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Analytical solutions of time and spatially dependent one-dimensional advection-diffusion equation

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ABSTRACT

Analytical solutions are obtained for a one-dimensional advection–diffusion equation with variable coefficients in a semi-infinite longitudinal domain. Three cases are considered. In the first one the solute dispersion is time dependent along a uniform flow and in the second case the dispersion and the velocity both are considered spatially dependent expressions, while in third case, dispersion and the velocity both have time and spatially dependent expressions in degenerate forms. In first and third cases the solutions may be used for different time dependent expressions. It has become possible by introducing new independent variables with the help of certain transformations.

Introduction

Distribution of solute through a medium is described by a partial differential equation of parabolic type. It is derived on the principle of conservation of mass and Fick's laws of diffusion. This equation is usually known as advection-diffusion equation. In one space dimension the linear advection-diffusion equation may be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(x,t)\frac{\partial C}{\partial x} - u(x,t)C \right] \tag{1}$$

where C is the solute concentration, x is space variable, t is time, $D(x,t)$ is solute dispersion and is called the dispersion coefficient if it is uniform and steady, and $u(x,t)$ is the velocity of the medium . If the medium is porous it satisfies the Darcy's law. It has wide applications in many disciplines like groundwater hydrology, chemical engineering bio sciences environmental sciences and petroleum engineering, where it helps understand the contaminant or pollutants concentration distribution behavior through an open medium like air, rivers, lakes or porous medium like aquifers, under ground oil reservoirs, for remedial processes. A non-exhaustive list must include the works of Ebach and White (1958), Banks and Ali (1964), Ogata (1970), Lai and Jurinak (1971) and Marino (1974), Al-Niami and Rushton (1977). Most of these works take into account the effects due to adsorption, first order decay, zero order production. Such solutions have been compiled by Lindstrom and Boersma (1989). Coming nearer to real problems, Shamir and Harleman (1967), Lin (1977), considered the layered porous media and non linear adsorption, Banks and Jerasate (1962), Hunt (1978), Kumar (1983) considered the porous media flow unsteady/ non-uniform.

Yates (1990) developed an analytical solution for describing the transport of dissolved substances in heterogeneous porous media with a distance dependent dispersion which may be used to characterize differences in the transport process relative to classical convection–dispersion equation for constant hydrodynamic dispersion in the porous medium. Some simple one-dimensional solutions are given by Tracy (1995), then by using a transformation the non-linear partial differential equation is converted to a linear one for a specific form of the moisture content vs. pressure head and relative hydraulic conductivity vs. pressure head curves which allows both two-dimensional and three-dimensional solutions to be derived.

Zoppou and Knight (1997) presented one-dimensional analytical solutions for the same equation in which they considered solute dispersion varying with square of position variable while velocity varies with the position variable.
Pang et al. (2003) applied the temporal moment solution for one dimensional advective-dispersive solute transport with linear equilibrium sorption and first order degradation for time pulse sources to analyze soil column experimental data. Wang et al. (2005) presented a numerical solution that is significantly more general than other semi-analytical solutions for governing equations describing advective–dispersive transport with multirate mass transfer between mobile and immobile domains. Su et al. (2005) presented a specific form of the Fokker–Planck equation with a time- and scale-dependent dispersivity for modelling solute transport in saturated heterogeneous porous media.

Sirin (2006) assumed pore flow velocity to be a non divergence – free, unsteady and non-stationary random function of space and time for ground water contaminant transport in a heterogeneous media. Kartha and Srivastava (2008) studied the effect of immobile water content on contaminant advection and dispersion in unsaturated porous media.

Kuntz and Grathwohl (2009) investigated transport and fate of reactive components in the unsaturated subsoil (vadose zone) using numerical simulation of steady–state and transient flow scenario. Moreira et al. (2009) presented a review of the GILTT solutions focusing the applications to pollutant dispersion in atmosphere. Guerrero et al. (2009) presented an analytical

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methodology by using change–of–variables in in combination of GITT, to solve advection–diffusion equation in finite domain for both transient and steady–state regimes. Cassol et al. (2009) presented an analytical solution for the two–dimensional atmospheric pollutant dispersion problem utilizing GITT, Laplace Transform and the matrix diagonalization. Zhang and Mi (2009) presented iterative functional differential equation with the delay depends on the argument of the unknown function and the state derivative. By reducing the equation with the Schröder transformation to another functional differential equation without iteration of the unknown function, we give existence of its local analytic solutions which extend the known results in related literature.

In the present paper, analytical solutions are obtained of advection–diffusion equation for uniform input source in a semi-infinite one dimensional uniform/non-uniform flow for time and spatially dependent dispersion. Also the analytical solution is obtained in a semi infinite domain for unsteady and non-uniform dispersion and velocity both being expressed in degenerate form.

Analytical Solutions

Let us write $D(x,t)=D_0 f_1(x,t)$ and $u(x,t)=u_0 f_2(x,t)$ in Eq. (1) as

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D_0 f_1(x,t)\frac{\partial C}{\partial x}-u_0 f_2(x,t)C\right), \quad (2)$$

where D_0 and u_0 are constants.

Let us introduce a new independent variable, X by following transformation (Jaiswal et al., 2009)

$$\frac{\partial X}{\partial x}=-\frac{1}{f_1(x,t)} \text{ or } X=-\int\frac{dx}{f_1(x,t)} \quad (3)$$

As a result Eq. (2) becomes

$$f_1(x,t)\frac{\partial C}{\partial t}=D_0\frac{\partial^2 C}{\partial X^2}+u_0\frac{\partial}{\partial X}\left(f_2(x,t)C\right)$$

Now three cases are considered. In the first case a time dependent dispersion along a uniform flow is considered and for case two spatially dependent dispersion and velocity are considered in which dispersion is squarely proportional to the velocity. While the third case deals with space and time dependent dispersion along a non-uniform and unsteady flow in a semi-infinite domain.

2.1 Unsteady dispersion along uniform flow

Let $f_1(x,t)=f(mt)$, and f2(x,t)=1, where m is a resistive coefficient whose dimension is inverse of that of the time variable t. $f(mt)$ is chosen such that for $m=0$ or $t=0$, $f(mt)=1$. Thus $f(mt)$ is an expression in the non-dimensional variable mt. Then from Eq. (3) we have

$$X=-\frac{x}{f(mt)} \quad (5)$$

Eq. (4) will become

$$f(mt)\frac{\partial C}{\partial t}=D_0\frac{\partial^2 C}{\partial X^2}+u_0\frac{\partial C}{\partial X} \quad (6)$$

Let us introduce a new time variable using the following transformation (Crank, 1975)

$$T=\int_0^t\frac{dt}{f(mt)} \quad (7)$$

The partial differential equation Eq. (6) reduces into that with constant coefficients as

$$\frac{\partial C}{\partial T}=D_0\frac{\partial^2 C}{\partial X^2}+u_0\frac{\partial C}{\partial X} \quad (8)$$

To proceed further with Eq. (8), let us assume following initial and boundary conditions for Eq. (1) in a semi-infinite longitudinal domain:

$$C(x,t=0)=0,\ x\ge 0, \quad (9)$$

$$C(x,t)=C_0,\ x=0,\ t>0, \quad (10a)$$

$$C(x,t)=0,\ x\to\infty,\ t\ge 0 \quad (10b)$$

These conditions in context of Eq. (5) and Eq. (7) will assume the form

$$C(X,T)=0,\ -\infty<X\le 0,\ T=0, \quad (11)$$

$$C(X,T)=C_0,\ X=0,\ T>0, \quad (12a)$$

$$C(X,T)=0,\ X\to-\infty,\ T\ge 0 \quad (12b)$$

Using

$$Z=-X, \quad (13)$$

the initial and boundary value problem becomes

$$\frac{\partial C}{\partial T}=D_0\frac{\partial^2 C}{\partial Z^2}-u_0\frac{\partial C}{\partial Z} \quad (14)$$

$$C(Z,T)=0,\ Z\ge 0,\ T=0, \quad (15)$$

$$C(Z,T)=C_0,\ Z=0,\ T>0 \quad (16a)$$

$$C(Z,T)=0,\ \ Z\to\infty,\ T\ge 0 \quad (16b)$$

Now introducing a new dependent variable by following transformation

$$C(Z,T)=K(Z,T)\exp\left[\frac{u_0}{2D_0}Z-\frac{u_0^{\,2}}{4D_0}T\right], \quad (17)$$

the set of Eqs. (14) to (16) reduces into

$$\frac{\partial K}{\partial T}=D_0\frac{\partial^2 K}{\partial Z^2} \quad (18)$$

$$K(Z,T)=0,\ T=0,\ Z\ge 0 \quad (19)$$

$$K(Z,T)=C_0\exp\left(\frac{u_0^2}{4D_0}T\right),\ Z=0,\ T>0; \quad (20a)$$

$$K(Z,T)=0,\ Z\to\infty,\ T\ge 0 \quad (20b)$$

Eq. (18) and Eq. (20b) are satisfied by a solution (Crank, 1984)

$$K(Z,T)=A\,\text{erfc}\left(\frac{Z}{2\sqrt{D_0 T}}\right), \quad (21)$$

where *A* is an arbitrary constant and may be obtained by using Eq. (20a). Further reusing Eq. (17) the desired solution is

$$C(Z,T)=C_0\exp\left(\frac{u_0}{2D_0}Z\right)\text{erfc}\left(\frac{Z}{2\sqrt{D_0 T}}\right), \quad (22)$$

where $Z = \dfrac{x}{f(mt)}$ and $T = \int_{t=0}^{t} \dfrac{dt}{f(mt)}$.

2.2 Spatially dependent dispersion along non-uniform flow

In this case let us consider following expressions for dispersion and velocity

$f_1(x,t) = (1+ax)^2$, and $f_2(x,t) = (1+ax)$,

where ax is a non-dimensional and a is non-zero real constant accounting for the variations in velocity and dispersion due to inhomogeneity. To have these variations small, let $0 < ax \le 1$. Different values of a will represent different media having different inhomogeneous nature. From the Eq. (3) we will have

$$X = \frac{1}{a(1+ax)} \quad (23)$$

Eq. (4) will become

$$\frac{\partial C}{\partial t} = a^2 D_0 X^2 \frac{\partial^2 C}{\partial X^2} + a u_0 X \frac{\partial C}{\partial X} - a u_0 C \quad (24)$$

To proceed further, let us assume following initial and boundary conditions for Eq. (1) in a longitudinal semi-infinite domain:

$$C(x,t) = 0, \; x \ge 0, \; t = 0 \quad (25)$$

$$C(x,t) = C_0, \; x = 0, \; t > 0 \quad (26a)$$

$$C(x,t) = 0, \; x \to \infty, \; t \ge 0 \quad (26b)$$

Using

$$Z = -\log aX \quad (27)$$

The initial and boundary value problem in terms of the new independent variables (Z,T) will become

$$\frac{\partial C}{\partial t} = a^2 D_0 \frac{\partial^2 C}{\partial Z^2} - (a u_0 - a^2 D_0) \frac{\partial C}{\partial Z} - a u_0 C \quad (28)$$

$$C(Z,t) = 0, \quad Z \ge 0, \; t = 0, \quad (29)$$

$$C(Z,t) = C_0, \; Z = 0, \; t > 0 \quad (30a)$$

$$C(Z,t) = 0, \; Z \to \infty, \; t \ge 0 \quad (30b)$$

Further using a new dependent variable by transformation

$$C(Z,t) = K(Z,t) \exp\left[\frac{w_0}{2a^2 D_0} Z - \left(\frac{w_0^2}{4a^2 D_0} + a u_0 \right) t \right],$$

$$w_0 = a u_0 - a^2 D_0, \quad (31)$$

the set of Eqs. (28) to (30) reduces into

$$\frac{\partial K}{\partial t} = a^2 D_0 \frac{\partial^2 K}{\partial Z^2} \quad (32)$$

$$K(Z,t) = 0, \; t = 0, \; Z \ge 0 \quad (33)$$

$$K(Z,t) = C_0 \exp\left\{ \left(\frac{w_0^2}{4a^2 D_0} + a u_0 \right) t \right\}, \; Z = 0, \; t > 0 \quad (34a)$$

$$K(Z,t) = 0, \; Z \to \infty, \; t \ge 0 \quad (34b)$$

Eq. (32) and Eq. (34b) are satisfied by a solution (Crank, 1984)

$$K(Z,t) = B \operatorname{erfc}\left(\frac{Z}{2a\sqrt{D_0 t}} \right), \quad (35)$$

where B is an arbitrary constant and may be eliminated from the general solution by using the condition (34a) and using back transformation Eq. (31) one may get the desired solution as

$$C(Z,t) = C_0 \exp\left(\frac{w_0}{2a^2 D_0} Z \right) \operatorname{erfc}\left(\frac{Z}{2a\sqrt{D_0 t}} \right) \quad (36)$$

where $w_0 = a u_0 - a^2 D_0$, $Z = \log\{(1+ax)\}$.

2.3 Dispersion and velocity being unsteady and spatially dependent

In this case let us consider following expressions for dispersion and velocity in degenerate forms

$f_1(x,t) = f(mt)(1+ax)^2$, and $f_2(x,t) = f(mt)(1+ax)$,

Eq. (4) becomes

$$\frac{1}{f(mt)} \frac{\partial C}{\partial t} = a^2 D_0 X^2 \frac{\partial^2 C}{\partial X^2} + a u_0 X \frac{\partial C}{\partial X} - a u_0 C \quad (37)$$

Let us introduce a new time variable using the following transformation (Crank, 1975)

$$T_n = \int_0^t f(mt) dt \quad (38)$$

The partial differential equation Eq. (37) becomes

$$\frac{\partial C}{\partial T_n} = a^2 D_0 X^2 \frac{\partial^2 C}{\partial X^2} + a u_0 X \frac{\partial C}{\partial X} - a u_0 C \quad (39)$$

Using Eq. (27) and introducing a new dependent variable by transformation

$$C(Z,T_n) = K(Z,T_n) \exp\left[\frac{w_0}{2aD_0} Z - \left(\frac{w_0^2}{4a^2 D_0} + a u_0 \right) T_n \right],$$

$$w_0 = a u_0 - a^2 D_0, \quad (40)$$

The Eq. (39) is reduced into diffusion equation in terms of $K(Z,T_n)$ may be written as

$$\frac{\partial K}{\partial T_n} = a^2 D_0 \frac{\partial^2 K}{\partial Z^2} \quad (41)$$

Let us considered same initial and boundary conditions which are considered in case first and second. Using Eqs. (3), (27), (38) and (40), the initial and boundary conditions in $K(Z,T_n)$ may be written as

$$K(Z,T_n) = 0, \; T_n = 0, \; Z \ge 0 \quad (42)$$

$$K(Z,T_n) = C_0 \exp\left\{ \left(\frac{w_0^2}{4a^2 D_0} + a u_0 \right) T_n \right\}, \; T_n > 0,$$

$$Z_0 = \log(f(mt)) \quad (43a)$$

$$K(Z,T_n) = 0, \; Z \to \infty, \; T_n \ge 0 \quad (43b)$$

The analytical solution of diffusion equation Eq. (41) with initial and boundary conditions Eqs. (42)-(43) in $K(Z,T_n)$ is obtained. Using back transformation Eq. (40), the analytical solution is,

$$C(Z,T_n) = C_0 \exp\left\{\frac{w_0}{2a^2 D_0}(Z - Z_0)\right\} \frac{\operatorname{erfc}\left(\dfrac{Z}{2a\sqrt{D_0 T_n}}\right)}{\operatorname{erfc}\left(\dfrac{Z_0}{2a\sqrt{D_0 T_n}}\right)}, \quad (44)$$

where $w_0 = au_0 - a^2 D_0$, $Z = \log\{f(mt)(1+ax)\}$,

$Z_0 = \log(f(mt))$ and $T_n = \int_0^t f(mt)dt$.

Illustration and Discussion

Pollution can be classified in many ways. On the basis of the medium of the environment where it occurs most, it can be classified as air pollution, soil pollution, surface water pollution and ground water pollution. Solute particles released from different sources in these media degrade their quality for variety of uses. Its source may be natural or anthropogenic. One type of the source of these pollutions is a point source. Stationary point sources include volcanoes, factories, electric power plants, mineral smelters, petroleum refineries and different small scale industries; while mobile point sources include all sorts of transport vehicles moving by road, rail or air. Groundwater pollution occurs due to infiltrations of wastes through rain water, from garbage disposal sites, septic tanks, mines, discharge from surface water bodies polluted due to industrial and municipal influents.

To show the analytical solutions graphically some admissible expressions for $f(mt)$ have been chosen. The expressions $f(mt)$ and corresponding expressions of new time variable, T and T_n for cases first and third are given below.

	$f(mt)$	T (case-1)	T_n (case-3)
(i)	$1+mt$	$\frac{1}{m}\log(1+mt)$	$t+\frac{1}{2}mt^2$
(ii)	$\frac{1}{1+mt}$	$t+\frac{1}{2}mt^2$	$\frac{1}{m}\log(1+mt)$
(iii)	e^{mt}	$\frac{1}{m}\left[1-e^{-mt}\right]$	$\frac{1}{m}\left[e^{mt}-1\right]$
(iv)	e^{-mt}	$\frac{1}{m}\left[e^{mt}-1\right]$	$\frac{1}{m}\left[1-e^{-mt}\right]$

The analytical solutions Eqs. (22), (36) and (44) are illustrated with the help of same set of input data to understand the concentration distribution behavior in the three cases, respectively. The set of input data are $C_0 = 1.0$, $u_0 = 1.14$ (km/yr), $D_0 = 1.25$ (km^2/yr) and $m = 0.1(yr)^{-1}$. All the figures are drawn at t(yr) = 0.1, 0.4, 0.7 and 1.0, respectively.

Fig. 1 depicts the distribution of concentration for $f(mt) = \exp(mt)$ at different time. Fig. 2 shows, the comparison of the concentration values for two functions $f(mt) = \exp(mt)$ and $1/(1+mt)$ at t(yr) = 0.7 and 1.0. The former expression is of increasing nature while the latter expression is of decreasing nature.

In Fig. 2, the concentration value of increasing function is higher than the decreasing function at a particular time. The figure shows that the trained of distribution of concentration is similar in all function of $f(mt)$.

The concentration values for other expressions of $f(mt)$ do not differ distinctly with the concentration values drawn in the figure hence are not drawn.

Figure 1. Concentration values obtained from Eq. (22) represented by four curves for the expression $f(mt) = \exp(mt)$ at different time t (yr).

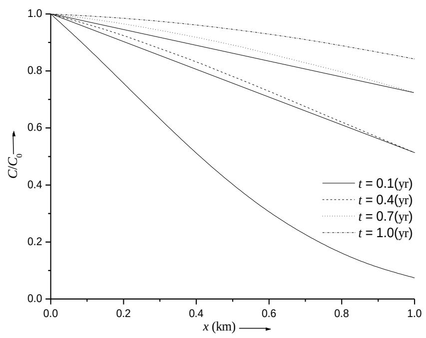

Fig. 1. Concentration values obtained from Eq. (22) represented by four curves for the expression $f(mt) = \exp(mt)$ at different time t (yr).

Figure 2. Compare the concentration values obtained from Eq. (22) represented by two solid curves for $f(mt) = \exp(mt)$ and two dotted curves for $f(mt) = 1/(1+mt)$ at time t = 0.7(yr) and 1.0 (yr) respectively.

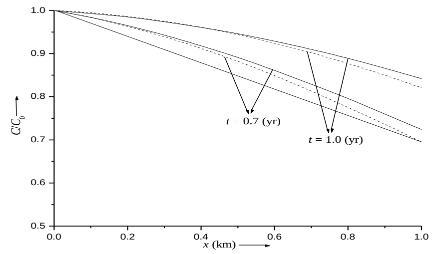

Fig 2. Compare the concentration values obtained from Eq. (22) represented by two solid and two dotted curves for $f(mt) = \exp(mt)$ and $f(mt) = (1+mt)^{-1}$ at time t = 0.7(yr) and 1.0 (yr) respt.

Fig. 3 and 4, shows the concentration values for spatially dependent dispersion and velocity for $a = 1.0(\text{km})^{-1}$, $a = 0.1(\text{km})^{-1}$, respectively. Similar pattern are shown in both the figures. Fig. 5 represents, the concentration distribution behavior of unit input concentration along a semi-infinite domain for analytical solution. Eq. (44) at $a = 1.0(\text{km})^{-1}$ for the function $f(mt) = \exp(mt)$, where $m = 0.1(yr)^{-1}$.

Figure 3. Concentration values obtained from Eq. (36) represented by four curves for $a = 1.0$ (km)$^{-1}$ at different time t (yr).

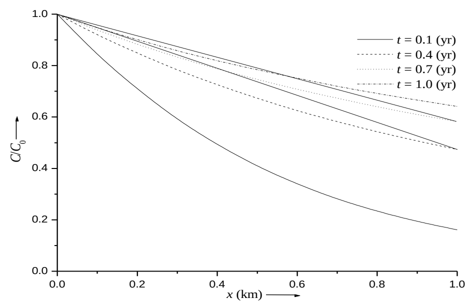

Fig 3. Concentration values obtained from (Eqn. (36)) represented by four curves for a = 1.0 at different time t.

Figure 4. Concentration values obtained from Eq. (36) represented by four curves for $a = 0.1$ (km)$^{-1}$ at different time t (yr).

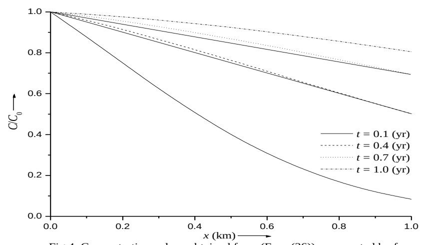

Fig 4. Concentration values obtained from (Eqn. (36)) represented by four curves for a = 0.1 at different time t.

Figure 5. Concentration values obtained from Eq. (44) represented by four curves for expression $f(mt) = \exp(mt)$ and $a = 1.0$ (km)$^{-1}$ at different time t (yr).

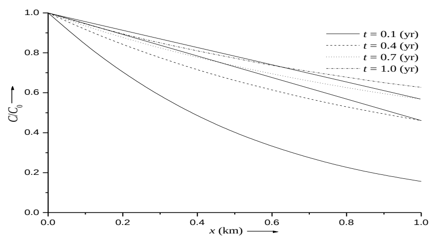

Fig 5. Concentration values obtained from Eq. (37) represented by four curves for $f(mt) = \exp(mt)$ and a = 1.0 at different time t.

Conclusion

In the present work the advection-diffusion equation in one-dimension are solved analytically uniform continuous input source in semi-infinite domain for time dependent solute dispersion along uniform velocity, spatially dependent dispersion along non-uniform velocity in which dispersion is proportional to square of velocity and dispersion and velocity both have time and spatially dependent expressions in degenerate forms, respectively. Usually advection-diffusion equation has been solved earlier for same expressions of dispersion and velocity, either both being time dependent or space dependent. It has been possible with the introduction of a new transformation in independent variable. The main feature of the solutions is that they can be used for variety of expressions for time dependent function $f(mt)$.

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References

1. Al-Niami, A. N. S., and Rushton, K. R., 1977. Analysis of flow against dispersion in porous media. Journal of Hydrology 33, 87–97.

2. Banks, R. B., and Ali, J., 1964. Dispersion and adsorption in porous media flow. Journal of Hydraulic Division 90, 13–31.

3. Banks, R. B., and Jerasate, S. R., 1962. Dispersion in unsteady porous media flow. Journal of Hydraulic Division 88, 1–21.

4. Cassol, M., Wortmann, S., Rizza, U., 2009. Analytic modeling of two-dimensional transient atmospheric pollutant dispersion by double GITT and Laplace Transform techniques. Environmental Modelling & Software 24, 144–151.

5. Crank, J., 1975. The mathematics of diffusion, Oxford University Press, London.

6. Crank, J., 1984. Free and moving boundary problems, Section 3.1, Clarendon Press, Oxford, London.

7. Ebach, E. H., and White, R., 1958. The mixing of fluids flowing through packed solids. J. Am. Inst. Chem. Engng 4, 161–164.

8. Guerrero, J.S.P., Pimentel, L.C.G., Skaggs, T.H., and van Genuchten, M.Th., 2009. Analytical solution of the advection-diffusion transport equation using a change–of–variable and integral transform technique. International Journal of Heat and Mass Transfer 52, 3297–3304.

9. Hunt, B., 1978. Dispersion calculations in nonuniform seepage. Journal of Hydrology 36, 261–277.

10. Jaiswal, D.K., Kumar, A., Kumar N. and Yadava, R.R., 2009. Analytical solutions for temporally and spatially dependent solute dispersion of pulse type input concentration in one dimensional semi-infinite media. Journal of Hydro environment Research 2, 254–263.

11. Kartha, S.A., Srivastava, R., 2008. Effect of immobile water content on contaminant transport in unsaturated zone. Journal of Hydro–environment Research 1, 206–215.

12. Kumar, N., 1983. Unsteady flow against dispersion in finite porous media. Journal of Hydrology 63, 345–358.

13. Kuntz, D., Grathwohl, P., 2009. Comparision of steady-state and transient flow conditions on reactive transport of contaminants in the vadose soil zone. Journal of Hydrology 369 , 225–233.

14. Lai, S. H., and Jurinak, J. J., 1971. Numerical approximation of cation exchange in miscible displacement through soil columns. Soil Sc. Soc. Am. Proc 35, 894– 899.

15. Lin, S. H., 1977. Nonlinear adsorption in layered porous media flow. Journal of Hydraulic Division 103, 951–958.

16. Lindstrom, F.T., Boersma, L., 1989. Analytical solutions for convective dispersive transport in confined aquifers with different initial and boundary conditions. Water Resources Research 25, 241-256.

17. Marino, M. A., 1974. Distribution of contaminants in porous media flow. Water Resources Research 10, 1013–1018.

18. Moreira, D.M., Vilhena, M.T., Buske, D., Tirabassi, T., 2009. The state-of-art of the GILTT method to simulate pollutant dispersion in the atmosphere. Journal of Hydrology 92, 1–17.

19. Ogata, A., 1970. Theory of dispersion in granular media. US Geo.l Sur. Prof. Paper 411–I, 34.

20. Pang, L., Gottz, M. and Close, M., 2003. Application of the method of temporal moment to interpret solute transport with sorption and degradation. J. Contam. Hydrol 60, 123–134.

21. Sirin, H., 2006. Ground water contaminant transport by non divergence – free, unsteady and non-stationary velocity fields. Journal of Hydrology 330, 564–572.

22. Shamir, U. Y., and Harleman, D. R. F., 1967. Dispersion in layered porous media. Journal of Hydraulic Division 95, 237–260.

23. Su, N., Sander, G. C., Liu, F., Anh, V., and Barry, D. A., 2005, Similarity solutions for solute transport in fractal porous media using a time- and scale-dependent dispersivity, Applied Mathematical Modelling, vol. 29 pp. 852–870.

24. Tracy, Fred T., 1995. 1-D, 2-D, 3-D analytical solutions of unsaturated flow in groundwater. Journal of Hydrology 170, 199–214.

25. Wang, P. P., Zheng, C., and Gorelick, S. M., 2005, A general approach to advective–dispersive transport with multirate

mass transfer, Advances in Water Resources, vol. 28, pp. 33–42.
26. Yates, S. R., 1990. An analytical solution for one-dimensional transport in heterogeneous porous media. Water Resources Research 26, 2331–2338.
27. Zhang, P., and Mi, L., 2009, Analytic solutions of a second order iterative functional differential equation, Applied Mathematics and Computation, 210(2), 277-283.
28. Zoppou, C. and Knight, J. H., 1997. Analytical solutions for advection and advection- diffusion equation with spatially variable coefficients. J. Hydraul. Engng 123, 144–148.