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On approximations of the series involving quadruple hypergeometric function k_{12}

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ABSTRACT

In this paper, we apply our generalized theorem on summability to derive some approximation formulae of the series involving quadruple hypergeometric function K_{12}. This work may be useful in the field for computation of the solution of hydrodynamical problems.

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Introduction

Exton (1972,1976) had defined following complete quadruple hypergeometric function

$$K_{12}(a,a,a,a,b_1,b_2,b_3,b_4,c_1,c_1,c_2,c_2;x,y,z,t)$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_{m+n}(c_2)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!} \tag{1.1}$$

The normalized Dirichlet integral in the dimensional space $R^n\subset C^{n+1}$ has been introduced due to Mathai and Houbold (2008) in the form

$$I(\mu)=\int_{R^n}d_\mu(x)$$

$$=\frac{1}{B(\mu)}\int_0^1\cdot(n)\cdot\int_0^1 x_1^{\mu_1}\ldots x_n^{\mu_n}(1-x_1-\cdots-x_n)^{\mu_{n+1}-1}dx_1\ldots dx_n \tag{1.2}$$

where, $x=(x_1,\cdots,x_n)\in R^n$ such that

$$0\le(x_1,\cdots,x_n)\le 1,\mu=(\mu_1,\cdots,\mu_n)\in C^{n+1},$$

$$B(\mu)=\frac{\Gamma(\mu_1)\cdots\Gamma(\mu_n)\Gamma(\mu_{n+1})}{\Gamma(\mu_1+\cdots+\mu_n+\mu_{n+1})},Re(\mu_i)>0,\forall i=1,2,\cdots,n,n+1.$$

A generalized Basanquet and Kasteman (1939) theorem of summability has been presented by Kumar and Yadav (2010) such that

Theorem-1

Suppose that $f_n(x_1,\cdots,x_r)$ is measurable in the region $x_1\in(0,\alpha_1),\cdots,x_r\in(0,\alpha_r)$, where, $\alpha_i>0,\forall i=1,2,\ldots,r$, then a necessary and sufficient condition for every probability density function $g(x_1,\cdots,x_r)$ defined in the region $x_1\in(0,\alpha_1),\cdots,x_r\in(0,\alpha_r)$ there exists

$$\sum_{n=0}^{\infty}\left|\int_0^{\alpha_1}\cdots\int_0^{\alpha_r}g(x_1,\cdots,x_r)f_n(x_1,\cdots,x_r)dx_1\ldots dx_r\right|\le\xi \tag{1.3}$$

$$\sum_{n=0}^{\infty}|f_n(x_1,\cdots,x_r)|\le\xi \tag{1.4}$$

where, ξ is an absolute constant for almost every $x_i\in(0,\alpha_i),\alpha_i>0,\forall i=1,2,\ldots,r$.

Again, Kumar, Pathan, Priyanka (2009) have derived following theorem:

Theorem-2

For

$$\alpha>0,\beta>0,\gamma>0,\delta>0,Re(\sigma_i)>Re(-\mu_i)>0,\forall i=1,2,3,4\ and$$

$$Re\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)>0,a,b=(b_1,b_2,b_3,b_4),c=(c_1,c_2),\mu=(\mu_1,\mu_2,\mu_3,\mu_4),$$

$h_1,h_2,h_3,and\ h_4\in C$, a function due to a weighted Dirichlet type integral formula exists

$$F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=\frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(a-\sum_{i=1}^{4}(\mu_i+\sigma_i))}$$

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$$\times \int_0^{\alpha}\int_0^{\beta}\int_0^{\gamma}\int_0^{\delta}(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1})^{a-\mu_1-\sigma_1-\mu_2-\sigma_2-\mu_3-\sigma_3-\mu_4-\sigma_4-1}$$

$$\times x^{\mu_1+\sigma_1-1}y^{\mu_2+\sigma_2-1}z^{\mu_3+\sigma_3-1}t^{\mu_4+\sigma_4-1}$$

$$\times K_{12}(a,a,a,a,b_1,b_2,b_3,b_4,c_1,c_1,c_2,c_2;x,y,z,t)\,dx\,dy\,dz\,dt \tag{1.5}$$

Provided that

$$0 \le x\alpha^{-1}+y\beta^{-1}+z\gamma^{-1}+t\delta^{-1} \le 1.$$

Then, for

$$max\{|h_1\alpha|,|h_2\beta|\} < 1 \text{ and } max\{|h_3\gamma|,|h_4\delta|\} < 1$$

there holds the degeneration formula

$$F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta) = F_3[\mu_1+\sigma_1,\mu_2+\sigma_2,b_1,b_2;c_1;h_1\alpha,h_2\beta]$$

$$F_3[\mu_3+\sigma_3,\mu_4+\sigma_4,b_3,b_4;c_2;h_3\gamma,h_4\delta] \tag{1.6}$$

Inequalities

Here, in our investigation, first we obtain some inequalities of the function

$$F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta).$$

Then make their applications to obtain approximation formulae of quadruple hypergeometric function K_{12}.

Theorem-3

For

$0<h_1\alpha<1, 0<h_2\beta<1, 0<h_3\gamma<1, 0<h_4\delta<1, c_1>\mu_1+\sigma_1>c_1-b_1>0, c_1>\mu_2+\sigma_2>c_1-b_2>0, c_2>\mu_3+\sigma_3>c_2-b_3>0 \text{ and } c_2>\mu_4+\sigma_4>c_2-b_4>0$

then there holds an inequality

$$F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)$$

$$< \frac{\Gamma(\mu_1+\sigma_1-c_1+b_1)\Gamma(\mu_2+\sigma_2-c_1+b_2)\Gamma(\mu_3+\sigma_3-c_2+b_3)\Gamma(\mu_4+\sigma_4-c_2+b_4)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}$$

$$(\Gamma(c_1))^2(\Gamma(c_1))^2(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_2\beta)^{c_1-\mu_2-\sigma_2-b_2}(1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}$$

$$(1-h_4\delta)^{c_2-\mu_4-\sigma_4-b_4}\times {}_2F_1\left[\begin{matrix}(c_1-1),\frac{(c_1+1)}{2};\\ \frac{(c_1-1)}{2};\end{matrix}\; -h_1h_2\alpha\beta\right]$$

$$\times {}_2F_1\left[\begin{matrix}(c_2-1),\frac{(c_2+1)}{2};\\ \frac{(c_2-1)}{2};\end{matrix}\; -h_3h_4\gamma\delta\right] \tag{2.1}$$

Proof:

Under the restrictions

$c>a_1>c-b_1>0, c>a_2>c-b_2>0, 0<x<1, 0<y<1$, Joshi and Arya (1991) have derived the inequality

$$F_3[a_1,a_2,b_1,b_2;c;x,y] < \frac{\Gamma(a_1+b_1-c)\Gamma(a_2+b_2-c)(\Gamma(c))^2}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)\Gamma(b_2)}$$

$$\times (1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2}\times {}_2F_1\left[\begin{matrix}(c-1),\frac{(c+1)}{2};\\ \frac{(c-1)}{2};\end{matrix}\; -xy\right] \tag{2.2}$$

In the right hand side of the equation (1.6) for both $F_3[.]$ functions under the restrictions

$0<h_1\alpha<1, 0<h_2\beta<1, 0<h_3\gamma<1, 0<h_4\delta<1, \mu_1+\sigma_1>c_1-b_1>0, \mu_2+\sigma_2>c_1-b_2>0, \mu_3+\sigma_3>c_2-b_3>0 \text{ and } \mu_4+\sigma_4>c_2-b_4>0, c_1>0, c_2>$,

apply the formula (2.2), we find the inequality (2.1).

Theorem 5.

For

$0<h_1\alpha<1,\ 0<h_2\beta<1 \text{ and } 0<h_3\gamma<1, 0<h_4\delta<1 \text{ and } \mu_1+\sigma_1>c_1-b_1>0, \mu_2+\sigma_2>c_1-b_2>0, \mu_3+\sigma_3>c_2-b_3>0 \text{ and } \mu_4+\sigma_4>c_2-b_4>0, c_1>c_2>0$

,then there holds an inequality

$$F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)$$

$$< \frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+b_2-c_1)\Gamma(\mu_3+\sigma_3+b_3-c_2)\Gamma(\mu_4+\sigma_4+b_4-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}$$

$$(\Gamma(c_1))^2(\Gamma(c_1))^2(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_2\beta)^{c_1-\mu_2-\sigma_2-b_2}(1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}$$

$$(1-h_4\delta)^{c_2-\mu_4-\sigma_4-b_4}(1+h_1h_2\alpha\beta)^{-c_1}(1+h_3h_4\gamma\delta)^{-c_2}(1-h_1h_2\alpha\beta)(1-h_3h_4\gamma\delta) \tag{2.3}$$

Proof:

The contiguous function relation for Gaussian hypergeometric function ${}_2F_1(.)$ in the notations (Rainville, (1971,p.53))

$F = {}_2F_1[a, b ; c; x]$, $F(a+) = {}_2F_1[a+1,b;c;x]$, $F(a-) = {}_2F_1[a-1,b;c;x]$, is given by

$$(1-x)\,F = F(b-) - c^{-1}(c-a)\times F(c+) \tag{2.4}$$

Multiply $((1+h_1h_2\alpha\beta)(1+h_3h_4\gamma\delta))$ in both sides of the inequality (2.1) and then in its right hand side use the contiguous function relation (2.4) and again solving it we obtain the inequality (2.3). 5. Approximation of quadruple hypergeometric function K_{12}

In this section, we use the inequalities obtained in the section 2 and approximate quadruple hypergeometric function K_{12}

Theorem 5

If

$x\in(0,\alpha), y\in(0,\beta), z\in(0,\gamma), \text{ and } t\in(0,\delta), \text{ such that } \alpha>0,\beta>0,\gamma>0,\delta>0, 0\le x\alpha^{-1}+y\beta^{-1}+z\gamma^{-1}+t\delta^{-1}\le 1, \text{ then for } \lambda>0, 0<h_1\alpha<1, 0<h_2\beta<1 \text{ and } 0<h_3\gamma<1, 0<h_4\delta<1, \mu_1+\sigma_1>c_1-b_1>0, \mu_2+\sigma_2>c_1-\lambda>0, \mu_3+\sigma_3>c_2-b_3>0 \text{ and } \mu_4+\sigma_4>c_2-\lambda>0, c_1>0,\ c_2>0, |T|<1$,

following approximation formula of quadruple hypergeometric function K_{12} holds :

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}K_{12}(a,a,a,a,b_1,\lambda+n,b_3,\lambda+n;c_1,c_1,c_2,c_2;h_1x,h_2y,h_3z,h_4t)T^n$$

$$< (1-T)^{-\lambda}\frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+\lambda-c_1)\Gamma(\mu_3+\sigma_3+b_3-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_3)}$$

$$\frac{\Gamma(\mu_4+\sigma_4+\lambda-c_2)}{\{\Gamma(\lambda)\}^2}(\Gamma(c_1))^2(\Gamma(c_2))^2(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}$$

$$\left(1-\frac{h_4\delta T}{1-T}\right)^{c_2-\mu_4-\sigma_4-\lambda}\left(1-\frac{h_2\beta T}{1-T}\right)^{c_1-\mu_2-\sigma_2-\lambda}\left(1+\frac{h_1h_2\alpha\beta T}{1-T}\right)^{-c_1}\left(1+\frac{h_3h_4\gamma\delta T}{1-T}\right)^{-c_2}$$

$$\left(1-\frac{h_1h_2\alpha\beta T}{1-T}\right)\left(1-\frac{h_3h_4\gamma\delta T}{1-T}\right)$$

$$\times {}_4F_3\left[\begin{matrix}c_1,c_2,\mu_2+\sigma_2+\lambda-c_1,\mu_4+\sigma_4+\lambda-c_2;\\ \lambda,1+c_1-\mu_1-\sigma_1-b_1,1+c_2-\mu_3-\sigma_3-b_3;\end{matrix}\right.$$

$$\left.\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2\beta h_4\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right]$$

(3.1)

Provided that

$$\left|\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2\beta h_4\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right| < 1.$$

Proof:

To prove above theorem, we consider the bilinear generating relation of Kumar,Pathan and Yadav(2009) given by, when |T|<1,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,\lambda+n,b_3,\lambda+n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n$$

$$=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2\beta h_4\delta T}{(1-T)^2}\right\}^n$$

$$F_3\left[\mu_1+\sigma_1,\mu_2+\sigma_2+n,b_1,\lambda+n;c_1+n;h_1\alpha,\frac{h_2\beta}{1-T}\right]$$

$$F_3\left[\mu_1+\sigma_1,\mu_2+\sigma_2+n,b_1,\lambda+n;c_1+n;h_1\alpha,\frac{h_2\beta}{1-T}\right]$$

(3.2)

Then, we follow the equation (1.6) and the theorem -4 in right hand side of the relation (3.2), we find that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,\lambda+n,b_3,\lambda+n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n$$

$$< (1-T)^{-\lambda}\frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+\lambda-c_1)\Gamma(\mu_3+\sigma_3+b_3-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_3)}$$

$$\frac{\Gamma(\mu_4+\sigma_4+\lambda-c_2)}{\{\Gamma(\lambda)\}^2}(\Gamma(c_1))^2(\Gamma(c_2))^2(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}$$

$$\left(1-\frac{h_4\delta T}{1-T}\right)^{c_2-\mu_4-\sigma_4-\lambda}\left(1-\frac{h_2\beta T}{1-T}\right)^{c_1-\mu_2-\sigma_2-\lambda}\left(1+\frac{h_1h_2\alpha\beta T}{1-T}\right)^{-c_1}\left(1+\frac{h_3h_4\gamma\delta T}{1-T}\right)^{-c_2}$$

$$\left(1-\frac{h_1h_2\alpha\beta T}{1-T}\right)\left(1-\frac{h_3h_4\gamma\delta T}{1-T}\right)$$

$${}_4F_3\left[\begin{matrix}c_1,c_2,\mu_2+\sigma_2+\lambda-c_1,\mu_4+\sigma_4+\lambda-c_2;\\ \lambda,1+c_1-\mu_1-\sigma_1-b_1,1+c_2-\mu_3-\sigma_3-b_3;\end{matrix}\right.$$

$$\left.\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2\beta h_4\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right]$$

(3.3)

Provided that

$$\left|\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2\beta h_4\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right| < 1.$$

Now in left hand side of (3.3) define the function $F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)$, by the theorem -2 and then make an appeal to the theorem -1, we get the approximation formula (3.1) of quadruple hyper geometric function K_{12} .

Example

Let in the region, $x\in(0,\alpha), y\in(0,\beta), z\in(0,\gamma), and\ t\in(0,\delta)$, such that $\alpha>0,\beta>0,\gamma>0,\delta>0, 0\le x\alpha^{-1}+y\beta^{-1}+z\gamma^{-1}+t\delta^{-1}\le 1$, the position of the particle is given by the sequence of function

$$f_n(x,y,z,t)=K_{12}(a,a,a,a,b_1,\lambda+n,b_3,\lambda+n;c_1,c_1,c_2,c_2;h_1x,h_2y,h_3z,h_4t)$$

$\forall\, n\in N_0=\{0,1,2,\dots\}$, Then there exists a convergent function $g(x,y,z,t)$ defined for $\lambda>0$, and |T| < 1, such that

$g(x,y,z,t)=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}f_n(x,y,z,t)T^n$, then

$g(x,y,z,t)$

$$< (1-T)^{-\lambda}\frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+\lambda-c_1)\Gamma(\mu_3+\sigma_3+b_3-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_3)}$$

$$\frac{\Gamma(\mu_4+\sigma_4+\lambda-c_2)}{\{\Gamma(\lambda)\}^2}(\Gamma(c_1))^2(\Gamma(c_2))^2(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}$$

$$\left(1-\frac{h_4\delta T}{1-T}\right)^{c_2-\mu_4-\sigma_4-\lambda}\left(1-\frac{h_2\beta T}{1-T}\right)^{c_1-\mu_2-\sigma_2-\lambda}\left(1+\frac{h_1h_2\alpha\beta T}{1-T}\right)^{-c_1}\left(1+\frac{h_3h_4\gamma\delta T}{1-T}\right)^{-c_2}$$

$$\left(1-\frac{h_1h_2\alpha\beta T}{1-T}\right)\left(1-\frac{h_3h_4\gamma\delta T}{1-T}\right)$$

$${}_4F_3\left[\begin{matrix}c_1,c_2,\mu_2+\sigma_2+\lambda-c_1,\mu_4+\sigma_4+\lambda-c_2;\\ \lambda,1+c_1-\mu_1-\sigma_1-b_1,1+c_2-\mu_3-\sigma_3-b_3;\end{matrix}\right.$$

$$\left.\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2\beta h_4\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right]$$

(4.1)

provided that

$$\left|\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2\beta h_4\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right| < 1.$$

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