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Exact solutions of schrödinger equation with woods–saxon plus rosen-morse potential

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ABSTRACT
With the newly improved ansaltz for the wave function and adopting the modified approximation scheme to evaluate the centrifugal term, we solve the Schrödinger equation with Woods-Saxon plus Rosen-Morse Potentials analytically, for arbitrary *l*-state. We also obtain the bound state energy spectrum and the unnormalized real and imaginary wave function.

Introduction

The exact analytic solutions of Schrödinger equation can only be obtained for few cases. In recent years many authors have solved the Schrödinger equation for different potentials [1-4]. Different methods have been deployed in solving the Schrödinger equation.

These methods include, Super Symmetric quantum mechanics (SUSTQM), [5], Nikiforov-Uvarov [6], algebraic approach [7-10], Asymptotic iteration method [11, 12], shape invariant [13-15] and factorization method [16-18]. Also in recent times, much attention has been paid on factorization method. This method reproduces accurate analytical solutions for many differential equations that are important in the applications to many problems in physics, such as the equation of Hermit, Laguerre, Legendre, Bessel and Jacobi [19-21].

However, the factorization method gives a complete analytical solutions of Schrödinger equation for Woods-Saxon, Poschl-Teller and harmonic potentials.

Recently, we use the factorization method and found the exact solution of Schrödinger equation for inverted Woods-Saxon and Manning-Rosen Potential [3].

Satisfied by the factorization method through comparisons with other methods, we are tempted to solve the time-independent Schrödinger equation for Woods-Saxon plus Rosen-Morse potential [22]. This potential plays important role in many different fields of Physics such as chemical and molecular Physics.

The organization of the paper is as follows: We present in section II, the Woods-Saxon plus Rosen-Morse Potential. In section III, we obtain the exact solutions of Woods-Saxon plus Rosen-Morse Potential. Finally, conclusions are given in the last section.

Woods-Saxon plus Rosen-Morse Potential

The generalized Woods-Saxon plus Morse Potential is given by

$$V(r)=\frac{V_0 e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)}-\frac{V_1 e^{-4\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}-V_2 Sech^2(\alpha r)-V_3 \tanh_q(\alpha r) \qquad (1)$$

where r is the radius of the nuclei, a is a parameter defined as $\alpha = 1/2a$, where a is a constant that are usually adjusted to the experimental value of nuclear interaction barrier and V_0, V_1, V_2, V_3 are potential depths of the nuclear.

In Figs. 1 and 2 we present a plot of generalized Woods-Saxon plus Rosen-Morse potential as a function of r for various parameters of α = 0.25, 0.5 and 0.75

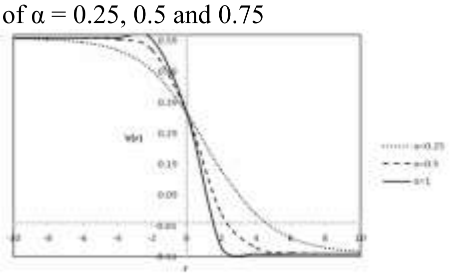

Fig 1: A plot of generalized Wood- Saxon plus Rosen- Morse potential

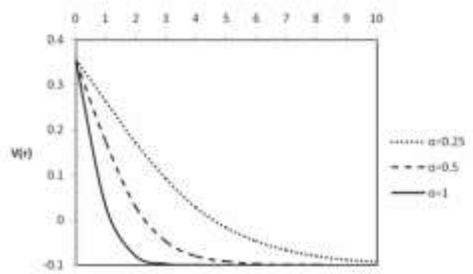

Fig 2: A plot of generalized Wood-Saxon plus Rosen-Morse potential for positive values of r

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Factorization Method

In spherical co-ordinate, the Schrödinger is written as

$$-\frac{\eta^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 Sin\theta}\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 Sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi(r,\theta,\varphi)+V(r)\psi(r,\theta,\varphi)$$
$$=E\psi(r,\theta,\varphi), \qquad (2)$$

where $V(r)$ is the Woods-Saxon plus Rosen Morse Potential of equation (1).

The exact solution of Eq. (2) is defined as

$$\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi) \qquad (3)$$

Substituting (3) in to (2), he obtain the following equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2\mu}{\eta^2}\left[E-\frac{V_0e^{-2\alpha r}}{1+qe^{-2\alpha r}}+\frac{V_1e^{-4\alpha r}}{(1+qe^{-2\alpha r})^2}+V_2Sech_q^2(\alpha r)+V_3\tanh_q(\alpha r)\right]R(r)-\frac{\lambda}{r^2}R(r)=0 \qquad (4)$$

$$\frac{d\Theta(\theta)}{d\theta^2}+Cot\theta\frac{d\Theta(\theta)}{d\theta}+\left[\lambda-\frac{m_l^2}{Sin^2\theta}\right]\Theta(\theta)=0 \qquad (5)$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m_l^2\Phi(\varphi)=0 \qquad (6)$$

where $\lambda=\lambda(\lambda+1)$ and λ is angular quantum number and the magnetic quantum number $m_\lambda=0,\pm I,\pm 2\Lambda\ \pm\lambda$. The solution of equations (5) and (6) are well known [7] and is usually given in terms of spherical harmonics function $Y_{\lambda m}(\theta,\varphi)$.

Solutions of the Radial Equation and Energy Eigenvalues

The radial part of the Schrödinger equation is given as

$$\frac{d^2R}{dr^2}+\frac{2dR}{dr}+\frac{2\mu}{\eta^2}\left[E-\frac{V_0e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)}+\frac{V_1e^{-4\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}+V_2Sech_q^2(\alpha r)+V_3\tanh_q(\alpha r)-\frac{\lambda}{r^2}\right]R(r)=0, \qquad (7)$$

where the deformed hyperbolic function are defined as

$$Sin_q(\alpha r)=\frac{e^{\alpha r}-qe^{-\alpha r}}{2}, Cosh_q(\alpha r)=\frac{e^{\alpha r}+qe^{-\alpha r}}{2}$$
$$\tanh_q=\frac{Sinh_q(\alpha r)}{Cos_q(\alpha r)} \qquad (8)$$

Equation(s) in its present form has no analytical solution for $\lambda\neq 0$, an approximation method has to be made [23]. We write the centrifugal term $1/r^2$ in equation (7) as

$$\frac{1}{r^2}\approx\frac{\alpha^2e^{-\alpha r}}{\left(1+qe^{-\alpha r}\right)^2}, \qquad (9)$$

which reduces to the improved approximation scheme [24] where q = -1. Substituting (9) into Eq. (7) yields

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2\mu}{\eta^2}\left[E-\frac{V_0e^{-2\alpha r}}{(1+qe^{-2\alpha r})}+\frac{V_1e^{-4\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}+V_2Sech_q^2(\alpha r)+V_3\tanh_q(\alpha r)-\frac{\lambda\alpha^2e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}\right]R(r)=0 \qquad (10)$$

Now using the common ansaltz $U(r)=R(r)/r$, Eq. (10) could be transformed into

$$\frac{d^2U(r)}{dr^2}+\frac{2\mu}{\eta^2}\left[E-\frac{V_0e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)}+\frac{V_1e^{-4\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}+V_2Sech_q^2(\alpha r)+V_3\tanh_q(\alpha r)-\frac{\lambda\alpha^2e^{-\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}\right]U(r)=0, \qquad (11)$$

where the last term provides a centrifugal potential, which together with second, third and fourth terms comprise the effective potential $V_{eff}(r)$. By using a new variable s = coth(αr), we can rewrite the Schrödinger equation of Eq. (11) as

$$\left(1-s^2\right)\frac{d^2U}{ds^2}-\frac{2sdU(s)}{ds}+\left[\frac{\varepsilon}{1-s^2}+\frac{\beta}{s(1-s^2)}-\frac{\gamma(s+1)}{s^2(1-s)}+\frac{\delta}{s^2}-\frac{\eta}{s(1-s^2)}+\frac{\xi}{s^2(s-1)^2}\right]U(s)=0, \qquad (12)$$

where the following dimensionless constants have been employed

$$\varepsilon=-\frac{\mu E}{2\eta^2\alpha^2}, \beta=\frac{\mu V_0}{4\eta^2\alpha^2 q},$$
$$\gamma=\frac{\mu V_1}{8\alpha^2\eta^2q^2}, \delta=\frac{\mu V_2}{2\eta^2\alpha^2}, \eta=\frac{\mu V_3}{2\eta^2\alpha^2}$$
$$\xi=\frac{\lambda(\lambda+1)}{8\eta^2 q} \qquad (13)$$

Equation (12) is the well known associated Jacobi. In order to solve Eq. (12) explicitly, we invoke the new ansatz for the wave function of the form [3].

$$U(s)=P(s)In\left(1+w(s)\right), \qquad (14)$$

where $P(s)$ is the associated Jacobi polynomial satisfying Eq. (12). Now substituting, Eq. (14) into Eq. (12) and after a little algebraic, we obtain

$$\left(1-s^2\right)P''(s)+\left[\frac{2\left(1-s^2\right)w'}{(1+w)In(1+w)}-2s\right]P'(s)$$

$$+\left[\frac{(1-s^2)w''}{(1+w)In(1+w)}-\frac{(1-s^2)w'^2}{(1+w)^2 In(1+w)}-\frac{2sw'}{(1+w)In(1+w)}+\frac{\varepsilon}{(1-s^2)}\right]P(s)+\left[\frac{\beta}{s(1-s)}-\frac{\gamma(s+1)}{s(1-s)}+\frac{\delta}{s^2}-\frac{\eta}{s(1-s^2)}+\frac{\xi}{s^2(1-s^2)}\right]P(s)=0 \tag{15}$$

Compare Eq. (15) with the standard associated Jacobi differential equation [25, 26],

$$(1-s^2)P''(s)+\left(\beta-\alpha-(\alpha+\beta+2)s\right)P'(s)+\left(n(\alpha+\beta+n+1)-\frac{\lambda(\alpha+\beta+\lambda+(\alpha-\beta)s)}{(1-s^2)}\right)P(s)=0. \tag{16}$$

we obtain

$$\frac{w'}{(1+w)In(1+w)}=\frac{\beta-\alpha}{(1-s^2)} \tag{17}$$

Solving Eq. (17) explicitly, we have

$$W(s)=N(1+s)^{\alpha-\beta}(1-s)^{\beta+\alpha}\left[1\pm i\left(1+S\right)^{\beta-\alpha}\left(1-s\right)^{-(\alpha+\beta)}\right] \tag{18}$$

where N, is the normalization constant. It is worthy to note at this point that in order to obtain equation (18) from Eq. (17), we perform integration by partial fraction and solving the resulting quadratic equation yields equation (18) by taking the limit $\frac{1}{(1+s)^{\frac{\alpha-\beta}{2}}(1-s)^{\frac{\beta-\alpha}{2}}}>>1.$

Equation (15) that we obtain from the new ansaltz can be condensed as the generalized associated Jacobi differential equation. The associated Jacobi function with variable s from Eq. (16) can be written as [25, 26],

$$P_{n,\lambda}^{(\alpha,\beta)}(s)=\frac{B_{n,\lambda}(\alpha,\beta)}{(1-s)^{n+1/2}(1+s)^{\beta+1/2}}\left(\frac{d}{ds}\right)^{n-1}\left[(1-s)^{\alpha+n}(1+s)^{\beta+n}\right] \tag{19}$$

where $B_{n,\lambda}(\alpha,\beta)$ is the normalization constant and *n,l* are non-negative integers define in the internal $0\le l\le n<\infty$.

The wave function *w(s)* in Eqn. (18) is a generalized wave function which comprises of the real and imaginary part and reduces to the known solution in the literature when the imaginary part *Im w(s) = 0*, [22].

The wave function can be obtained from Eq. (14) as

$$R(r)=\frac{U(r)}{r}=\frac{P_{n,l}^{(\alpha,\beta)}(r)}{r}In(1+w(r)) \tag{20}$$

Thus, the final form of the radial wave function can be written in terms of the Jacobi polynomials resulting

$$R(r)=\frac{C_nP_{n,l}^{\alpha,\beta}(r)}{r}In\left[1+\left\{(1+s)^{\alpha-\beta}(1-s)^{\beta+\alpha}\left(1\pm i(1+s)^{\beta-\alpha}(1-s)^{-(\alpha+\beta)}\right)\right\}\right], \tag{21}$$

If we Taylor expand the terms in the logarithm to first order, we get

$$R(r)=\frac{C_nP_{n,l}^{\alpha,\beta}(r)}{r}\left(1+Coth\,\alpha r\right)^{\alpha-\beta}\left(1-Coth(\alpha r)\right)^{\beta+\alpha}\left[1\pm i\left(1+Coth(\alpha r)\right)^{\beta-\alpha}\left(1-Coth(\alpha r)\right)^{-(\alpha+\beta)}\right] \tag{22}$$

where C_n is the new normalization constant. Further comparison of Eq. (15) with Eq. (16) gives the energy spectrum for the Woods-Saxon plus Rosen Morse potential as

$$E_{n,l}=-\frac{\eta^2\alpha^2}{2m}\left[(\alpha+l)^2-l(l+1)\right] \tag{23}$$

Discussion

The potential of equation (1) is a generalized potential consisting of the generalized Woods-Saxon and the generalized Rosen-Morse potentials.

In Figure 3, we present the plot of the Woods-Saxon potential with r, for different values of α= 0.25, 0.5 and 0.75 respectively. The exact solution of this potential is given in [7]. We depict a similar plot in figure 4 for α= 0.25, 0.5 and 0.75 where the range of r lies between 0 ≤ r ≤ 10.

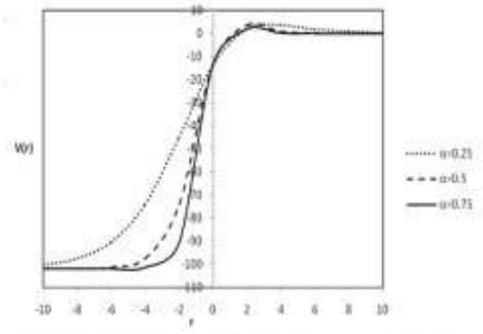

Fig 3: Variation of Woods-Saxon potential with r for various parameter of α=0.25,0.5,0.75

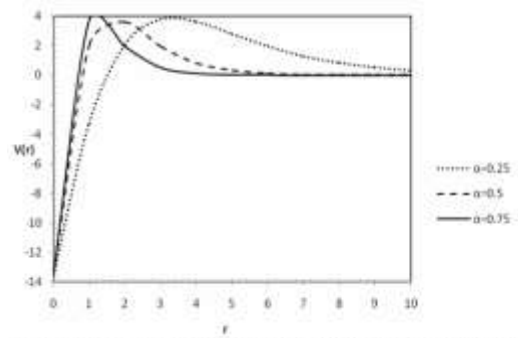

Fig 4: Variation of generalized Woods-Saxon potential with r for various parameter of α=0.25,0.5,0.75 for positive values of r

We display in figure 5, the graph of Rosen-Morse potential as a function of r. Here we restrict the range of r between -10 ≤ r ≤ 10 for various parameter of α= 0.25, 0.5 and 0.75 respectively. The shape of the Rosen-Morse potential may be viewed as a good candidate for common quark potential of QCD traits reported in [2] and [27]. We also have a similar plot for the Rosen-Morse in figure 6 but with the range of r restricted to 0 ≤ r ≤ 10.

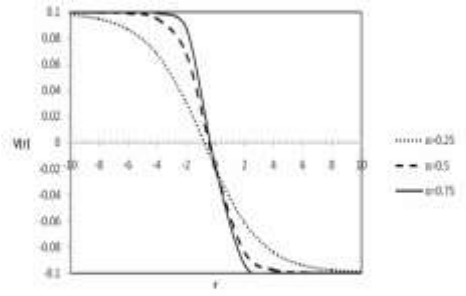

Fig 5: A graph of Rosen-Morse potential as a function of r for various parameter of α=0.25,0.5,0.75

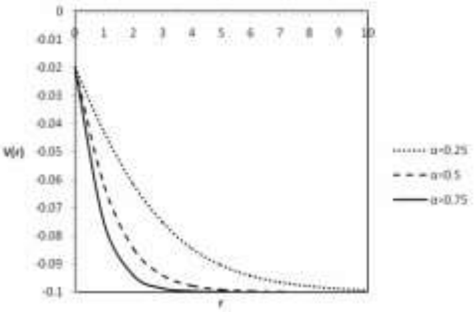

Fig 6: Agraph of Rosen-Morse potential as a function of r for various parameter α=0.25,0.5,0.75 for positive values of r

The effective potential $V_{eff}(r)$ which is the sum of the potential U(r) and the centrifugal term is plotted as a function of r in figure 7 for l = 0, 1, 2, 3 corresponding to the s, p, d, f-states with deformation parameter q=1. We display a similar plot of $V_{eff}(r)$ versus in figure 8 for q= -1. The effective potential is also plotted as a function of r in figure 9 for s, p, and d-states respectively with deformation parameter q= -0.4 for comparison.

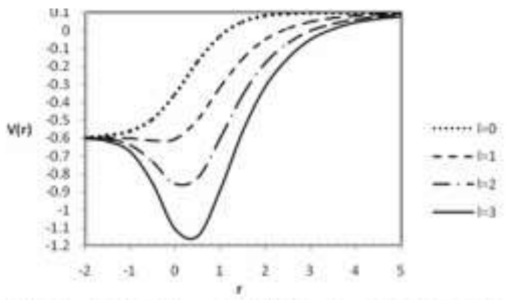

Fig 7: The plot of effective potential $V_{eff}(r)$ with r for l=0,1,2,3 and q=1

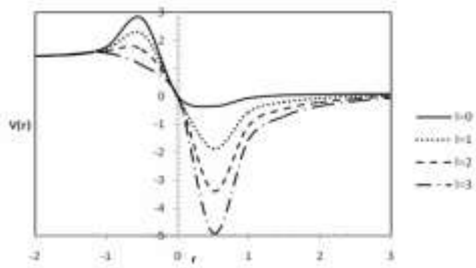

Fig 8: The plot of effective potential V_{eff} with r for l=0,1,2,3 and q=-1

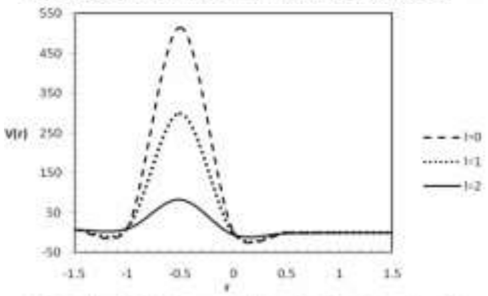

Fig 9: The plot of effective potential V_{eff} with r for l=0,1,2 and q=-0.4

Conclusion

In this paper, we solve the Schrodinger equation analytically with Woods-Saxon potential using a new ansatz for the wavefunction. We discuss the shape of these potentials and interestingly the Rosen-Morse potential capture the essentials of QCD quark-gluon dynamics. We obtain the eigenfunction and the corresponding eigenvalues for the potential and expressed the wavefunction in terms of the Jacobi polynomials. It is shown that our results reduce to those obtained in literatures when the imaginary part tends to zero.

References

1. A.N Ikot and L.E Akpabio, Approximate Analytical Solution of the Schrodinger equation with Hulthen potential for arbitrary l-state, International Review of Physics, 4, 224 (2010).
2. A.N Ikot and L.E Akpabio, Approximate Solution of the Schrodinger equation with Rosen-Morse potential including the Centrifuge term, Applied Physics Research, 2, 202 (2010).
3. A.N Ikot, L.E Akpabio and E.B Umoren, Exact Solution of Schrodinger equation with inverted Woods-Saxon and Manning-Rosen potential, J. Sci. Res, to appear, doi:10.3329/jsr.v3il.5310.
4. A.D Antia, A.N Ikot and L.E Akpabio, Exact Solutions of Schrodinger equation with Manning-Rosen potential plus a ring-shaped like potential by Nikiforov-Uvarov method, Euro. J. Sci. Res. 46, 107 (2010)
5. A.T Nikiforov and V.B Uvarov, Special Functions of mathematical physics, (Basel, Birklauser, 1988)
7. S. Flugge, PracticalQuatum Mechanics, vol. 1 (Springer, Berlin, 1994).
8. L.D Landau and L.M Lifshitz, Quantum Mechanics (Pergamon Press, London, 1958).
9. D. Rapp, Quantum Mechanics (Holt, Rinheart and Wilson Inc, New York, 1970.
10. M.R Sedare and E. Karimi, Algebraic Approach to the Hulthen Potential, arxiv: quant-ph/0703267.
11. O. Ozer, Eigenvalue Problems of non-Hermitan Systems via Improved Asymptotic Iteration method, Chin. Phys. Lett. 25, 3111 (2008).
12. O. Bayrak, G. Kocak and I. Boztosun, Any l-state solution on the Hulthen potential by the asymptotic interaction method,arxiv;Math-ph/0609010
13. M.R. Setare and O.Hatami, exact solutions of the Dirax Equation for an Electron in a magnetic Field with shape Invariant Method, Chin.Phys.Leth. 25, 3848 (2008).
14. H. Motavali and A. Rostami, A new Mathematical purposes for generation of shape invariant potentials and optical medium using Supersymmetric quantum machanics, Progress in Electromagnetic Research c, 1, 131 (2008).
15. M.R Niazian, A Super-symmetry Approach to Hydrogen-like Atoms under constant Electromagnet Field, African Physical Review, 4; 008, 58 (2010).
16. S.H Dong, Factorization Method in Quantum Mechanics, (Springer, Dordrecht, (2007).
17. J. Sadeghi and B. Pourhassan, Exact Solution of the Non-Central Modified Krfatzer potential plus a ring-shaped like potential by the Factorization Method, Electronic Journal of Theoretical Physics, 17, 193 (2008).
18. S.A.S Ahmed and L. Buragohan, Exactly solved potentials generated from the Manning-Rosen potential using Extended Transformation Method, Electronic Journal of Theoretical Physics, 23 ,145, (2010).
19. A.R Amani, J. Sadeghi and T Barzegor, The Stability of Tachyonic Field System by Factorization Method, Applied

Sciences, 11, 1 (2009).
20. S.M Ikhdair and R. Sever, Polynomial Solution of Non-Central potentials, Int. J. Theor. Phys. 46: 2384 (2009).
21. C. Quesne, Solvable Rational potentials and Exceptional Orthogonal Polynomial in Super-symmetric Quantum Mechanics, Sigma, 5: 084, 1 (2009).
22. S. Meyur, and S. Debnath, Eigen Spectra for Woods-Saxon potential, Lat. Am. J. Phys. Educ. 4, 637 (2010).
23. Y. Xu, S. He and C.S Jia, Approximate Analytical Solution of the Klein-Gordon equation with the Posche-Teller potential including the Centrifugal term, phys. Scr. 81, 4501: 1 (2010).
24. G. F Wei, X. Y Liu and W.L Chen, The Relativistic Scattering States of the Hulthen potential with an Improved new approximate Scheme to the Centrifugal Term, Int. J. Theor. Phys. 48; 1649 (2009).
25. M.R Pahlavai, J. Sadeghi and M. Ghezelbash, App. Sci. 11, 106 (2009).
26. A.D Polyamin and A.V Manzhirov, Handbook of Integral Equations (CRC Press Boca Rodan, 1998).
27. C.B. Compean and M. Kirchbach, Angular Momentum Dependent Quark Potential of QCD and Dynamical O(4) Symmetry, Proceeding of the Mini-Workshop Progress in Quark Models, Bled, Slovenia, July 10-17, 2006.