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## **Discrete Mathematics**



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Elixir Dis. Math. 34C (2011) 2468-2476

# Cordial labeling for middle graph of some graphs

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This paper is aimed to discuss cordial graphs in the context of middle graph of a graph. We

present here cordial labeling for the middle graphs of path, crown, star and tadpole.

#### ARTICLE INFO

Article history: Received: 12 March 2011; Received in revised form: 22 April 2011; Accepted: 27 April 2011;

### Keywords

Cordial labeling, Cordial graph, Middle graph. AMS Subject classification number(2010): 05C78.

#### Introduction

We begin with simple, finite and undirected graph G = (V(G), E(G)). In the present work |V(G)| and |E(G)|

denote the number of vertices and edges in the graph G respectively. For all other terminology and notations we follow Harary[1]. We will give brief summary of definitions which are useful for the present investigations.

**Definition** -1.1: If the vertices of the graph are assigned values subject to certain conditions then is known as *graph labeling*.

An extensive survey on graph labeling we refer to Gallian[2]. According to Beineke and Hegde[3] graph labeling serves as a frontier between number theory and structure of graphs. A detailed study of variety of applications of graph labeling is reported in Bloom and Golomb [4].

**Definition** -1.2 : Let G be a graph. A mapping  $f:V(G) \rightarrow \{0,1\}$  is called *binary vertex labeling* of G and f(v) is called the *label* of the vertex v of G under f.

For an edge e = uv, the induced edge labeling  $f^*: E(G) \to \{0,1\}$  is given by  $f^*(e) = |f(u) - f(v)|$ . Let  $v_f(0), v_f(1)$  be the number of vertices of G having labels 0 and 1 respectively under f and let  $e_f(0), e_f(1)$  be the number of edges having labels 0 and 1 respectively under  $f^*$ . **Definition** -1.3 : A binary vertex labeling of a graph G is called a *cordial labeling* if  $|v_f(0) - v_f(1)| \le 1$  and  $|e_f(0) - e_f(1)| \le 1$ . A graph G is *cordial* if it admits cordial labeling. The concept of cordial labeling was introduced by Cahit[5] and he proved that every tree is cordial. In the same paper he proved that  $K_n$  is cordial if and only if  $n \le 3$ . Ho et

Tele: E-mail addresses: samirkvaidya@yahoo.co.in, viholprakash@yahoo.com al.[6] proved that unicyclic graph is cordial unless it is  $C_{4k+2}$ . Andar et al.[7] has discussed cordiality of multiple shells. Vaidya et al.[8],[9],[10],[11] have also discussed the cordiality of various graphs.

**Definition** -1.4: The *middle graph* M(G) of a graph G is the graph whose vertex set is V(G)UE(G) and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

In the present investigations we prove that the middle graphs of path, crown (The Crown  $(C_n \in K_1)$  is obtained by joining a single pendant edge to each vertex of  $C_n$ ), star and tadpole (Tadpole T(n,l) is a graph in which path  $P_l$  is attached to any one vertex of cycle  $C_n$ ) admit cordial labeling. **Main Results** 

**Theorem - 2.1:** The middle graph M(G) of an Eulerian graph G is Eulerian and  $|E(M(G))| = \frac{\sum_{i=1}^{n} d(v_i)^2 + 2e}{2}$ 

**Proof:** Let G be an Eulerian graph. If  $v_1, v_2, v_3, ..., v_n$  are vertices of G and  $e_1, e_2, e_3, ..., e_q$  are edges of G then  $v_1, v_2, v_3, ..., v_n, e_1, e_2, ..., e_q$  are the vertices of M(G). Then it is obvious that if  $d(v_i)$  is even in G then it remains even in M(G).

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ABSTRACT

Now it remains to show that  $d(e_i)$  is even in M(G). For that if v' and v'' are the vertices adjacent to any vertex  $e_i$  then  $d(e_i) = (d(v')-1) + (d(v'')-1)$ = d(v') + d(v'') - 2 which is even as both d(v') and d(v'') are even for  $1 \le i \le q$ . Therefore M(G) is an Eulerian graph. It is also obvious that the  $d(v_i)$  number of edges are incident with each vertex  $v_i$  of q = M(G)

G which forms a complete graph  $K_{d(v_i)}$  in M(G).

Now if the total number of edges in M(G) be denoted as  $|E(M(G))|_{\text{then}}$ 

$$|E(M(G))| = d(v_1) + d(v_2) + d(v_3) + \dots + d(v_n) + |E(K_{d(v_1)})| + |E(K_{d(v_2)})| + |E(K_{d(v_3)})| + \dots + |E(K_{d(v_n)})|$$

$$= d(v_1) + d(v_2) + d(v_3) + \dots + d(v_n) + \frac{d(v_1)(d(v_1) - 1)}{2} + \frac{d(v_2)(d(v_2) - 1)}{2} + \dots + \frac{d(v_n)(d(v_n) - 1)}{2}$$

$$= \frac{d(v_1)^2}{2} + \frac{d(v_2)^2}{2} + \dots + \frac{d(v_n)^2}{2} + \frac{d(v_1)}{2} + \frac{d(v_2)}{2} + \dots + \frac{d(v_n)}{2}$$

$$= \frac{\sum_{i=1}^n d(v_i)^2 + \sum_{i=1}^n d(v_i)}{2}$$
But 
$$\sum_{i=1}^n d(v_i) = 2e$$

$$|E(M(G))| = \frac{\sum_{i=1}^n d(v_i)^2 + 2e}{2}$$
proved.

**Corollary - 2.2 :** The middle graph M(G) of any graph G is

$$E(M(G)) \models \frac{\sum_{i=1}^{n} d(v_i)^2 + 2e}{2}$$

not cordial when  $\equiv 2(mod 4)$ 

**Proof:** By Theorem 2.1, for M(G) of any graph G,

$$|E(M(G))| = \frac{\sum_{i=1}^{n} d(v_i)^2 + 2e}{2}.$$

Then as proved by Cahit[5] an Eulerian graph with  $e \equiv 2 \pmod{4}$  is not cordial.

**Theorem - 2.3 :**  $M(P_n)$  is a cordial graph.

**Proof:** If  $v_1, v_2, ..., v_n$  and  $e_1, e_2, ..., e_n$  are respectively the vertices and edges of  $P_n$  then  $v_1, v_2, ..., v_n, e_1, e_2, ..., e_n$  are the vertices of  $M(P_n)$ .

To define  $f: V(M(P_n)) \rightarrow \{0,1\}$ , we consider following four cases.

**Case 1:**  $n \text{ is odd}, n = 2k + 1, k = 1, 3, 5, 7, \dots$ In this case  $|V(M(P_n))| = 2n - 1, |E(M(P_n))| = 2n + 2k - 3$  We label the vertices as follows.

$$f(v_{2i-1}) = 0 \quad \text{for} \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(v_{2i}) = 1 \quad \text{for} \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(e_{4i-3}) = 1 \quad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(e_{4i-2}) = 1 \quad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(e_{4i-1}) = 0 \quad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor$$

In view of the above defined labeling pattern we have  $v_f(0) + 1 = v_f(1) = n \quad e_f(0) = e_f(1) + 1 = n + k - 1$ 

**Case 2:**  $n_{\text{odd}}$ , n = 2k + 1, k=2,4,6...In this case  $|V(M(P_n))| = 2n - 1$ ,  $|E(M(P_n))| = 2n + 2k - 3$ We label the vertices as follows.

$$f(v_{2i-1}) = 0 \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$f(v_{2i}) = 1 \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(e_{4i-3}) = 0$$

$$f(e_{4i-2}) = 0$$

$$f(e_{4i-1}) = 1$$

$$f(e_{4i}) = 1$$

$$1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor$$

In view of the above defined labeling pattern we have  $v_f(0) = v_f(1) + 1 = n \quad e_f(0) = e_f(1) + 1 = n + k - 1$ 

**Case 3:**  $n \text{ even, } n = 2k \text{ }_{k=1,3,5,7,\dots}$ In this case  $|V(M(P_n))| = 2n-1$ ,  $|E(M(P_n))| = 2n+2k-4$ We label the vertices as follows.

$$\begin{cases} f(v_{2i-1}) = 0\\ f(v_{2i}) = 1 \end{cases} 1 \le i \le \frac{n}{2}$$

$$f(e_{4i-3}) = 0 \quad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor + 1$$

$$f(e_{4i-2}) = 0 \quad 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor$$

$$\begin{cases} f(e_{4i-1}) = 1 \\ f(e_{4i}) = 1 \end{cases} \ \ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor$$

**Case 4:** n even,  $n = 2k \text{ k}_{=2,4,6,...}$ In this case  $|V(M(P_n))| = 2n-1$ ,  $|E(M(P_n))| = 2n+2k-4$ We label the vertices as follows.

$$\begin{cases} f(v_{2i-1}) = 0\\ f(v_{2i}) = 1 \end{cases} 1 \le i \le \frac{n}{2}$$

$$\begin{cases} f(e_{4i-3}) = 0\\ f(e_{4i-2}) = 0 \end{cases} 1 \le i \le \frac{n}{4}$$

$$f(e_{4i-1}) = 1 \text{ for } 1 \le i \le \frac{n}{4}$$

$$f(e_{4i}) = 1 \text{ for } 1 \le i \le \frac{n}{4} - 1$$

In above two cases we have  $v_f(0) = v_f(1) + 1 = n$  $e_{f}(0) = e_{f}(1) = n + k - 2$ 

Thus in all the four cases f satisfies the condition for cordial labeling. That is,  $M(P_n)$  is a cordial graph.

**Illustration - 2.4 :** In the following Figure 2.1  $M(P_7)$  and its cordial labeling is shown.



Figure 2.1  $M(P_7)$  and its cordial labeling Theorem - 2.5 : The middle graph of crown is a cordial graph. **Proof:** Consider the crown  $C_n \in K_1$  in which  $v_1, v_2, \dots, v_n$ be the vertices of cycle  $C_n$  and  $v'_1, v'_2, ..., v'_n$  be the pendant vertices attached at each vertex of  $C_n$ . Let  $e_1$ ,  $e_2$ , ...,  $e_n$  and  $e'_1$ ,  $e'_2$ , ...,  $e'_n$  are vertices corresponding to edges of  $C_n \in K_1$  in  $M(C_n \in K_1)$ 

To define ,  $f: V(M(C_n \in K_1)) \rightarrow \{0,1\}$  we consider following three cases.

**Case 1:**  $n_{\text{is odd,}} n = 2k + 1, k = 2,4,6,\dots$  $|V(M(C_n \in K_1))| = 4n$  and this In case  $|E(M(C_n \in K_1))| = 6n + 2 \left| \frac{n}{2} \right| + 1$ 

We label the vertices as follows.

$$\begin{cases} f'(v_{2i-1}) = 0\\ f(v_{2i}) = 1 \end{cases} \\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ f(v_{n}) = 1 \\ f(v_{2i-1}) = 1 \\ f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \\ for \\ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ f(v_{2l-\frac{n}{4} \rfloor + 2i}) = 1 \\ f(v_{2l-\frac{n}{4} \rfloor + 2i+1}) = 0 \\ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ f(e_{2i-1}) = 1 \\ f(e_{2i}) = 0 \\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ (e_{n}) = 0 \\ (e_{n}) = 0 \\ for \\ (e_{2i-1}) = 1 \\ for \\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ +1 \\ (e_{2i}) = 1 \\ for \\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ he above defined pattern \\ f(e_{2i-1}) = 0 \\ for \\ f(e_{2i-1}) = 1 \\ f(e$$

In view of th

$$v_{f}(0) = v_{f}(1) = 2n, \quad e_{f}(0) + 1 = e_{f}(1) = 3n + \left\lfloor \frac{n}{2} \right\rfloor + 1$$
  
**Case 2:**  $n$  is odd,  $n = 2k + 1, k = 1, 3, 5, 7, \dots$   
In this case  $|V(M(C_{n} \in K_{1}))| = 4n$  and  $|E(M(C_{n} \in K_{1}))| = 6n + 2\left\lfloor \frac{n}{2} \right\rfloor + 1$   
We label the vertices as follows

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We label the vertices as follo

$$\begin{cases} f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i}) = 1 \\ f(v'_{2\lfloor \frac{n}{4} \rfloor + 2i+1}) = 0 \end{cases} 1 \le i \le \lfloor \frac{n}{4} \rfloor + 1$$

Now label the remaining vertices as in case 1. In view of the above defined pattern we have

$$v_{f}(0) = v_{f}(1) = 2n e_{f}(0) = e_{f}(1) + 1 = 3n + \lfloor \frac{n}{2} \rfloor + 1$$
  
**Case 3:** *n* is even,  $n = 2k$ ,  $k = 2, 3, ...$   
In this case  $|V(M(C_{n} \in K_{1}))| = 3n$  and  $|E(M(C_{n} \in K_{1}))| = 7n$ 

We label the vertices as follows.

$$\begin{cases} f(v_{2i-1}) = 0\\ f(v_{2i}) = 1 \end{cases} 1 \le i \le \frac{n}{2}$$

$$f(v'_i) = 1_{\text{for } 1 \le i \le n}$$

$$f(e_i) = 0_{\text{for } 1 \le i \le n}$$

$$f(e'_{2i-1}) = 1_{\text{for } 1 \le i \le \frac{n}{2}}$$

$$1 \le i \le \frac{n}{2}$$

In view of the above defined pattern we have

$$v_f(0) = v_f(1) = \frac{3n}{2}, e_f(0) = e_f(1) = 3n + \frac{n}{2}$$

Thus in all the cases described above f admits cordial labeling for the graph under consideration. That is, middle graph of the crown is a cordial graph.

**Illustration - 2.6 :** In the following Figure 2.2 cordial labeling for  $M(C_7 \in K_1)$  is shown.



**Figure 2.2**  $M(C_7 \in K_1)$  and its cordial labeling **Theorem - 2.7 :**  $M(K_{1,n})$  is a cordial graph.

**Proof:** Let v,  $v_1$ ,  $v_2$ , ...,  $v_n$  be the vertices of star  $K_{1,n}$  with v as an apex vertex and  $e_1$ ,  $e_2$ , ...,  $e_n$  be the vertices in  $M(K_{1,n})$  corresponding to the edges  $e_1$ ,  $e_2$ , ...,  $e_n$  in  $K_{1,n}$ . To define  $f:V(M(K_{1,n})) \rightarrow \{0,1\}$ , we consider following two cases.

**Case 1:**  $n = 2k + 1, k = 2, 3, 4, \dots$ 

In this case 
$$|V(M(K_{1,n}))| = 2n+1$$
,  $|E(M(K_{1,n}))| = 2n(\lfloor \frac{k}{2} \rfloor + 1)$ 

or  $|E(M(K_{1,n}))| = 2n(\lfloor \frac{k}{2} \rfloor + 1) + 2k + 1$  depending upon k = 2, 4, 6, 8... or k = 3, 5, 7, 9...

$$f(e_{2i-1}) = 0, \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$$
  

$$f(e_{2i}) = 1, \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$
  

$$f(v_{n-i}) = p_i, \text{ where } p_i = 1, \text{ if } i \text{ is even,}$$
  

$$= 0, \text{ if } i \text{ is odd, } 0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor - 1$$

$$f(v_{n-\left\lfloor\frac{k}{2}\right\rfloor-i}) = f(e_{n-\left\lfloor\frac{k}{2}\right\rfloor-i}), \ 0 \le i \le n - \left\lfloor\frac{k}{2}\right\rfloor - 1$$
$$f(v) = 1$$

Using above pattern if k = 2, 3, 6, 7.... then  $v_f(0) + 1 = v_f(1) = n + 1$  and if k = 4, 5, 8, 9.... then  $v_e(0) = v_e(1) + 1 = n + 1$ .

If 
$$k = 2, 4, 6, 8...$$
 then  $e_f(0) = e_f(1) = n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)$  and if  $k = 3, 5, 7, ...$  then  $e_f(0) = e_f(1) + 1 = n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + k + 1$   
**Case 2:**  $n = 2k, k = 2, 3, 4, ...$   
In this case  $|V(M(K_{1,n}))| = 2n + 1$ ,  $|E(M(K_{1,n}))| = 2n\left(\frac{k}{2} + 1\right) - k$   
or  $|E(M(K_{1,n}))| = 2n\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + 2\left\lfloor \frac{k}{2} \right\rfloor - 1$  depending upon  $k = 2, 4, 6, 8...$  or  $k = 3, 5, 7, 9...$   
 $f(e_{2i-1}) = 0, \quad 1 \le i \le \frac{n}{2}$   
 $f(e_{2i}) = 1, \quad 1 \le i \le \frac{n}{2}$   
 $f(v_{n-i}) = p_i$ , where  $p_i = 0$ , if  $i$  is even,  
 $= 1$ , if  $i$  is odd,  $0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor - 1$   
 $f\left(v_{n-\lfloor \frac{k}{2} \rfloor - i\right) = f\left(e_{n-\lfloor \frac{k}{2} \rfloor - i}\right), \quad 0 \le i \le n - \lfloor \frac{k}{2} \rfloor - 1$ 

Using above pattern if k = 2, 3, 6, 7.... then  $v_f(0) = v_f(1) + 1 = n + 1$  and if k = 4, 5, 8, 9.... then  $v_f(0) + 1 = v_f(1) = n + 1$ .

If 
$$k = 2, 4, 6, 8...$$
 then  $e_f(0) = e_f(1) = n(\frac{k}{2}+1) - \frac{k}{2}$  and if  $k = 3, 5, 7, ...$  then  $e_f(0) = e_f(1) + 1 = n(\lfloor \frac{k}{2} \rfloor + 1) + \lfloor \frac{k}{2} \rfloor$ .

Also note that for n = 2 we have  $v_f(0) = v_f(1) + 1 = 3$  and  $e_f(0) + 1 = e_f(1) = 3$ .

Thus in all the cases described above f admits cordial labeling for  $M(K_{1,n})$ . That is,  $M(K_{1,n})$  admits cordial labeling. **Illustration - 2.8** : In the following *Figure* 2.3 cordial labeling of  $M(K_{1,6})$  is shown.



Figure 2.3  $M(K_{1,6})$  and its cordial labeling

**Theorem - 2.9 :** M(T(n, l+1)) is a cordial graph.

**Proof:** Consider the tadpole T(n, l+1) in which  $V_1, V_2, ..., V_n$ be the vertices of cycle  $C_n$  and  $v'_1, v'_2, v'_3, ..., v'_l$  be the vertices of the path attached to the cycle  $C_n$ . Also let  $e_1, e_2$ , ...,  $e_n$  and  $e'_1$ ,  $e'_2$ , ...,  $e'_l$  be the vertices in M(T(n,l+1))corresponding to the edges of cycle  $C_n$  and path  $P_n$ respectively in T(n, l+1).

To define  $f: V(M(T(n, l+1))) \rightarrow \{0, 1\}$ , we consider the following cases. Case 1: n is odd

**Subcase 1:** n = 2k+1, k = 2, 4, 6, ..., and <math>l = 2j. *j* = 2, 4, 6, .....

In this subcase 
$$|V(M(T(n, l+1)))| = 2n+2l$$
,  
 $|E(M(T(n, l+1)))| = 2\left|\frac{n}{2}\right| + 2n+2l+6$ 

$$\begin{aligned} f(v_{(1,i+1)}) &= 2\left\lfloor \frac{1}{2} \right\rfloor^{+2i+2i+6} \\ & f(v_{2i-1}) = 1 \\ f(v_{2i}) = 0 \end{aligned} \right\} 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ f(e_{4i-3}) &= 0, \ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor + 1 \\ & 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ f(e_{4i-2}) &= 0, \\ f(e_{4i-1}) = 1 \\ f(e_{4i}) = 1 \end{aligned} \right\} 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ f(v_n) &= f(v_1') = 1 \\ (\text{when } v_1' \text{ is attached to } v_n) \\ f(v_{2i+1}') &= 0 \\ f(v_{2i+1}') &= 1 \\ 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor \\ f(e_{4i-3}') &= f(e_{4i-2}') = 0 \\ f(e_{4i-1}') &= f(e_{4i}') = 1 \end{aligned} \right\} 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor \end{aligned}$$

In view of the above defined labeling pattern

$$v_{f}(0) = v_{f}(1) = n + l, \quad e_{f}(0) = e_{f}(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3$$
  
**Subcase 2:**  $n = 2k + 1, \quad k = 2, 4, 6, .... \text{ and } l = 2j$   
 $j = 3, 5, 7, ....$   
In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$   
 $|E(M(T(n, l + 1)))| = 2\left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 8$   
 $f(e'_{n-1}) = 0, f(e'_{n}) = 1$   
 $f(v_{n}) = 1, f(v'_{1}) = 1 \text{ (when } {v'_{1}} \text{ is attached to } {v_{1}} \text{ )}$ 

remaining vertices are labeled as in subcase 1.

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In view of the above defined labeling pattern  

$$v_f(0) = v_f(1) = n+l$$
,  $e_f(0) = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n+l+4$   
For  $l = 2$  we have  $e_f(0) = e_f(1) = 11$ .  
**Subcase 3:**  $n = 2k+1$ ,  $k = 2,4,6,...$  and  $l = 2j+1$ ,  
 $j = 1,3,5,7,...$   
In this subcase  $|V(M(T(n,l+1)))| = 2n+2l$ ,  
 $|E(M(T(n,l+1)))| = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n+2l+5$   
 $f(v_n) = f(v'_1) = 1$  (when  $v'_1$  is attached to  $v_n$ )  
 $f(v'_{2i}) = 0$ ,  $1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor + 1$   
 $f(v'_{2i+1}) = 1, 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor$   
 $f(e'_{4i-3}) = f(e'_{4i-2}) = 1$   
 $f(e'_{4i-1}) = 0$   
 $1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1$   
 $f(e'_{4i}) = 0, 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$ 

remaining vertices are labeled as in subcase 1 of case (2). Using above pattern we have

$$e_{f}(0) + 1 = e_{f}(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3$$
  
For  $l = 1$  we have  $e_{f}(0) + 1 = e_{f}(1) = 10$ .  
Subcase 4:  $n = 2k + 1$ ,  $k = 2, 4, 6, ...$  and  $l = 2j + 1$ ,  
 $j = 2, 4, 6, ...$   
In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l + 1)))| = 2\left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 7$   
 $f(v_{n}) = f(v'_{1}) = 1$  (when  $v'_{1}$  is attached to  $v_{n}$ )  
 $f(v'_{2i}) = 0$ ,  $1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor + 1$   
 $f(v'_{2i+1}) = 1$ ,  $1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor$   
 $f(e'_{4i-3}) = 1$ ,  $1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1$   
 $f(e'_{4i-3}) = 1$ ,  $1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1$   
 $f(e'_{4i-1}) = f(e'_{4i}) = 0$   
 $\left\{ 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$ 

remaining vertices are labeled as in subcase 1 of case (2) Using above pattern we have . .

$$v_f(0) = v_f(1) = n + l$$
,  $e_f(0) + 1 = e_f(1) = \lfloor \frac{n}{2} \rfloor + n + l + 4$ 

Subcase 5: n = 2k + 1, k = 1, 3, 5, 7, ... and l = 2j, j = 2, 4, 6, ....In this subcase |V(M(T(n, l+1)))| = 2n + 2l,  $|E(M(T(n, l+1)))| = 2\left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 6$   $f(v_{2i-1}) = 1$   $f(v_{2i}) = 0$   $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$   $f(e_{4i-3}) = f(e_{4i-2}) = 0$   $f(e_{4i-1}) = 1$   $1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor$   $f(v_n) = f(v_1') = 1$  (when  $v'_1$  is attached to  $v_n$ )  $f(v'_{2i}) = 1$   $f(v'_{2i+1}) = 0$   $1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor$  $f(e'_{4i}, o) = f(e'_{4i}, o) = 0$ 

$$\begin{cases} f(e'_{4i-3}) = f(e'_{4i-2}) = 0\\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{cases} \\ 1 \le i \le \frac{l}{4}$$

Using above pattern we have

 $\begin{aligned} v_{f}(0) &= v_{f}(1) = n + l \quad e_{f}(0) = e_{f}(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 3 \\ \text{Subcase 6:} \quad n = 2k + 1, \quad k = 1, 3, 5, 7, \dots \quad \text{and} \quad l = 2j, \\ j &= 3, 5, 7, \dots \end{aligned}$ In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l, \\ |E(M(T(n, l + 1)))| &= 2 \left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 8 \\ f(v_{n}) &= f(v_{1}') = 1_{\text{(when }} v_{1}' \text{ is attached to } v_{n}) \\ \quad f(v_{2i}') &= 0 \\ f(v_{2i+1}') &= 1 \\ 1 \le i \le \frac{l}{2} \\ f(e_{4i-3}') &= f(e_{4i-2}') = 0 \\ f(e_{4i-1}') &= f(e_{4i}') = 1 \\ 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor \\ \quad f(e_{4i-1}') &= 0, f(e_{4i}') = 1 \end{aligned}$ 

remaining vertices are labeled as in subcase 5 of case (2). Using above pattern we have

$$v_f(0) = v_f(1) = n + l$$
,  $e_f(0) = e_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 4$   
For  $l = 2$  we have  $e_f(0) = e_f(1) = 8$ .

Subcase 7: 
$$n = 2k + 1$$
,  $k = 1, 3, 5, 7, ...$  and  $l = 2j + 1$ ,  
 $j = 1, 3, 5, ...$   
In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l + 1)))| = 2\left\lfloor \frac{n}{2} \right\rfloor + 2n + 2l + 5$   
 $f(v_n) = f(v'_1) = 1$  (when  $v'_1$  is attached to  $v_n$ )  
 $f(v'_{2i}) = 0$ ,  $1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor + 1$   
 $f(v'_{2i+1}) = 1$ ,  $1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor$   
 $f(e'_{4i-3}) = f(v'_{4i-2}) = 1$   
 $f(e'_{4i-1}) = 0$   
 $1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$   
 $f(e'_{4i}) = 0$ ,  $1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$ 

remaining vertices are labeled as in subcase 5 of case (2). Using above pattern we have

$$\begin{aligned} v_{f}(0) &= v_{f}(1) = n + l \quad e_{f}(0) = e_{f}(1) + 1 = \lfloor \frac{n}{2} \rfloor + n + l + 3 \\ \text{For } l = 1 \text{ we have } e_{f}(0) = e_{f}(1) + 1 = 7 \\ \text{.} \end{aligned}$$
Subcase 8:  $n = 2k + 1, \quad k = 1,3,5,7,... \quad \text{and} \quad l = 2j + 1, \\ j = 2,4,6,.... \\ \text{In this subcase } |V(M(T(n,l+1)))| = 2n + 2l, \\ |E(M(T(n,l+1)))| = 2 \lfloor \frac{n}{2} \rfloor + 2n + 2l + 7 \\ f(v_{n}) = f(v_{1}') = 1_{(\text{when } v_{1}' \text{ is attached to } v_{n})} \\ f(v_{2i}') = 0, \quad 1 \le i \le \lfloor \frac{l}{2} \rfloor + 1 \\ f(v_{2i+1}') = 1, \quad 1 \le i \le \lfloor \frac{l}{2} \rfloor \\ f(e_{4i-3}') = 1, \quad 1 \le i \le \lfloor \frac{l}{4} \rfloor + 1 \\ f(e_{4i-1}') = f(e_{4i}') = 0 \\ \end{vmatrix}$ 

remaining vertices are labeled as in subcase 5 of case (2). Using above pattern we have

$$v_{f}(0) = v_{f}(1) = n + l, \quad e_{f}(0) = e_{f}(1) + 1 = \left\lfloor \frac{n}{2} \right\rfloor + n + l + 4$$
  
Case 2: n is even  
Subcase 1:  $n = 2k, \quad k = 2, 4, 6, \dots$  and  $l = 2j, \quad j = 2, 4, 6, \dots$   
In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ 

In this subcase |V(M(T(n,l+1)))| = 3n+2l+5

f

$$\begin{cases} f(v_{2i-1}) = 1\\ f(v_{2i}) = 0 \end{cases} 1 \le i \le \frac{n}{2}$$

$$\begin{cases} f(e_{4i-3}) = f(e_{4i-2}) = 0\\ f(e_{4i-1}) = f(e_{4i}) = 1 \end{cases} 1 \le i \le \frac{n}{4}$$

$$(v'_{1}) = 1 \text{ (when } v'_{1} \text{ is attached to } v_{1})$$

$$\begin{cases} f(v'_{2i}) = 0\\ f(v'_{2i+1}) = 1 \end{cases} 1 \le i \le \frac{l}{2}$$

$$\begin{cases} f(e'_{4i-3}) = f(e'_{4i-2}) = 0\\ f(e'_{4i-1}) = f(e'_{4i}) = 1 \end{cases} \\ 1 \le i \le \frac{l}{4} \end{cases}$$

Using above pattern we have

$$v_{f}(0) = v_{f}(1) = n + l \quad e_{f}(0) = e_{f}(1) + 1 = \frac{3n}{2} + l + 3$$
  
Subcase 2:  $n = 2k$ ,  $k = 2, 4, 6, ...$  and  $l = 2j$ ,  
 $j = 3, 5, 7, ....$ 

In this subcase |V(M(T(n, l+1)))| = 2n+2l, |E(M(T(n, l+1)))| = 3n+2l+7

$$f(v'_{1}) = 1_{\text{(when }} v'_{1} \text{ is attached to } v_{1})$$

$$f(e'_{4i-3}) = f(e'_{4i-2}) = 0$$

$$f(e'_{4i-1}) = f(e'_{4i}) = 1$$

$$1 \le i \le \lfloor \frac{l}{4} \rfloor$$

 $f(e'_{n-1}) = 0$ ,  $f(e'_n) = 1$ 

remaining vertices are labeled as in subcase 1 of case (2). Using above pattern we have

$$\begin{split} v_{f}(0) &= v_{f}(1) = n + l \quad e_{f}(0) + 1 = e_{f}(1) = \frac{3n}{2} + l + 4 \\ \text{For } l &= 2 \text{ we have } e_{f}(0) + 1 = e_{f}(1) = 10 \\ \text{Subcase } 3: \quad n = 2k , \quad k = 2, 4, 6, \dots \quad \text{and} \quad l = 2j + 1, \\ j &= 1, 3, 5, 7, \dots \\ \text{In this subcase } |V(M(T(n, l + 1)))| = 2n + 2l \\ |E(M(T(n, l + 1)))| = 3n + 2l + 4 \\ f(v_{1}') &= 1 (\text{when } v_{1}' \text{ is attached to } v_{1}) \\ |E(M(T(n, l + 1)))| &= 3n + 2l + 4 \\ f(v_{2i}') &= 0, \quad 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor + 1 \\ f(v_{2i+1}') &= 1, \quad 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor \\ f(e_{4i-3}') &= f(e_{4i-2}') = 1 \\ f(e_{4i-1}') &= 0 \\ \end{split} \right\} \\ 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1 \end{split}$$

$$f(e'_{4i}) = 0, \quad 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 1 of case (2). Using above pattern we have

$$v_{f}(0) = v_{f}(1) = n + l \quad e_{f}(0) = e_{f}(1) = \frac{3n}{2} + l + 2$$
  
For  $l = 1$  we have  $e_{f}(0) = e_{f}(1) = 8$ .  
Subcase 4:  $n = 2k$ ,  $k = 2, 4, 6, ....$  and  $l = 2j + 1$ ,  
 $j = 2, 4, 6, ....$   
In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l + 1)))| = 3n + 2l + 6$   
 $f(v'_{1}) = 1$  (when  $v'_{1}$  is attached to  $v_{1}$ )  
 $f(v'_{2i}) = 0, \quad 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor + 1$   
 $f(v'_{2i+1}) = 1, \quad 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor$   
 $f(e'_{4i-3}) = 1, \quad 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1$   
 $f(e'_{4i-2}) = 1$   
 $f(e'_{4i-1}) = f(e'_{4i}) = 0$   
 $\left\{ 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$ 

remaining vertices are labeled as in subcase 1 of case (2). Using above pattern we have

$$v_{f}(0) = v_{f}(1) = n + l \quad e_{f}(0) = e_{f}(1) = \frac{3n}{2} + l + 3$$
  
Subcase 5:  $n = 2k$ ,  $k = 3, 5, 7, \dots$  and  $l = 2j$ ,  $j = 2, 4, 6, \dots$ .  
In this subcase  $|V(M(T(n, l + 1)))| = 2n + 2l$ ,  
 $|E(M(T(n, l + 1)))| = 3n + 2l + 5$   
 $f(v_{2i-1}) = 1$ ,  $i \neq n$ 

$$\begin{cases} f(v_{2i}) = 0 \\ f(v_{2i}) = 0 \\ \end{cases} \begin{cases} 1 \le i \le \frac{n}{2} \\ 1 \le i \le \frac{n}{2} \\ f(e_{4i-3}) = f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = f(e_{4i}) = 1 \\ \end{cases} \\ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ 1 \le i \le \left\lfloor \frac{n}{4} \right\rfloor \\ f(e_{1i-1}) = 0, f(e_n) = 1 \\ f(v_1) = 1 \\ (when v_1 is attached to v_1) \\ f(v_{2i}) = 0 \\ f(v_{2i+1}) = 1 \\ 1 \le i \le \frac{l}{2} \\ f(e_{4i-3}) = f(e_{4i-2}) = 0 \\ f(e_{4i-1}) = f(e_{4i}) = 1 \\ \end{cases} \\ 1 \le i \le \frac{l}{4} \end{cases}$$

Using above pattern we have

$$\begin{aligned} v_{f}(0) &= v_{f}(1) = n + l \quad e_{f}(0) + 1 = e_{f}(1) = \frac{3n}{2} + l + 3 \\ \text{Subcase 6: } n = 2k \quad k = 3, 5, 7, \dots \text{ and } l = 2j \quad j = 3, 5, 7, \dots \\ \text{In this subcase} & |V(M(T(n, l + 1)))| = 2n + 2l \\ |E(M(T(n, l + 1)))| &= 3n + 2l + 7 \\ f(v_{1}') &= 1 \text{ (when } v_{1}' \text{ is attached to } v_{1}) \\ f(e_{4i-3}') &= f(e_{4i-2}') = 0 \\ f(e_{4i-1}') &= f(e_{4i}') = 1 \\ \end{aligned} \end{aligned}$$

 $f(e'_{n-1}) = 1$   $f(e'_n) = 0$ 

remaining vertices are labeled as in subcase 5 of case (2). Using above pattern we have

 $v_{f}(0) = v_{f}(1) = n + l, \quad e_{f}(0) + 1 = e_{f}(1) = \frac{3n}{2} + l + 4$ For l = 2 we have  $e_{f}(0) + 1 = e_{f}(1) = 13$ Subcase 7:  $n = 2k, \quad k = 3, 5, 7, ..., \text{ and } l = 2j + 1, j = 1, 3, 5, 7, ..., |V(M(T(n, l + 1)))| = 2n + 2l$ 

In this subcase 
$$|V(M(I(n, l+1)))| = 2n + 2$$
  
 $|E(M(T(n, l+1)))| = 3n + 2l + 4 + 4 \lfloor \frac{j}{2} \rfloor$   
 $f(v'_{1}) = 0$  (when  $v'_{1}$  is attached to  $v_{2}$ )  
 $f(v'_{2i}) = 1$ ,  $1 \le i \le \lfloor \frac{l}{2} \rfloor + 1$   
 $f(v'_{2i+1}) = 0$ ,  $1 \le i \le \lfloor \frac{l}{2} \rfloor$   
 $f(e'_{4i-3}) = f(e'_{4i-2}) = 0$   
 $f(e'_{4i-1}) = 1$   
 $f(e'_{4i}) = 1$ ,  $1 \le i \le \lfloor \frac{l}{4} \rfloor$   
 $f(e'_{4i}) = 1$ ,  $1 \le i \le \lfloor \frac{l}{4} \rfloor$ 

remaining vertices are labeled as in subcase 5 of case (2). Using above pattern we have

$$v_f(0) = v_f(1) = n + l, \quad e_f(0) = e_f(1) = \frac{3n}{2} + l + 2 + 2\left\lfloor \frac{J}{2} \right\rfloor$$
  
For  $l = 1$  we have  $e_f(0) = e_f(1) = 11$ 

**Subcase** 8: n = 2k, k = 3, 5, 7, ... and l = 2j+1, j = 2, 4, 6, ...

In this subcase 
$$|V(M(T(n, l+1)))| = 2n+2l$$
  
 $|E(M(T(n, l+1)))| = 3n+2l+2+4\left\lfloor \frac{j}{2} \right\rfloor$   
 $f(v'_1) = 0$  (when  $v'_1$  is attached to  $v_2$ )

$$f(v'_{2i}) = 1, \quad 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor + 1$$

$$f(v'_{2i+1}) = 0, \quad 1 \le i \le \left\lfloor \frac{l}{2} \right\rfloor$$

$$f(e'_{4i-3}) = 0, \quad 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$f(e'_{4i-3}) = 0, \quad 1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor + 1$$

$$f(e'_{4i-1}) = f(e'_{4i}) = 1$$

$$1 \le i \le \left\lfloor \frac{l}{4} \right\rfloor$$

remaining vertices are labeled as in subcase 5 of case (2). Using above pattern we have

$$v_f(0) = v_f(1) = n + l$$
,  $e_f(0) = e_f(1) = \frac{3n}{2} + l + 1 + 2 \left\lfloor \frac{j}{2} \right\rfloor$ 

Thus in all the cases described above f admits cordial labeling for M(T(n, l+1)). That is, M(T(n, l+1)) admits cordial labeling.

**Illustration - 2.10 :** In the following Figure 2.4 cordial labeling of M(T(6,5)) is shown.



# Figure 2.4 M(T(6,5)) and its cordial labeling Concluding Remarks

Labeling of discrete structure is a potential area of research due to its diversified applications. We discuss here cordial labeling in the context middle graph of a graph. We contribute six new results to the theory of cordial labeling. It is possible to investigate analogous results for various families of graph and in the context of different graph labeling problems which is the open area of research.

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