

In order to establish our result we assume the following conditions for $\alpha \in(0,1)$ :
$(\mathbf{H} 1) k: D:=\{(t, s) \in J \times J: s \leq t\} \times \mathcal{B}_{h} \rightarrow X_{\alpha} \quad$ is continuous and there exists a constant $L_{1}>0$ such that for all $(t, s) \in D, x, y \in \mathcal{B}_{h}$,
(i) $\left.\int_{0}^{t}[k(t, s, x)-k(t, s, y)] d s\right|_{\alpha} \leq L_{1}\|x-y\|_{\mathcal{B}_{h}}$,
(ii) $\left|\int_{0}^{t} k\left(t, s, x_{s}\right) d s\right|_{\alpha} \leq L_{1}\left(1+\left\|x_{t}\right\|_{\mathcal{B}_{h}}\right)$.
(H2) $f: J \times \mathcal{B}_{h} \times X_{\alpha} \rightarrow X_{\alpha}$ is continuous, and there exist positive constants $L_{2}, L_{3}$ such that for each $\left(t, x_{i}, y_{i}\right) \in J \times \mathcal{B}_{h} \times X_{\alpha}, \mathrm{i}=1,2$.
$\left|f\left(t, x_{1}, y_{1}\right)-f\left(t, x_{2}, y_{2}\right)\right|_{\alpha} \leq L_{2}\left(\left\|x_{1}-x_{2}\right\|_{\mathcal{B}_{h}}+\left|y_{1}-y_{2}\right|_{\alpha}\right)$.
(H3)The function $f: J \times \mathcal{B}_{h} \times X_{\alpha} \rightarrow X_{\alpha}$; $(t, \phi, x) \rightarrow f(t, \phi, x)$ is continuous with respect to $\phi$ and $x$ for a.e. $t \in J$ and is strongly measurable with respect to $t$ for any $(\phi, x) \in \mathcal{B}_{h} \times X_{\alpha}$. Forpositive number $r>0$, there exists a function $\alpha_{r} \in C\left(J, R_{+}\right)$with $\sup _{t \in J} \alpha_{r}(t)<+\infty$ such that,
$\left.\left.\sup \{\mid f, \phi, x)\right|_{\alpha}:\|\phi\|_{\mathbb{B}_{h}} \leq r_{y}|x| \leq r\right\} \leq \alpha_{r}(t)$, for a.e. $t \in J$
and

$$
\lim \inf _{r \rightarrow+\infty} \frac{\sup _{t \in J} \alpha_{r}(t)}{r}=\beta<\infty
$$

Our first existence result for the problem (1.1)-(1.2) is based on the Banach fixed point theorem.
Theorem 3.1 Assume that the conditions (H1) - (H2) and Lemma 2.2 are satisfied. Further, if $\phi \in X_{\alpha}$ and

$$
\begin{equation*}
\frac{u M b^{q}}{\Gamma(q+1)} L_{2}\left(1+L_{1}\right)<1 \tag{3.1}
\end{equation*}
$$

then the system (1.1)-(1.2) has a unique mild solution on the interval $(-\infty, b]$.
Proof: In order to obtain the existence of mild solutions for the system (1.1) - (1.2). Transform it into a fixed point problem.
We consider the operator $\Phi: \mathcal{B}_{h}^{\prime} \rightarrow \mathcal{B}_{h}^{l}$ defined by

$$
\Phi x(t)=\left(\begin{array}{l}
\phi(t), t \in(-\infty, 0]  \tag{3.2}\\
S_{q}(t) \phi(t)+\int_{0}^{s}(t-s)^{-1} T_{q}(t-s) f\left(s_{s} x_{v} \int_{0}^{s} k\left(s_{1}, \tau_{1}\right) d d r\right) d s, t \in J .
\end{array}\right.
$$

For $\tilde{\phi} \in \mathcal{B}_{h}$, we define $\tilde{\phi}$ by

$$
\tilde{\phi}(t)=\left(\begin{array}{ll}
\phi(t), & t \in(-\infty, 0] \\
S_{q}(t) \phi(t), & t \in J
\end{array}\right.
$$

then $\tilde{\phi} \in \mathcal{B}_{h}^{\prime}$. Let $x(t)=y(t)+\tilde{\phi}(t),-\infty<t \leq b$. It is easy to see that $x$ satisfies (2.1) if and only if $y$ satisfies $y_{0}=0$ and

$$
y(t)=\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s) f\left(s, y_{s}+\tilde{\phi}_{s}, \int_{0}^{s} k\left(s, \tau, y_{\mathrm{r}}+\tilde{\phi}_{\mathrm{T}}\right) d \tau\right) d s
$$

Let $\mathcal{B}_{h}^{\prime \prime}=\left\{y \in \mathcal{B}_{h}^{\prime}: y_{0}=0 \in \mathcal{B}_{h}\right\}$. For any $y \in \mathcal{B}_{h}^{\prime \prime}$,
$\|y\|_{b}=\left\|y_{0}\right\|_{\mathbb{B}_{h}}+\sup \{|y(s)|: 0 \leq s \leq b\}$

$$
\stackrel{\sup \{|y(s)|: 0 \leq s \leq b\}}{ }
$$

thus $\left(\mathcal{B}_{h}^{\prime \prime}\|\cdot\|_{b}\right)$ is a Banach space. Set $B_{r}=\left\{y \in \mathcal{B}_{h}^{\prime \prime}:\|y\|_{b} \leq r\right\} \quad$ for some $r \geq 0$, then $B_{r} \subseteq \mathcal{B}_{h}^{\prime \prime}$ is uniformly bounded, and for $y \in B_{r}$, from Lemma 2.2, we have

$$
\begin{align*}
& \left\|y_{t}+\tilde{\phi}_{t}\right\|_{\mathcal{B}_{h}} \leq\left\|y_{t}\right\|_{\mathcal{B}_{h}}+\left\|\tilde{\phi}_{t}\right\|_{\mathcal{B}_{h}} \\
& \leq l \sup _{s \in 0, t]}|y(s)|+\left\|y_{0}\right\|_{\mathcal{B}_{h}}+l \sup _{s \in 0, t]}|\tilde{\phi}(s)|+\left\|\tilde{\phi}_{0}\right\|_{\mathcal{B}_{h}} \\
& \leq l\left(r+M_{1}|\phi(0)|\right)+\|\phi\|_{\mathcal{B}_{h}}=r^{\prime} . \tag{3.3}
\end{align*}
$$

Define the operator $\bar{\Phi}: \mathcal{B}_{h}^{\prime \prime} \rightarrow \mathcal{B}_{h}^{\prime \prime}$ by

Then, the operator $\Phi$ has a fixed point is equivalent to $\bar{\Phi}$ has a fixed point, and so we turn to proving that $\bar{\Phi}$ has a fixed point. Now, we shall show that the operator $\bar{\Phi}$ is a contraction map on $\mathcal{B}_{h}^{\prime \prime}$. In fact, for each $y, \bar{y} \in \mathcal{B}_{h}^{\prime \prime}, t \in J$, we have

$$
|(\bar{\Phi} y)(t)-(\bar{\Phi} \bar{y})(t)|_{\alpha}
$$

$\leq \frac{M b^{q}}{\Gamma(q+1)} L_{2}\left[\left\|y_{s}-\bar{y}_{s}\right\|_{\mathcal{B}_{h}}+\left|\int_{0}^{s} k\left(s, \tau, y_{\mathrm{\tau}}\right) d \tau-\int_{0}^{s} k\left(s, \tau, \bar{y}_{\mathrm{\tau}}\right) d \tau\right|_{\alpha}\right]$

$$
\leq \frac{M b^{q}}{\Gamma(q+1)} \mathrm{L}_{2}\left(1+L_{1}\right)\left\|y_{s}-\bar{y}_{s}\right\|_{\mathcal{B}_{h}}
$$

$$
\begin{aligned}
& \leq \frac{M b^{q}}{\Gamma(q+1)} L_{2}\left(1+L_{1}\right)\left[\operatorname{limp}_{s \in[0, b]}|y(s)-\bar{y}(s)|_{\alpha}+\left\|y_{0}\right\|_{\mathcal{P}_{h}}+\left\|\bar{y}_{0}\right\|_{\mathcal{B}_{h}}\right] \\
& \quad \leq\left[\frac{l M b^{q}}{\Gamma(q+1)} L_{2}\left(1+L_{1}\right)\right]_{s \in[0, b]}|y(s)-\bar{y}(s)|_{\alpha^{*}}
\end{aligned}
$$

Taking supremum over $t$, we have
$\|(\bar{\Phi} y)(t)-(\bar{\Phi} \bar{y})(t)\|_{\alpha} \leq\left[\frac{l M b^{q}}{\Gamma(q+1)} L_{2}\left(1+L_{1}\right)\right]\|y-\bar{y}\|_{\alpha^{*}}$
From (3.1), we see that $\bar{\Phi}$ is a contraction. Therefore, the system (1.1) - (1.2) has a unique mild solution on the interval $(-\infty, b]$.

Now, we give another existence result for the system (1.1) - (1.2) by means of Schauder's fixed point theorem.

Theorem 3.2 . Suppose that the assumptions (H1) - (H3) are satisfied. Then the system (1.1) - (1.2) has atleast one mild solution on $I$, provided that

$$
\begin{equation*}
\frac{M b^{q}}{\Gamma(q+1)} \beta l\left(1+L_{1}\right)<1 \tag{3.5}
\end{equation*}
$$

Proof: Let $\bar{\Phi}: \mathcal{B}_{h}^{\prime \prime} \rightarrow \mathcal{B}_{h}^{\prime \prime}$ be defined as (3.4). Now we will prove that $\bar{\Phi}$ has a fixed point by using Lemma 2.3. We proceed in the following four steps.
Step 1. $\bar{\Phi}\left(B_{r}\right) \subseteq B_{r}$ for some $r>0$.
We claim that there exists a positive integer $r$, such that $\bar{\Phi}\left(B_{r}\right) \subseteq B_{r}$. If it is not true, then for each positive number $r$, there exists a function $y^{r}(\cdot) \in B^{r}$, but $\Phi\left(y^{r}\right) \notin B_{r}$. That is $\left|\left(\bar{\Phi} y^{r}\right)(t)\right|>r$ for some $t \in J$.
However, on the other hand, we have from (H1)-(H3) and (3.3),

$$
r<\left|\bar{\Phi} y^{r}(t)\right|_{\alpha}
$$

$\leq \frac{q M}{\mathrm{r}(q+1)} \int_{0}^{t}(t-s)^{q-1}\left|f\left(s, y_{s}^{r}+\tilde{\phi}_{s}, \int_{0}^{s} k\left(s, \tau, y_{\tau}^{r}+\tilde{\phi}_{\tau}\right)\right) d \tau\right|_{\alpha} d s$
$\leq \frac{M b^{q}}{\Gamma(q+1)} \sup _{t \in J} \alpha_{r^{*}}(t)$,
where $r^{*}=L_{1}+\left(1+L_{1}\right) r^{\prime}$ and $r^{\prime}=l\left(r+M_{1}|\phi(0)|\right)+\|\phi\|_{\mathcal{P}_{h^{\prime}}}$.
By assumption (H3), it is easy to obtain $\lim _{r \rightarrow \infty} \inf \frac{r^{r}}{r}=l$.
Dividing both sides of (3.6) by r , and taking $r \rightarrow \infty$, we have

$$
\begin{gathered}
1 \leq \frac{M b^{q}}{\Gamma(q+1)} \cdot \liminf _{r \rightarrow \infty}\left(\frac{\sup \alpha_{r}(t)}{r^{*}} \cdot \frac{r^{*}}{r}\right) \\
=\frac{M b^{q}}{\Gamma(q+1)} \beta l\left(1+L_{1}\right)
\end{gathered}
$$

That is

$$
\frac{M b^{q}}{\Gamma(q+1)} \beta l\left(1+L_{1}\right) \geq 1
$$

This contradicts (3.5). Hence for some positive number $r$, $\bar{\Phi}\left(B_{r}\right) \subseteq B_{r}$.
Step 2. $\bar{\Phi}: \mathcal{B}_{h}^{\prime \prime} \rightarrow \mathcal{B}_{h}^{\prime \prime}$ is continuous.
Let $\left\{y^{(n)}(t)\right\}_{0}^{\infty} \subseteq \mathcal{B}_{h}^{\prime \prime}$, with $y^{(n)} \rightarrow y$ in $\mathcal{B}_{h}^{\prime \prime}$. Then, there exists a number $r>0$ such that $\left|y^{(n)}(t)\right| \leq r$ for all $n$ and a.e. $t \in J$, so $y^{(n)} \in B_{r}$ and $y \in B_{r}$. By Hypothesis (H3) we have for a.e. $(t, s) \in D$,

$$
f\left(s_{1} y_{s}^{(\mathrm{s})}+\tilde{\phi}_{s}, \int_{0}^{s} k\left(s, \tau, y_{\mathrm{T}}^{(\mathrm{n})}+\tilde{\phi}_{\mathrm{T}}\right) d r\right) \rightarrow f\left(\left(s, y_{s}+\tilde{\phi}_{s},\right]_{0}^{z} k\left(s, \tau, y_{\mathrm{T}}+\tilde{\phi}_{\mathrm{T}}\right) d r\right) \text { and }
$$

since

$$
\begin{aligned}
& \leq 2(t-s)^{(q-1)} \alpha_{r} \cdot(s),
\end{aligned}
$$

where the function

We have by the Dominated convergence theorem,

$$
\begin{aligned}
& \left\|\bar{\Phi} y^{(n)}-\bar{\Phi} y\right\|_{\mathcal{B}_{\text {h }}}^{s} \\
& \leq\left\|T_{q}(t-s)\right\|(X) \\
& \left.\| \int_{0}^{t}(t-s)^{r-1}\left[f\left(s y_{2}^{(n)}+\tilde{\phi}_{v}\right)_{0}^{s} k\left(s, \tau_{2}\right)_{\tau}^{(n)}+\tilde{\phi}_{\tau}\right) d t\right) \\
& \left.-f\left(s_{s} y_{3}+\tilde{\phi}_{\mathrm{i}} \int_{0}^{3} k\left(s, \tau_{\mathrm{T}} y_{\mathrm{T}}+\tilde{\phi}_{\mathrm{T}}\right) d \mathrm{t}\right)\right] \| d s \\
& \leq \frac{M q}{\Gamma(q+1)}(X) \\
& \| \int_{0}^{t}(t-s)^{r-1}\left[f\left(s y_{2}^{(m)}+\tilde{\phi}_{p} \int_{0}^{s} k\left(s_{1} \tau_{s} j_{t}^{(n)}+\tilde{\phi}_{r}\right) d t\right)\right. \\
& \left.-f\left(s_{\mathrm{r}} y_{\mathrm{S}}+\tilde{\phi}_{\mathrm{i}} \int_{0}^{s} k\left(s_{1} \tau_{J_{T}}+\tilde{\phi}_{\mathrm{T}}\right) d t\right)\right] \| d s \\
& \rightarrow 0 \text { as } n \rightarrow \infty,
\end{aligned}
$$

which proves that the operator $\bar{\Phi}$ is continuous.
Step 3. $\bar{\Phi}$ maps $B_{r}$ into an equicontinuous family.
Let $y \in B_{r}$ and $r_{1}, r_{2} \in J$. Then if $0<r_{1}<r_{2} \leq b$, in view of (H3) and (3.3), we have

$$
\begin{aligned}
& \left\|(\bar{\Phi} y)\left(r_{1}\right)-(\bar{\Phi} y)\left(r_{2}\right)\right\|_{\alpha} \\
& \left.\leq \| \int_{0}^{p_{1}^{-s}}\left[\left(r_{1}-s\right)^{q-1} T_{q}\left(r_{1}-s\right)-\left(r_{2}-s\right)^{r^{-1}} T_{q}\left(r_{2}-s\right)\right]\left(s_{j} y_{2}+\tilde{\phi}_{v} \int_{0}^{s} k\left(s_{1} r_{s}\right)_{T}+\tilde{\phi}_{2}\right) d r\right) d s \|_{\alpha} \\
& +\left\|\int_{r_{1}-\varepsilon}^{\tau_{1}}\left[\left(r_{1}-s\right)^{\alpha-1} T_{q}\left(r_{1}-s\right)-\left(r_{2}-s\right)^{\alpha-1} T_{q}\left(r_{2}-s\right)\right) f\left(s_{1} y_{2}+\tilde{\phi}_{2} \int_{0}^{s} k\left(s_{s} \tau_{2} y_{\tau}+\tilde{\phi}_{t}\right) d r\right) d s\right\|_{a}^{a} \\
& +\left\|\int_{s_{1}}^{r_{1}}\left(r_{2}-s\right)^{q-1} T_{Q}\left(r_{2}-s\right) f\left(s_{1} y_{s}+\tilde{\phi}_{p} \int_{0}^{s} k\left(s_{1} r_{1} y_{\mathrm{T}}+\tilde{\phi}_{\mathrm{T}}\right) d t\right) d s\right\|_{a} \\
& \leq \int_{0}^{r_{1}-\varepsilon}\left\|\left(r_{1}-s\right)^{q-1} T_{q}\left(r_{1}-s\right)-\left(r_{2}-s\right)^{q-1} T_{q}\left(r_{2}-s\right)\right\| \alpha_{r} \cdot(s) d s \\
& +\int_{r_{1}-\varepsilon}^{r_{1}}\left\|\left(r_{1}-s\right)^{q-1} T_{q}\left(r_{1}-s\right)-\left(r_{2}-s\right)^{q-1} T_{q}\left(r_{2}-s\right)\right\| \alpha_{r^{*}}(s) d s \\
& +\int_{r_{1}}^{r_{2}}\left\|\left(r_{2}-s\right)^{q-1} T_{q}\left(r_{2}-s\right)\right\| \alpha_{r^{*}}(s) d s .
\end{aligned}
$$

The right-hand side is independent of $y \in B_{r}$ and tends to zero as $r_{2}-r_{1} \rightarrow 0$ with $\varepsilon$ sufficiently small, since the compactness of $T_{q}(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus, $\bar{\Phi}$ maps $B_{r}$ into an equicontinuous family. The equicontinuities for the cases $r_{1}<r_{2} \leq 0$ and $r_{1}<0<r_{2}$ are obvious.
Step 4. $\bar{\Phi}$ maps $B_{r}$ into a precompact set in $X$.
Let $0<t \leq b$ be fixed. For $\varepsilon \in(0, t)$ and $\forall \varepsilon_{1}>0$, define the operator $\bar{\Phi}_{s_{s} s_{1}}$ on $B_{r}$ by the formula
$\left(\bar{\Phi}_{\varepsilon_{s} \varepsilon_{1}} y\right)(t)=$
$\int_{0}^{\infty} \xi_{q}(\theta) T\left(t^{q} \theta\right) \phi(t) d \theta+q \int_{0}^{t-\varepsilon} \int_{s_{1}}^{\infty} \theta(t-s)^{q-1} \xi_{q}$

$$
\begin{aligned}
& (\times) T\left((t-s)^{q} \theta\right) f\left(s, y_{s}, \int_{0}^{t} k\left(t, \tau_{s} y_{\tau}\right) d \tau\right) d s d \theta \\
& =T\left(\varepsilon^{q} \varepsilon_{1}\right) \int_{0}^{\infty} \xi_{q}(\theta) T\left(\left(t^{q} \theta\right)-T\left(\varepsilon^{q} \varepsilon_{1}\right)\right) \phi(t) d \theta \\
& +T\left(\varepsilon^{q} \varepsilon_{1}\right) q \int_{0}^{t-s} \int_{\varepsilon_{1}}^{z} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left((t-s)^{q} \theta\right) f\left(s_{1} y_{s} \int_{0}^{t} k\left(t_{0} \tau_{j} y_{\mathrm{T}}\right) d \tau\right) d s d \theta \\
& =T\left(\varepsilon^{q} \varepsilon_{1}\right) \int_{0}^{\infty} \xi_{q}(\theta) T\left(\left(t^{q} \theta\right)-T\left(\varepsilon^{q} \varepsilon_{1}\right)\right) \phi(t) d \theta \\
& +T\left(\varepsilon^{q} \varepsilon_{1}\right) q \int_{0}^{t-s} \int_{\varepsilon_{1}}^{=} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left((t-s)^{q} \theta\right)- \\
& \left.T\left(\varepsilon^{q} \varepsilon_{1}\right)\right) f\left(s_{s} y_{b} \int_{0}^{t} k\left(t, \tau_{j} y_{\mathrm{T}}\right) d \tau\right) d s d \theta_{j}
\end{aligned}
$$

where $y \in B_{r}$. Then from the compactness of $T\left(\varepsilon^{q} \varepsilon_{1}\right)\left(\varepsilon^{q} \varepsilon_{1}>0\right)$, we obtain that the set $V_{\varepsilon_{0} \varepsilon_{1}}=\left\{\left(\bar{\Phi}_{\varepsilon_{s} \varepsilon_{1}} y\right)(t): y \in B_{r}\right\}$ is relatively compact in $X$ for all $\varepsilon \in(0, t)$ and $\varepsilon_{1}>0$. Moreover, for each $y \in B_{r}$, we have that
$\|\left(\bar{\Phi} y(t)-\left(\bar{\Phi}_{\varepsilon_{k}, \varepsilon_{1}} y\right)(t) \|_{\alpha}\right.$
$=\| \int_{0}^{\infty} \xi_{q}(\theta) T\left(t^{q} \theta\right) \phi(t) d \theta+\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s) f\left(s, y_{z^{\prime}} \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d s$
$-\int_{0}^{x} \zeta_{q}(\theta) T\left(t^{q} \theta\right) d \theta-q \int_{0}^{t-z} \int_{\varepsilon_{1}}^{\pi} \theta(t-s)^{q-1} T\left((t-s)^{q} \theta\right) f\left(s_{1} y_{s} \int_{0}^{t} k\left(t \tau_{1} \tau_{1} y_{\tau}\right) d r\right) d s d \theta I_{a}$
$\leq q \| \int_{0}^{t-s}(t-s)^{q-1} \int_{0}^{\infty} \theta \xi_{q}(\theta) I\left(t^{t} \theta\right) d \theta f\left(s_{1} y_{s} \int_{0}^{t} k\left(t, \tau_{j} y_{r}\right) d r\right) d s+\int_{t-s}^{t}(t-s)^{q-1}$ $\int_{0}^{x} \theta \xi_{q}(\theta) T\left(t^{q} \theta\right) f\left(s, y_{s} \int_{0}^{t} k\left(t_{,} \tau_{y} y_{\mathrm{r}}\right) d \tau\right) d s+\int_{0}^{t-z} \int_{z_{1}}^{x} \theta(t-s)^{q-1} \xi_{q}(\theta) r\left((t-s)^{q} \theta\right)$

$$
(\times) f\left(s, y_{s}, \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d \theta d s \|_{\alpha}
$$

$\leq q \| \int_{0}^{t-s} \int_{0}^{\infty} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left(t^{q} \theta\right) f\left(s, y_{s}, \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d \theta d s$ $+\int_{t-\varepsilon}^{t} \int_{0}^{\infty} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left(t^{q} \theta\right) f\left(s, y_{s} \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d \theta d s$ $-\int_{0}^{t-s} \int_{\varepsilon_{1}}^{\infty} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left(t^{q} \theta\right) f\left(s, y_{s}, \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d \theta d s \|_{\alpha}$ $\leq q \| \int_{0}^{t} \int_{0}^{\varepsilon_{1}} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left((t-s)^{q} \theta\right) f\left(s, y_{s^{\prime}} \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d \theta d s$
$+\int_{t-\varepsilon}^{t} \int_{\varepsilon_{1}}^{\infty} \theta(t-s)^{q-1} \xi_{q}(\theta) T\left((t-s)^{q} \theta\right) f\left(s, y_{s} \int_{0}^{t} k\left(t, \tau, y_{\tau}\right) d \tau\right) d \theta d s \|_{\alpha}$ Therefore,

$$
\|\left(\bar{\Phi} y(t)-\left(\bar{\Phi}_{s_{z} \varepsilon_{1}} y\right)(t) \| \rightarrow 0 \text { as } \varepsilon_{y} \varepsilon_{1} \rightarrow 0^{+}\right.
$$

and there are relatively compact sets arbitrarily close to the set $V(t)=\left\{(\bar{\Phi} y)(t): y \in B_{r}\right\}$ hence the set $V(t)$ is relatively compact in $X$.

Thus, by the Arzela-Ascoli theorem $\bar{\Phi}$ is a compact operator and by Schauder's fixed point theorem there exists a fixed point $y\left({ }^{\prime}\right)$ for $\bar{\Phi}$ on $B_{r}$. Hence $x(t)=y(t)+\tilde{\phi}(t)$, $t \in(-\infty, b]$ is a fixed point of the operator $\Phi$ which is a mild solution of the problem (1.1)-(1.2). The proof is now completed.

## An Example

Consider the following semilinear fractional functional integrodifferential equations of the form

$$
\begin{align*}
& D_{t}^{q} z(t, y)=\frac{\partial^{2}}{\partial y^{2}} z(t, y)+\int_{-\infty}^{t} a_{1}(t, y, s-t) Q(z(s, y)) d s \\
& \quad+\int_{0}^{t} \int_{\infty}^{s} k(s-\tau) Q_{2}(z(\tau, y)) d \tau d s, \quad y \in[0, \pi], \quad t \in[0, b] \tag{4.1}
\end{align*}
$$

$z(t, 0)=z(t, \pi)=0, \quad t \geq 0$,
$z(t, y)=\phi(t, y), \quad t \in(-\infty, 0], y \in[0, \pi]$,
where $D_{t}^{q}$ is a Caputo fractional partial derivative of order $0<\alpha<1$ and $\phi \in \mathcal{B}_{h}$. Let us take $X=L^{2}([0, \pi])$ with the norm $\|\cdot\|_{L^{2}}$ and define $A: X \rightarrow X$ by $A w=w^{s}$ with the doma
$D(A)=\left\{w \in X: w, w^{\prime \prime}\right.$ are absolutely continuous, $\left.w^{\prime \prime} \in X, w(0)=w(\pi)=0\right\}$.
Then

$$
A w=\sum_{n=1}^{\infty} n^{2}<w_{,} w_{n}>w_{n}, \quad w \in D(A)
$$

where $w_{n}(s)=\sqrt{\frac{2}{\pi}} \sin n s, n=1,2, \ldots$. is the orthogonal set of eigen vectors of $A$. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ in $X$ and is given by

$$
T(t) w=\sum_{n=1}^{\infty} e^{-n^{2} t}<w_{,} w_{n}>w_{n}, \quad w \in X
$$

For every $w \in X,(-A)^{\frac{-1}{2}} w=\sum_{n=1}^{\infty} n<w_{s}, w_{n}>w_{n}$ and $\left\|(-A)^{\frac{-1}{2}}\right\|=1$. The operator $(-A)^{\frac{1}{2}}$ is given by

$$
(-A)^{\frac{-1}{2}} w=\sum_{n=1}^{\infty} n<w_{,} w_{n}>w_{n}
$$

on the space
$D\left((-A)^{\frac{1}{2}}\right)=\left\{w(\cdot) \in X, \sum_{n=1}^{\infty} n<w, w_{n}>w_{n} \in X\right\}$.
Since the analytic semigroup $T(t)$ is compact [22].
Let $h(s)=e^{2 s}, s<0$ then $l=\int_{-\infty}^{0} h(s) d s=\frac{1}{2}$ and define

$$
\|\phi\|_{h}=\int_{-\infty}^{0} h(s) \sup _{\theta \in[s, 0]}|\phi(\theta)|_{L^{z}} d s
$$

Hence for $(t, \phi) \in[0, b] \times \mathcal{B}_{h}$, where

$$
\phi(\theta) y=\phi(\theta, y),(\theta, y) \in(-\infty, 0] \times[0, \pi]
$$

Set

$$
z(t)(y)=z(t, y),
$$

$f\left(t, \phi, B_{1} \pi\right)(y)=\int_{-\infty}^{0} a_{1}(t, y, \theta) Q_{1}(\pi(\theta)(y)) d \theta+B_{1} \phi(y)$
where
$B_{1} \pi(y)=\int_{0}^{t} \int_{-\infty}^{0} k(s-\theta) Q_{2}(\phi(\theta)(y)) d \theta d s$.
Then, the system (4.1)-(4.3) is the abstract formulation of the system (1.1) - (1.2). Further, we can impose some suitable conditions on the above defined functions to verify the assumptions on Theorem 3.2. We can conclude that system (4.1) - (4.3) has at least one mild solution on $I$.

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