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Applied Mathematics

Elixir App. Math. 33 (2011) 2316-2318

Elixir ISSN: 2229-712X

Two summation formulae based on half argument involving contigious relation

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ARTICLE INFO

Article history:

Received: 1 March 2011;

Received in revised form: 24 March 2011;

Accepted: 31 March 2011;

Keywords

Contiguous relation ,

Recurrence relation ,

Gauss second summation theorem.

ABSTRACT

The main objective of this paper is to establish two summation formulae based on half argument involving Contiguous Relation. The results derived in this paper are of general character.

A.M.S. Subject Classification (2000) : 33C05 , 33C45 , 33C60 , 33C70

Introduction

The Pochhammer's symbol

$$(\alpha , k) = (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2).....(\alpha+k-1); & \text{if } k=1,2,3.... \\ 1 & ; \text{ if } k=0 \\ k! & ; \text{ if } \alpha=1 \end{cases} \qquad (1)$$

Generalized Gaussian Hypergeometric function of one variable

$${}_AF_B(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots b_B;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k.....(a_A)_k\, z^k}{(b_1)_k(b_2)_k.....(b_B)_k\, k!}$$

or

$${}_AF_B((a_A);(b_B);z) \equiv {}_AF_B((a_j)^A_{j=1};(bj)^B_{j=1};z) = \sum_{k=0}^{\infty} \frac{((a_A))_k\, z^k}{((b_B))_k\, k!} \qquad (2)$$

where the parameters b_1 , b_2 , ….,b_B are neither zero nor negative integers and A , B are non negative integers.

Contiguous Relations

[Andrews p.363(9.16) , E.D. p.51(10), H.T.F.I. p.103(32)]

$$(a-b){}_2F_1(a,b;c;z)=a\,{}_2F_1(a+1,b;c;z)-b\,{}_2F_1(a,b+1;c;z) \qquad (3)$$

[Abramowitz p.558(15.2.19)]

$$(a-b)(1-z)\,{}_2F_1(a, b ; c; z) = (c-b)\,{}_2F_1(a,b-1; c; z) + (a-c)\,{}_2F_1(a-1,b; c; z) \qquad (4)$$

Recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z) \qquad (5)$$

A New Summation Formula [2]

$${}_2F_1\left(a, b ; \frac{a+b-1}{2}; \frac{1}{2}\right) = 2^{b-1} \frac{\Gamma(\frac{a+b-1}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} \left\{ \frac{b+a-1}{a-1} \right\} + 2\, \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})} \right] \qquad (6)$$

Main Summation Formulae

For both the results $a \neq b$

For $a<1$ and $a>11$

$${}_2F_1\left(a, b ; \frac{a+b-11}{2}; \frac{1}{2}\right) = 2^{b-1} \frac{\Gamma(\frac{a+b-11}{2})}{(a-b)\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})} \left\{ \frac{10395a-19524a^2+12139a^3-3480a^4+505a^5-36a^6+a7-10395b+25415a^2b-16848a^3b+6227a^4b-624a^5b}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right.\right.$$

$$+\frac{65a^6b+19524b^2-25415ab^2+7722a^3b^2-1716a^4b^2+429a^5b^2-12139b^3+16848ab^3-7722a^2b^3+429a^4b^3}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$\left. + \frac{3480b^4-6227ab^4+1716a^2b^4-429a^3b^4-505b^5+624ab^5-429a^2b^5+36b^6-65ab^6-b^7}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right\}$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})} \left\{ \frac{-15468a+20972a^2-7432a^3+2056a^4-40a^5+12a^6+15468b-11752a^2b+8944a^3b-1092a^4b}{(a-10)(a-8)(a-6)(a-4)(a-2)} \right.$$

$$+\frac{208a^5b-20972b^2+11752ab^2-1144a^3b^2+572a^4b^2+7432b^3-8944ab^3+1144a^2b^3-2056b^4+1092ab^4}{(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$\left.\left. +\frac{-572a^2b^4+140b^5-208ab^5-12b^6}{(a-10)(a-8)(a-6)(a-4)(a-2)} \right\}\right] \qquad (7)$$

For $a<1$ and $a>12$

$${}_2F_1\left(a, b ; \frac{a+b-12}{2}; \frac{1}{2}\right) = 2^{b-1} \frac{\Gamma(\frac{a+b-12}{2})}{(a-b)\Gamma(b)} \left[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-11}{2})} \left\{ \frac{46080a-84864a^2+61568a^3-15496a^4+3276a^5-182a^6}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right.\right.$$

$$+\frac{13a^7-46080b+28416ab+51584+51584a^2b-42848a^3b+19604a^4b-1820a^5b+273a^6b+56448b^2-87168ab^2}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{15184a^2b^2+10296a^3b^2-3146a^4b^2+1001a^5b^2-25984b^3+37280ab^3-21528a^2b^3+1144a^3b^3+429a^4b^3+5880b^4}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$\left. + \frac{-10948ab^4+3094a^2b^4-1001a^3b^4-700b^5+868ab^5-637a^2b^5+42b^6-77ab^6-b^7}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right\}$$

$$+\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-12}{2})} \left\{ \frac{46080a-56448a^2+25984a^3-5880a^4+700a^5-42a^6+a^7-46080b-28416ab+87168a^2b-37280a^3b}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \right.$$

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$$+\frac{10948a^4b-868a^5b+77a^6b+84864b^2-51584ab^2-15184a^2b^2+21528a^3b^2-3094a^4b^2+637a^5b^2-61568b^3}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{42848ab^3-10296a^2b^3-1144a^3b^3+1001a^4b^3+15496b^4-19604ab^4+3146a^2b^4-429a^3b^4-3276b^5+1820ab^5}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-1001a^2b^5+182b^6-273ab^6-13b^7}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}\}] \qquad (8)$$

C derivations of summation formulae (7) to (8):

Derivation of (7): Substituting $c = \frac{a+b-11}{2}$ and $z = \frac{1}{2}$ in equation (4) , we get

$$(a-b)\ {}_2F_1(a, b; \tfrac{a+b-11}{2}; \tfrac{1}{2}) = (a-b-11)\ {}_2F_1(a, b-1; \tfrac{a+b-11}{2}; \tfrac{1}{2})$$
$$+ (a-b+11)\ {}_2F_1(a-1, b; \tfrac{a+b-11}{2}; \tfrac{1}{2})$$

Now with the help of the derived result from equation (6) , we get

$$\text{L.H.S} = (a-b-11)\, 2^{b-2}\frac{\Gamma(\frac{a+b-11}{2})}{(a-b+1)\Gamma(b-1)}\Big[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}$$

$$\{\frac{-10395+8184a+8129a^2-7920a^3+2255a^4-264a^5+11a^6+19524b-25158ab+3256a^2b+4268a^3b-1276a^4b}{(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{154a^5b-12139b^2+17824ab^2-5566a^2b^2+297a^4b^2+3480b^3-5316ab^3+1848a^2b^3-132a^3b^3-505b^4+680ab^4}{(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{-275a^2b^4+36b^5-54ab^5-b^6}{(a-9)(a-7)(a-5)(a-3)(a-1)}\}+\frac{\Gamma(\frac{b-1}{2})}{\Gamma(\frac{a-10}{2})}$$

$$\{\frac{-16413+1292a+10779a^2-5816a^3+1025a^4-84a^5+a^6+35696b}{(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-18970ab-6384a^2b+5028a^3b-960a^4b+54a^5b-27665b^2+20680ab^2-1298a^2b^2-1144a^3b^2+275a^4b^2+10208b^3}{(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-8140ab^3+1584a^2b^3+132a^3b^3-1991b^4+1452ab^4-297a^2b^4+176b^5-154ab^5-11b^6}{(a-10)(a-8)(a-6)(a-4)(a-2)}\}\Big]$$

$$+\ \mathbf{(a-b+11)\ 2^{b-1}}\frac{\Gamma(\frac{a+b-11}{2})}{(a-b-1)\Gamma(b)}$$

$$\Big[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})}\{\frac{16413-35696a+27665a^2-10208a^3+1991a^4-176a^5+11a^6-1292b}{(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{18970ab-20680a^2b+8140a^3b-1452a^4b+154a^5b-10779b^2+6384ab^2+1298a^2b^2-1584a^3b^2+297a^4b^2+5816b^3}{(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-5028ab^3+1144a^2b^3-132a^3b^3-1025b^4+960ab^4-275a^2b^4+84b^5-54ab^5-b^6}{(a-10)(a-8)(a-6)(a-4)(a-2)}\}$$

$$+\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}\{\frac{10395-19524a+12139a^2-3480a^3+505a^4-36a^5+a^6-8184b+25158ab-17824a^2b+5316a^3b-680a^4b}{(a-11)(a-9)(a-7)(a-5)(a-3)}$$

$$+\frac{54a^5b-8129b^2-3256ab^2+5566a^2b^2-1848a^3b^2+275a^4b^2+7920b^3-4268ab^3+132a^3b^3-2255b^4+1276ab^4}{(a-11)(a-9)(a-7)(a-5)(a-3)}$$

$$+\frac{-297a^2b^4+264b^5-154ab^5-11b^6}{(a-11)(a-9)(a-7)(a-5)(a-3)}\}\Big]$$

$$= 2^{b-1}\frac{\Gamma(\frac{a+b-11}{2})}{\Gamma(b)}\Big[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}(a-b-11)(b-1)$$

$$\{\frac{-10395+8184a+8129a^2-7920a^3+2255a^4-264a^5+11a^6+19524b-25158ab+3256a^2b+4268a^3b-1276a^4b}{(a-b+1)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{154a^5b-12139b^2+17824ab^2-5566a^2b^2+297a^4b^2+3480b^3-5316ab^3+1848a^2b^3-132a^3b^3-505b^4+680ab^4}{(a-b+1)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{-275a^2b^4+36b^5-54ab^5-b^6}{(a-b+1)(a-9)(a-7)(a-5)(a-3)(a-1)}\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})}(a-b-11)$$

$$\{\frac{-16413+1292a+10779a^2-5816a^3+1025a^4-84a^5+a^6+35696b}{(a-b+1)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-18970ab-6384a^2b+5028a^3b-960a^4b+54a^5b-27665b^2+20680ab^2-1298a^2b^2-1144a^3b^2+275a^4b^2+10208b^3}{(a-b+1)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-8140ab^3+1584a^2b^3+132a^3b^3-1991b^4+1452ab^4-297a^2b^4+176b^5-154ab^5-11b^6}{(a-b+1)(a-10)(a-8)(a-6)(a-4)(a-2)}\}\Big]$$

$$+2^{b-1}\frac{\Gamma(\frac{a+b-11}{2})}{(a-b-1)\Gamma(b)}$$

$$\Big[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})}(a-b+11)\{\frac{16413-35696a+27665a^2-10208a^3+1991a^4-176a^5+11a^6-1292b}{(a-b-1)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{18970ab-20680a^2b+8140a^3b-1452a^4b+154a^5b-10779b^2+6384ab^2+1298a^2b^2-1584a^3b^2+297a^4b^2+5816b^3}{(a-b-1)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-5028ab^3+1144a^2b^3-132a^3b^3-1025b^4+960ab^4-275a^2b^4+84b^5-54ab^5-b^6}{(a-b-1)(a-10)(a-8)(a-6)(a-4)(a-2)}\}$$

$$+\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}(a-b+11)\{\frac{10395-19524a+12139a^2-3480a^3+505a^4-36a^5+a^6-8184b+25158ab-17824a^2b+5316a^3b-680a^4b}{(a-b-1)(a-11)(a-9)(a-7)(a-5)(a-3)}$$

$$+\frac{54a^5b-8129b^2-3256ab^2+5566a^2b^2-1848a^3b^2+275a^4b^2+7920b^3-4268ab^3+132a^3b^3-2255b^4+1276ab^4}{(a-b-1)(a-11)(a-9)(a-7)(a-5)(a-3)}$$

$$+\frac{-297a^2b^4+264b^5-154ab^5-11b^6}{(a-b-1)(a-11)(a-9)(a-7)(a-5)(a-3)}\}\Big]$$

On simplification , we get the result (7)

Similarly , we can prove the other result (8).

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