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Existence results for some partial functional integrodifferential systems with state-dependent delay

Nadaf NabisabYamanurasab[1] and M. Mallika Arjunan[2]

[1]Department of Mathematics, Anjuman Engineering College, Bhatkal-581320, Karnataka, India.

[2]Department of Mathematics, Karunya University, Karunya Nagar, Coimbatore-641 114, TamilNadu, India.

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ABSTRACT

This paper is concerned with the existence of solutions for some partial functional integrodifferential equations with state-dependent delay in Banachspaces. The results are obtained by using Leray-Schauder's alternative fixed point theorem. Finally, an example is provided to illustrate the main results.

Introduction

In this paper, we prove the existence of mild solutions for the following partial functional integrodifferential equations with state–dependent delay:

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x(t-\rho(x(t)))),$$

$$t \in J = [0, b], \qquad (1.1)$$

$$x(t) = \phi(t), t \in [-r, 0], \qquad (1.2)$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators. Here $B(t)$, $t \in J$ is a bounded linear operators, the initial data $\phi : [-r, 0] \to X$ is a continuous function, ρ is positive bounded continuous function on X and r is the maximal delay defined by

$$r = \sup_{x \in X} \rho(x).$$

The nonlinear integrodifferential equation with resolvent operators served as an abstract formulation of partial integrodifferential equations which arises in many physical phenomena $[11, 12]$. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. However, the resolvent operator does not satisfy the semigroup properties, see for instance $[15, 17]$.

Functional differential equations with state–dependent delay appear frequently in applications as models of equations and for this reason the study of this type of equation has received great attention in last few years, see for instance $[1, 2, 3, 5-10]$ and the references therein. For more details on differential equations with state–dependent delay, we refer the reader to the handbook by Canada et al.$[19]$.

Recently, much attention has been paid to the existence results for the partial functional differential equations with state–dependent delay such as $[9, 17]$and the references therein. Motivated by the works$[15, 18]$, the main aim of this paper is to establish some existence results for the problem $(1.1)-(1.2)$by using resolvent operators and Leray–Schauder's alternative fixed point theorem with semigroup theory. Our main results can be seen as a generalization of the works in $[15, 18]$ and the above mentioned partial functional differential equations with state–dependent delay.

This paper is organized as follows. In section 2, we recall some notations, definitions and preliminary facts which are used throughout this paper. In section 3, we use the Leray–Schauder's alternative fixed point theorem to prove the existence of mild solutions for the problem$(1.1)-(1.2)$. Section 4 is reserved for an example.

Preliminaries

In this section, we give somenotations, definitions and some results on resolvent operator that will be used to develop the main results.

$C[J, X]$is the Banach space of all continuous functions from Jinto X with the norm$\|x\|_\infty = \sup\{|x(t)| : t \in J\}$.

$B(X)$denotes the Banach space of bounded linear operators from X into X with the norm$\|N\|_{B(X)} = \sup\{|N(y)| : |y| = 1\}$.

$L^1(J, X)$denotes the Banach space of measurable functions $y : J \to X$ which are Bochner integrable and is normed by$\|y\|_{L^1} = \int_0^b |y(t)|\, dt$,for all $y \in L^1(J, X)$.

Tele:
E-mail addresses: nadaf_nabisab@yahoo.com, arjunphd07@yahoo.co.in

Let $(X, \|\cdot\|)$ be the Banach space, the notation $L(X,Y)$ stands for the Banach space ofall linear bounded operators from X into Y, and we abbreviate this notation to $L(X)$ when $X=Y$. R(t), $t>0$ is compact, analytic resolvent operator generated by A.

Assume that

(A_1) A is a densely defined, closed linear operator in a Banach space $(X, \|\cdot\|)$ and generates a C_o –semigroup $T(t)$. Hence D(A) endowed with graph norm $|x| = \|x\| + \|Ax\|$ is a Banach space which will be denoted by $(Y, \|\cdot\|)$.

$(A_2)\{B(t): t \in J\}$is a family of continuous linear operators from $(Y, \|\cdot\|)$ into $(X, \|\cdot\|)$. Moreover, there is an integrable function $c: [0,b] \to R^+$ such that for each $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}(J,X)$ and

$$\left\|\frac{d}{dt}B(t)y\right\| \le c(t)|y|, \quad y \in Y, \qquad t \in J.$$

Definition 2.1A family $\{R(t): t \ge 0\}$ of continuous linear operators on$X$ is called a resolvent operator for

$$\frac{dx}{dt} = Ax(t) + \int_0^t B(t-s)x(s)ds,$$

if:

$(R_1) R(0) = I(\text{the identity operator on } X)$,

(R_2)For all $x \in X$, the map $t \to R(t)x$ is continuous from J to X,

(R_3) For all $t \in J$, $R(t)$ is continuous linear operator on Y, and for all $y \in Y$, the map $t \to R(t)y$ belongs to $C(J,Y) \cap C'(J,X)$ and satisfies

$$\frac{d}{dt}R(t)y = AR(t)y + \int_0^t B(t-s)R(s)yds$$

$$= R(t)Ay + \int_0^t R(t-s)B(s)yds.$$

To prove the main results, we need the following theorem.

Theorem 2.1([15]) Let the assumptions (A_1) and (A_2)besatisfied. Then there exists a constant $H = H(b)$ such that

$\|R(t+h) - R(h)R(t)\|_{L(X)} \le Hh$,for

all$0 \le h \le t \le b$,

where$L(X)$ denotes the Banach space of continuous linear operators on X.

Next, if the C_0 –semigroup $T(\cdot)$ generated by A is compact (that is, T (t) is a compact operator for all t >0), then the corresponding resolvent operator R(·) is also compact (that is, R(t) is a compact operator for all t >0) and is operator norm continuous (or continuous in the uniform operator topology) for t >0.

To conclude this section, we recall the following well-known result.

Theorem2[4 ,Theorem 6.5.4][Leray – Schauder Alternative]. Let X be a Banach space and $C \subset X$ be convex with $0 \in C$. Let $F: C \to C$ be completely continuous operator. Then either F has a fixed point orthe set $\mathcal{E} = \{x \in C: x = \lambda F(x),\ 0 < \lambda < 1\}$is unbounded.

Existence Results

In this section, we shall present and prove our main results. First, we define the mild solution for the problem $(1.1)-(1.2)$.

Definition 3.1 Let $R(t)$ be a resolvent operator of equation (1.1), we call $x \in C(J,X)$ a mild solution of the problem $(1.1)-(1.2)$ if it satisfy

$$x(t) = R(t)\phi(0) + \int_0^t R(t-s)\ f(s, x(s-\rho(x(s))))ds.$$

In order to prove the main results, we list the following hypotheses.

(H_1) The function $f: J \times X \to X$ is Caratheodory that means that f is measurable with respect to the first argument and continuous with respect to the second argument.

(H_2) There exist constants$M_1, \ M_2 > 0$ such that $\|R(t)\| \le M_1$, $\|B(t)\| \le M_2$ for all $t \ge 0$

(H_3) There exists a function $p \in L^1(J, R^+)$ and a continuous nondecreasing function $\psi: [0,\infty) \to [0,\infty)$such that $|f(t,u)| \le p(t)\psi(\|u\|)$ for every $t \in J$ and for each $u \in X$.

Theorem 3.1 Assume that hypotheses $(H_1)-(H_3)$ hold. Then the problem $(1.1)-(1.2)$has atleast one mild solution on $[-r,b]$

.**Proof:**Transform the problem $(1.1)-(1.2)$ into a fixed point problem. Consider the operator $F: C([-r,b],X) \to C([-r,b],X)$defined by

$$F(x)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ R(t)\phi(0) + \int_0^t R(t-s)\ f(s, x(s-\rho(x(s))))ds, & t \in J. \end{cases}$$

We claim that the operator satisfies the conditions of the Theorem 2.2. The proof will be given in several steps.

Step 1. F is continuous.

Let $\{x_n\}$ be a sequence in X such that$x_n \to x$.Then

$$\|F(x_n)(t) - F(x)(t)\|$$

$$= \left\| \int_0^t R(t-s) f\left[\left(s, x_n\left(s-\rho(x_n(s))\right)\right) - f\left(s, x\left(s - \rho(x(s))\right)\right)\right]ds \right\|$$

$$\le \int_0^t \|R(t-s)\| \left\| f\left[\left(s, x_n\left(s-\rho(x_n(s))\right)\right) - f\left(s, x\left(s - \rho(x(s))\right)\right)\right]\right\| ds$$

$$\le M_1 \int_0^t \left\| f\left[\left(s, x_n\left(s-\rho(x_n(s))\right)\right) - f\left(s, x\left(s - \rho(x(s))\right)\right)\right]\right\| ds.$$

From hypothesis (H_1), the continuity of ρ and by Lebesgue dominated convergence theorem, the right hand side of the above inequality tends to zero as $n \to \infty$.

Thus

$$\|F(x_n)(t) - F(x)(t)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2. F maps bounded sets into bounded sets.

It is enough to show that for any $q > 0$ there exists a positive constant δ such that for each $x \in B_q = \{x \in X: \|x\|_\infty \le q\}$, we have $F(x) \in B_\delta$. For each $t \in J$, we have

$$\|F(x)(t)\| \le \|R(t)\|\|\phi(0)\| + \int_0^t \|R(t-s)\| \left\|f\left(s, x\left(s-\rho(x(s))\right)\right)\right\| ds$$

$$\le M_1\|\phi(0)\| + M_1 \int_0^t p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right) ds$$

$$\le M_1\|\phi(0)\| + M_1\psi(q)\int_0^b p(s)ds = \delta.$$

Step 3. F maps bounded sets into equicontinuous sets in $\mathbf{C([-r, b], X)}$.

We consider B_q as in Step 2 and let $\epsilon > 0$ be given. Now, let $\tau_1, \tau_2 \in J$ with $\tau_2 < \tau_1$. Then we have

$$\|F(x)(\tau_2) - F(x)(\tau_1)\| \le \|R(\tau_2) - R(\tau_1)\|\|\phi(0)\|$$

$$+ \left\|\int_0^{\tau_2} [R(\tau_2 - s) - \int_0^{\tau_1} R(\tau_1 - s)] f\left(s, x\left(s-\rho(x(s))\right)\right) ds\right\|$$

$$\le \|R(\tau_2) - R(\tau_1)\|\|\phi(0)\|$$

$$+ \int_0^{\tau_1-\epsilon} \|R(\tau_2 - s) - R(\tau_1 - s)\| \left\|f\left(s, x\left(s-\rho(x(s))\right)\right)\right\| ds$$

$$+ \int_{\tau_1-\epsilon}^{\tau_2} \|R(\tau_2 - s) - R(\tau_1 - s)\| \left\|f\left(s, x\left(s-\rho(x(s))\right)\right)\right\| ds$$

$$+ \int_{\tau_1}^{\tau_2} \|R(\tau_2 - s)\| \left\|f\left(s, x\left(s-\rho(x(s))\right)\right)\right\| ds$$

$$\le \|R(\tau_2) - R(\tau_1)\|\|\phi(0)\|$$

$$+ \int_0^{\tau_1-\epsilon} \|R(\tau_2 - s) - R(\tau_1 - s)\| p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right) ds$$

$$+ \int_{\tau_1-\epsilon}^{\tau_2} \|R(\tau_2 - s) - R(\tau_1 - s)\| p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right) ds$$

$$+ M_1 \int_{\tau_1}^{\tau_2} p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right) ds$$

$$\le \|R(\tau_2) - R(\tau_1)\|\|\phi(0)\|$$

$$+ \psi(q) \int_0^{\tau_1-\epsilon} \|R(\tau_2 - s) - R(\tau_1 - s)\| p(s) ds$$

$$+ \psi(q) \int_{\tau_1-\epsilon}^{\tau_2} \|R(\tau_2 - s) - R(\tau_1 - s)\| p(s) ds$$

$$+ M_1\psi(q)\int_{\tau_1}^{\tau_2} p(s) ds.$$

From the Theorem 2.1, we deduce that the right hand side of the above inequality tends to zero as $\tau_2 - \tau_1$ goes to 0 and ϵ sufficiently small. Thus F maps bounded sets into equicontinuous sets in $C([-r, b], X)$. Here we consider the case $0 < \tau_1 < \tau_2$, since the other cases $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2$ are very simple.

Step 4. F maps $\mathbf{B_q}$ into precompact set in X.

Let $0 \le t \le b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_q$, we define the operators

$$F_\epsilon(x)(t) = R(t)\phi(0) + \int_0^{t-\epsilon} R(t-s)\, f(s, x(s-\rho(x(s))))ds$$

$$= R(t)\phi(0) + R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)\, f(s, x(s-\rho(x(s))))ds$$

and

$$\widetilde{F_\epsilon}(x)(t) = R(t)\phi(0) + \int_0^{t-\epsilon} R(t-s)\, f(s, x(s-\rho(x(s))))ds.$$

From the Theorem 2.1 and the compactness of the operator $R(\epsilon)$, the set $\widetilde{F_\epsilon}(t) = \{F_\epsilon(x)(t): x \in B_q\}$ is relatively compact in X for every ϵ, $0 < \epsilon < t$. Moreover for each $x \in B_q$ and by Theorem 2.1, we have

$$\left\|F_\epsilon(x)(t) - \widetilde{F_\epsilon}(x)(t)\right\|$$

$$\le \left\|\int_0^{t-\epsilon} R(\epsilon)R(t-s-\epsilon)\, f(s, x(s-\rho(x(s))))ds - \int_0^{t-\epsilon} R(t-s)\, f(s, x(s-\rho(x(s))))ds\right\|$$

$$\le \int_0^{t-\epsilon} \|R(\epsilon)R(t-s-\epsilon) - R(t-s)\|_{L(X)} \|f(s, x(s-\rho(x(s))))\| ds$$

$$\le \epsilon H\int_0^{t-\epsilon} \|f(s, x(s-\rho(x(s))))\| ds.$$

So the set $\widetilde{F_\epsilon}(t) = \{F_\epsilon(x)(t): x \in B_q\}$ is relatively compact in X by using total boundedness. Applying this idea again and observing that

$$\|F(x)(t) - F_\epsilon(x)(t)\|$$

$$\le \left\|\int_0^{t} R(t-s)\, f(s, x(s-\rho(x(s))))ds - \int_0^{t-\epsilon} R(t-s)\, f(s, x(s-\rho(x(s))))ds\right\|$$

$$\le \left\|\int_{t-\epsilon}^{t} R(t-s)\, f(s, x(s-\rho(x(s))))ds\right\|$$

$$\le \int_{t-\epsilon}^{t} \|R(t-s)\| \|f(s, x(s-\rho(x(s))))\| ds$$

$$\leq M_1 \int_{t-\epsilon}^{t} p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds$$

$\leq M_1\psi(q)\int_{t-\epsilon}^{t} p(s)ds.$

Therefore, the set $F(t) = \{F(x)(t): x \in B_q\}$ is totally bounded. Hence $F(t)$ is relatively compact in X. Now, by using the Arzela-Ascoli Theorem, we can conclude that F is completely continuous.

Step 5. A priori bounds on solutions

Now it remains to show that the set

$\mathcal{E} = \{x \in C : x = \lambda F(x), \quad 0 < \lambda < 1\}$

is bounded.

Let $x \in \mathcal{E}$. Then for each $t \in J$,

$x(t) = \lambda R(t)\phi(0) + \lambda\int_0^t R(t-s)\, f(s, x(s-\rho(x(s))))ds.$

By the hypothesis (H_3) and for each $t \in J$, we have

$$\|x(t)\| \leq \|R(t)\|\|\phi(0)\| + \int_0^t \|R(t-s)\|\left\|f\left(s, x\left(s-\rho(x(s))\right)\right)\right\|ds$$

$\leq M_1\|\phi(0)\| + M_1\int_0^t p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds.$

Since $-r \leq s-\rho(x(s)) \leq s$ for each $s \in J$ and consider the function μ defined by

$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\},\ 0 \leq t \leq b.$

For $t \in [0, b]$, we have

$\mu(t) \leq M_1\|\phi(0)\| + M_1\int_0^t p(s)\psi(\mu(s))ds.$

Let us take the right hand side of the above inequality as $v(t)$, then

$v(0) = M_1\|\phi(0)\| = c,\ \mu(t) \leq v(t)$ and

$v'(t) = M_1 p(t)\psi(\mu(t)),\ t \in J.$

Using the nondecreasing character of ψ, we get

$v'(t) \leq M_1 p(t)\psi(v(t)), t \in J.$

This implies that for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq M_1\int_0^b p(s)ds < \int_{v(0)=c}^{\infty} \frac{ds}{\psi(s)}.$$

This implies that, there exists a constant Λ such that $v(t) \leq \Lambda,\ t \in J$, and hence $\|x\|_\infty \leq \mu(t) \leq v(t) \leq \Lambda$, $t \in J$ where Λ depends only on b and on the functions $p(\cdot)$ and $\psi(\cdot)$. As a consequence of Theorem 2.2, we deduce that F has a fixed point which is a mild solution of problem $(1.1) - (1.2)$.

Example

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $X = L^2(\Omega)$. Consider the following partial functional differential equation in X :

$$\frac{\partial u(t,\xi)}{\partial t} = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_0^t b(t-s,\xi)\frac{\partial^2 u(s,\xi)}{\partial \xi^2}ds + \theta(t)\left|u\left(t-\tau\left(u(t,\ \xi)\right),\ \xi\right)\right|,$$

for

$0 \leq t \leq b,\ \xi \in \Omega, \qquad (4.1)$

$u(t,\xi) = 0$, for $\xi \in \Omega$ and $0 \leq t \leq b$, (4.2)

$u(t,\xi) = u_0(t,\xi)$, for $-\tau_{max} \leq t \leq 0$ and $\xi \in \Omega$, (4.3)

where

$b(t,\xi) \in C'([0,b] x\, \overline{\Omega}), u_0 \in C^2([-\tau_{max},\ 0]\, x\, \Omega, \mathbb{R}^n)$, and θ is a continuous function from $[0,b]$ to $\mathbb{R}$. The delay function τ is a bounded positive continuous function in $\mathbb{R}^n$. Let τ_{max} be the maximal delay, which is defined by

$\tau_{max} = \sup_{x \in R} \tau(x).$

Set

$$A = \frac{\partial^2}{\partial \xi^2}, D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$(B(t)u)(\xi) = b(t,\xi)\frac{\partial^2 u(\xi)}{\partial \xi^2}, 0 \leq t \leq b, u \in D(A), \xi \in \Omega,$

$f\left(t, u\left(t-\rho\left(u(t)\right)\right)\right)(\xi) = \theta(t)\left|u\left(t-\tau\left(u(t,\xi)\right),\xi\right)\right|, 0 \leq t \leq b,$

$\xi \in \Omega, u \in X.$

Then the system $(4.1) - (4.3)$ is the abstract formulation of the system $(1.1) - (1.2)$. Further, we can impose some suitable conditions on the above defined functions to verify the assumptions on Theorem 3.1. We can conclude that the system $(4.1) - (4.3)$ has at least one mild solution on J.

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