



Recognition of alternating group A_{16} by its noncommuting graph

Deqin Chen¹, Shitian Liu¹ and Wujie Shi²

¹School of Science Sichuan University of Science & Engineering, Zigong, 643000, China

²School of Mathematics, Suzhou University Suzhou, Jiangsu, 215006, P.R.China.

ARTICLE INFO

Article history:

Received: 2 March 2011;

Received in revised form:

20 April 2011;

Accepted: 25 April 2011;

Keywords

Alternating group,

Noncommuting graph,

Element order,

Simple group.

Subject Classification(2000): 20D10,
20D20.

ABSTRACT

Abdollahi, Akbari, and Maimani put forward a conjecture named AAM's Conjecture as follows. If M is a finite nonabelian simple group and G is a group such that $\nabla(G) = \nabla(M)$, then $G \cong M$. In this note, we prove that if G is a finite nonabelian simple group with $\nabla(G) = \nabla(A_{16})$, then $G \cong A_{16}$, where A_{16} is the alternating group of degree 16.

© 2011 Elixir All rights reserved.

Introduction

Generally, there is an intimate relation between group and graph, and in many occasions properties of graphs give rise to some properties of groups and vice versa. Given a finite group G , we construct its noncommuting graph $\nabla(G)$ as follows. The vertex set of $\nabla(G)$ is $G \setminus Z(G)$, where $Z(G)$ is the center of G , and two vertices are adjacent by an edge whenever they do not commute (see [1,8,9]).

Given a graph X , we denote the vertices and edges of X by $V(X)$ and $E(X)$, respectively. Two graphs X and Y are said to be isomorphic if there exists a bijective map $\phi: V(X) \rightarrow V(Y)$ such that x and y are adjacent in X if and only if $\phi(x)$ and $\phi(y)$ are adjacent in Y . It is easy to see that if $X \cong Y$, then $|V(X)| = |V(Y)|$ and $|E(X)| = |E(Y)|$.

In 2006, A. Abdollahi, S. Akbari, and H. R. Maimani put forward a conjecture in [aam06] as follows.

AAM's Conjecture: If M is a finite nonabelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$.

In [12], it has been proved that AAM's Conjecture is true for all finite simple groups with nonconnected prime graphs and A_{10} , where A_{10} is the alternating group of degree 10 and also for projective general linear groups $\mathrm{PGL}(2, p)$ for odd prime p (see [6]), for special linear group $\mathrm{SL}(2, q)$ (see [2]), for projective special linear groups $L_2(q)$ and $L_3(q)$ (see [11] and [13], respectively).

In this paper, we will show that AAM's Conjecture is true for the alternating group A_{16} of degree 16.

Elementary Results

For a group G , we denote by $\omega(G)$ the set of orders of its elements. The set $\omega(G)$ is closed and partially ordered by the divisibility relation. Hence, it is uniquely determined by $\mu(G)$, the subset of its elements which is maximal under the divisibility relation.

Lemma 2.1 [4]

$$\mu(A_{16}) = \{2, 11, 2^2, 3, 3 \cdot 11, 5 \cdot 7, 2^2 \cdot 3^2, 3 \cdot 13, 2 \cdot 3 \cdot 7, 3^{25}, 5 \cdot 11, 2^2 \cdot 3 \cdot 5, 3^2 \cdot 7, 2^2 \cdot 3 \cdot 7\}$$

Proof.

From Lemma 4 of [16], it is easy to get the result. \square

Lemma 2.2 [15] Let G be a finite nonabelian simple group.

(1) If $\pi(G) \subseteq \{2, 3, 5\}$, then G is isomorphic to one of the following groups:

$$U_4(2), A_5, A_6.$$

(2) If $7 \in \pi(G) \subseteq \{2, 3, 5, 7\}$. Then G is isomorphic to one of the following groups:

$$U_4(2), L_2(7), L_2(8), U_3(3), L_2(49), U_3(5), L_3(4), J_2, U_4(3), S_4(7), S_6(2), O_8^+(2), \text{ and } A_i, i = 5, \Lambda, 9.$$

(3) If $11 \in \pi(G) \subseteq \{2, 3, 5, 7, 11\}$, then G is isomorphic to one of the following groups:

$$L_2(11), M_{11}, M_{12}, U_5(2), M_{22}, A_{11}, \mathrm{McL}, \mathrm{HS}, A_{12} \text{ and } U_6(2).$$

(4) If $13 \in \pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}$, then G is isomorphic to one of the following groups:

$$L_3(3), L_2(25), U_3(4), S_4(5), L_4(3), {}^2F_4(2), L_2(3), L_2(27), G_2(3), {}^3D_4(2), Sz(8), L_2(64), U_4(5), L_3(9),$$

Tele:

E-mail addresses: chendeqin@suse.edu.cn,

liust@suse.edu.cn, wjshi@suda.edu.cn

© 2011 Elixir All rights reserved

$S_6(3)$, $O_7(3)$, $G_2(4)$, $S_4(8)$, $O_8^+(3)$, $L_5(2)$, A_{13} , A_{14} , A_{15} , $L_6(3)$, Suz, A_{16} and Fi_{22} .

In the following, let g be an element of a finite group G . We denote by g^G the conjugacy class of G containing g . Also, we denote by $|g^G|$ the size of the conjugacy class g^G . The great common divisor of two numbers a, b is denoted by $\gcd(a, b)$.

Lemma 2.3 [8, Lemma 2] Let G be a finite group such that $Z(G)=1$. If H is a group such that $\nabla(H) \cong \nabla(G)$, Then $|Z(H)|(|C_G(G_i)|-1)$ and $|Z(H)|(|G_i^g|-1)$ for every $g_i \in G^*$, where $G^* = G \setminus \{1\}$ and $1 \leq i \leq G^*$. In particular, if one of the following two conditions holds:

$$(1) \quad \gcd\left\{|C_G(g_1)|-1, |C_G(g_2)|-1, \Lambda, \left|C_G\left(g_{|G^*|}\right)\right|-1\right\}=1,$$

or

$$(2) \quad \gcd\left\{|g_1^G|-1, |g_2^G|-1, \Lambda, \left|C_G\left(g_{|G^*|}^G\right)\right|-1\right\}=1,$$

then $H=G$.

Lemma 2.4 [12, Lemma 2.4] Let G and H be finite groups. If $\nabla(H) \cong \nabla(G)$, then

$$|C_G(x) \setminus Z(H)| = |C_G(\phi(x)) \setminus Z(G)|$$

for all $x \in H \setminus Z(H)$, where ϕ is a graph isomorphism from $\nabla(H)$ to $\nabla(G)$.

Lemma Let G be a finite nonabelian simple group such that $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}$. Then G is isomorphic to one of the group listed as Table 1.

Proof. From [3], [5, Tables 5.1, p. 170-171] and Lemma 2.2, G is isomorphic to one of the groups as Table 1 comparing the orders of G . \square

In the sequel a completely reducible group will be called a CR -group. The center of a CR -group is the direct product of the abelian factors in the decomposition. Hence a CR -group is centerless, i.e., it has trivial center, if and only if it is a direct product of nonabelian simple groups. The following lemma determines the structure of the automorphism group of a centerless CR -group. In the following Lemma, we denote the automorphism of G by $\text{Aut}(G)$ and the outer automorphism of G by $\text{O}(G)$.

Lemma 2.6 [10, Theorem 3.3.20] Let R be a finite centerless CR -group and write $R = R_1 \times \Lambda \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i , H_i and H_j are not isomorphic if $i \neq j$. Then $\text{Aut}(R) = \text{Aut}(R_1) \times \Lambda \times \text{Aut}(R_k)$ and

$\text{Aut}(R_i) \cong \text{Aut}(H_i) \wr S_{n_i}$, where in this wreath product

$\text{Aut}(H_i)$ appears in its right regular representation and the symmetric group S_{n_i} in its nature permutation representation. Moreover, the isomorphisms induce isomorphism $\text{O}(R) = \text{O}(R_1) \times \Lambda \times \text{O}(R_k)$ and $\text{O}(R_i) \cong \text{O}(H_i) \wr S_{n_i}$.

Main Results

Theorem Let A_{16} be the alternating group of degree 16. If G is a finite group with $\nabla(G) \cong \nabla(A_{16})$, then $G \cong A_{16}$.

Proof. Let $M := A_{16}$. Now we prove that $G \cong M$. We divide the proof into the following lemmas.

Lemma If $\nabla(G) \cong \nabla(M)$, then $|G| = |M|$. In particular, $Z(G) = 1$.

Proof. By Lemma 2.5, it is sufficient to find elements u, v, w in M such that $\gcd\{|C_M(u)|-1, |u^M|-1\}=1$ or $\gcd\{|C_M(v)|-1, |v^M|-1\}=1$. By Corollary 2.1, $\mu(M) =$

$$\{2 \cdot 11, 2^2 \cdot 3, 3 \cdot 11, 5 \cdot 7, 2^2 \cdot 3^2, 3 \cdot 13, 2 \cdot 3 \cdot 7, 3^2 \cdot 5, 5 \cdot 11, 2^2 \cdot 3 \cdot 5, 3^2 \cdot 7, 2^2 \cdot 3 \cdot 7\}.$$

Therefore there exist elements x, y, z in M such that $o(x) = 2^2 \cdot 3 \cdot 7$, $o(y) = 5 \cdot 11$ and $o(z) = 3 \cdot 13$ respectively. Since $|M| = 2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $\{5 \cdot 13, 7 \cdot 11, 7 \cdot 13, 11 \cdot 13\} \cap \omega(M) = \emptyset$, by Lemma 2.1, it follows that $|C_M(x)| = 2^i \cdot 3^j \cdot 7^k$, $|C_M(y)| = 5^l \cdot 11$ and $|C_M(z)| = 3^m$, where $2 \leq i \leq 14$, $1 \leq j \leq 6$, $1 \leq k \leq 2$, $1 \leq l \leq 3$ and $1 \leq m \leq 6$.

In the following, let x, y, z denote those elements mentioned above.

$$(1) \quad \text{See Table 2: } Z_x = |C_M(x)|; \quad Z'_x = Z_x - 1; \\ N_x := \frac{|M|}{Z_x}; \quad N'_x = N_x - 1.$$

Obviously, by Tables 2, 3 and 4, $\gcd\{|C_M(x)|-1, |x^M|-1\}=1$,

$$\gcd\{|C_M(y)|-1, |y^M|-1\}=1 \quad \text{and}$$

$$\gcd\{|C_M(z)|-1, |z^M|-1\}=1. \quad \text{By Lemma 2.3, we have that}$$

$$Z(G) \gcd\{|C_M(x)|-1, |x^M|-1\}=1,$$

$$Z(G) \gcd\{|C_M(y)|-1, |y^M|-1\}=1 \quad \text{and}$$

$$Z(G) \gcd\{|C_M(z)|-1, |z^M|-1\}=1, \quad \text{and so}$$

$Z(G) = 1 = Z(M)$. By the definition of noncommuting graph, we have that $|G \setminus Z(G)| = |G \setminus Z(M)|$. Thus $|G| = |M|$. \square

Lemma 3.2 If $\nabla(G) = \nabla(M)$, then $7 \cdot 13 \notin \omega(G)$ and $11 \cdot 13 \notin \omega(G)$.

Proof. Let ϕ be a graph isomorphism from $\nabla(M)$ to $\nabla(G)$. If $7 \cdot 13 \in \omega(G)$, then there exists an element $x \in G$ such that $o(x) = 7$. Therefore, $7 \cdot 13 | C_G(x)$. By Lemmas 2.4 and 3.1, $7 \cdot 13 | C_M(\phi(x))$. If $2 \in \pi(o(\phi^{-1}(x)))$, then there exists a nature number i such that $x_1 := (\phi^{-1}(x))^i \in M$ is of order 2. Then $7 \cdot 13 | C_M(x_1)$, too. Let $y_1 \in C_M(x_1)$ such that $o(y_1) = 13$. Hence $o(x_1 y_1) = 2 \cdot 13$. Thus $2 \cdot 13 \in \omega(M)$, which is a contradiction since 2 is not adjacent to 13 in $\Gamma(M)$. Therefore $2 \notin \pi(o(\phi^{-1}(x)))$. By a similar argument, we have that $\{3, 5, 7, 11, 13\} \cap \pi(o(\phi^{-1}(x))) = \emptyset$, too. Thus $\pi(M) \cap \pi(o(\phi^{-1}(x))) = \emptyset$. It follows that $\phi^{-1}(x) = 1$, a contradiction.

By a similar argument as done above, we get that $11 \cdot 13 \in \omega(G)$.

Lemma 3.3 If $\nabla(G) \cong \nabla(M)$, then G is nonsoluble.

Proof. Suppose that G is soluble. Since $|G| = |M| = 2^{14} \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, by Lemma 3.1, it follows that G has a Hall $\{11, 13\}$ -subgroup H of order $11 \cdot 13$. Therefore H is a cyclic subgroup, which implies that $11 \cdot 13 \in \omega(H) \subseteq \omega(G)$. This is a contradiction by Lemma 3.2.

Lemma 3.4 Let K be the maximal normal soluble subgroup of G . Then K is a $\{2, 3, 5\}$ -subgroup.

Proof. Suppose that $\{p, q, r\} = \{7, 11, 13\}$. We will prove this with three cases.

Case 1: $\{p, q, r\} \subseteq \pi(K)$.

Let T be a Hall $\{p, r\}$ -subgroup of K . Obviously, T is a cyclic subgroup of order $p \cdot r$, since $|G| = |M|$ by Lemma 3.1. Thus $11 \cdot 13 \in \omega(H) \subseteq \omega(G)$, a contradiction.

Case 2: $\{p, q\} \subseteq \pi(K)$ but $r \notin \pi(K)$.

Let T be a Hall $\{p, q\}$ -subgroup of K . It is easy to get that T is a cyclic subgroup of order $p \cdot q$. If $\{p, q\} \neq \{7, 13\}$, then $p \cdot q \in \omega(H) \subseteq \omega(G)$, a contradiction. Then $\{p, q\} = \{7, 13\}$, and K is a $\{2, 3, 5, 7, 13\}$ -subgroup. Let R_p be the Sylow p -subgroup of K . Then from Frattini argument $G = KN_G(R_p)$. Therefore the normalizer $N_G(R_p)$ contains an element x of order 11. Obviously, $\langle x \rangle R_p$ is a cyclic subgroup of order $p \cdot 11$. Hence $p \cdot 11 \in \omega(G)$, a contradiction.

Case 3: $r \in \pi(K)$ but $\{p, q\} \cap \pi(K) = \emptyset$.

Then K is a $\{2, 3, 5, r\}$ -subgroup. Let R_r be the Sylow r -subgroup of K . By Frattini argument, we have $G = KN_G(R_r)$. Therefore, the normalizer $N_G(R_r)$ contains

two elements x and y of order p and q , respectively. Obviously, $\langle x \rangle R_r$ and $\langle y \rangle R_r$ are cyclic subgroups of order $p \cdot r$ and $q \cdot r$ respectively. It is clear that $p \cdot r \in \omega(G)$ and $q \cdot r \in \omega(G)$, a contradiction. \square

In the following, let $Soc(G)$ be the socle of G , which is the subgroup generated by the set of all minimal normal subgroups of G .

Lemma 3.5 Let $\bar{G} := G/K$ and $S := Soc(G)$. Then the followings are true:

- (1) If $r \in \{7, 11, 13\}$ and $r \notin \pi(S)$, then $r \notin \pi(Aut(S))$.
- (2) If $S \neq M$, then $\pi(S) = \{2, 3, 5\}$.
- (3) S is isomorphic to one of the following groups: $S_4(3)$, A_5 , A_6 , $A_5 \times A_5$, $A_5 \times A_5 \times A_5$, $A_6 \times A_6$, $A_5 \times A_6$, $A_5 \times A_5 \times A_6$, $S_4(3) \times A_5$ or M .

Proof. Since G is nonsoluble by Lemma 3.3, we have that S is a centerless CR -group. But $S = P_1 \times P_2 \times \Lambda \times P_m$, where P_i is finite non-abelian simple group ($i = 1, 2, \Lambda, m$). Since $|G| = |M|$ by Lemma 3.1, P_i is isomorphic to one of the groups listed as Table 1.

In the following, we assume that $\{p, q, r\} = \{7, 11, 13\}$. Then $\{p, q, r\} \subseteq \pi(\bar{G})$ by Lemma 3.4.

- (1) Assume the first assertion to be false, then $r \nparallel Aut(S)$. Since $Inn(S) \cong S$, we have that $r \nparallel Inn(S)$. It follows that $r \nparallel Out(S)$ since $Out(S) = Aut(S)/Inn(S)$, but $Out(S) = Out(S_1) \times Out(S_2) \times \Lambda \times Out(S_k)$, where the group S_j ($j = 1, 2, \Lambda, k$) is direct products of some isomorphic copies of the simple groups belonging to the set P_1, P_2, Λ, P_m such that $S = P_1 \times P_2 \times \Lambda \times P_m = S_1 \times S_2 \times \Lambda \times S_k$. Therefore $r \nparallel Out(S_j)$ for some j such that $1 \leq j \leq k$. Suppose that S_j is a direct product of t isomorphic copies of a simple group P_i , where $P_i \in \{P_1, P_2, \Lambda, P_m\}$. It follows from Lemma 2.11 that $Out(S_j) = Out(P_i)^t \cdot t!$. Since $\pi(P_i) \subseteq \pi(S) \subseteq \{2, 3, 5, p, q\}$ and $\pi(Out(P_i)) \subseteq \{2, 3\}$ or $\pi(Out(P_i)) = 1$ by Lemma 2.5, we have $r \nparallel Out(P_i)$ and so $r \nparallel t!$. It means that $t \geq r \geq 5$. Since $\{7, 11, 13\} \cap \pi(P_i) = \emptyset$ by Lemma 2.5, it means that there exists at least a prime $u \in \{7, 11, 13\}$ such that $u \mid P_i$, and so $u^5 \nparallel S_j \nparallel S \nparallel G$, a contradiction. Hence $r \notin \pi(Aut(S))$, as desired.
- (2) We will prove by the following three cases.

- Case 1: $\{p, q\} \subseteq \pi(S)$ but $r \notin \pi(S)$.

It is obvious that $r \notin \pi(Aut(S))$ by the first assertion. Thus $r \in \pi(C_{\bar{G}}(S))$ since $\bar{G}/C_{\bar{G}}(S) \leq Aut(S)$. It follows that $\{p \cdot r, q \cdot r\} \subseteq \omega(G)$, a contradiction.

Case 2: $r \in \pi(S)$ but $\{p, q\} \cap \pi(S) = \emptyset$.

It is easy to get that $\{p, q\} \cap \pi(Aut(S)) = \emptyset$ by the first assertion. Since $\bar{G}/C_{\bar{G}}(S) \leq Aut(S)$, $\{p, q\} \subseteq \pi(C_{\bar{G}}(S))$ and so $\{p \cdot r, q \cdot r\} \subseteq \omega(G)$, a contradiction.

Case 3: $\{p, q, r\} \subseteq \pi(S)$.

Since $|G| = |M|$, it follows that $G \cong M$, by Lemma 2.5, which contradicts the hypothesis.

(3) By the second assertion, we have $\pi(S) = \{2, 3, 5\}$ if $S \neq M$. Thus by Lemma 2.5, S is isomorphic to one of the groups: $S_4(3)$, A_5 , A_6 , $A_5 \times A_5$, $A_5 \times A_5 \times A_5$, $A_6 \times A_6$, $A_5 \times A_6$, $A_5 \times A_5 \times A_6$, $S_4(3) \times A_5$ or M . \square

Lemma 3.6 Let G be one of the groups as Lemma 3.5(3). Then the followings are true.

$$(1) |S| \left| \frac{|\bar{G}|}{C_{\bar{G}}(S)} \right| |Aut(S)|$$

(2) If $S \neq M$, then $\{7, 11, 13\} \subseteq \pi(C_{\bar{G}}(S))$ and $C_{\bar{G}}(S)$ is nonsolvable.

Proof. (1) Since $S \cap C_{\bar{G}}(S) = Z(S) = 1$, we have that $S \cong S / S \cap C_{\bar{G}}(S) \cong C_{\bar{G}}(S) / C_{\bar{G}}(S)$

$$\leq \bar{G} / C_{\bar{G}}(S) \underset{\sim}{<} Aut(S). \text{ Thus } |S| \left| \frac{|\bar{G}|}{C_{\bar{G}}(S)} \right| |Aut(S)|.$$

(2) If $S \cong A_5 \times A_5$, $A_5 \times A_5 \times A_5$, $A_6 \times A_6$, $A_5 \times A_6$, $A_5 \times A_5 \times A_6$, $S_4(3) \times A_5$, then

$$|Aut(S)| = |Aut(A_5)|^2 \cdot |S_2| = 2^9 \cdot 3^2 \cdot 5^2,$$

$$|Aut(A_5)|^3 \cdot |S_3| = 2^{13} \cdot 3^4 \cdot 5^3, \quad |Aut(A_6)|^2 \cdot |S_2| = 2^{11} \cdot 3^4 \cdot 5^2,$$

$$|Aut(S_4(3))| \cdot |Aut(A_5)| = 2^{10} \cdot 3^5 \cdot 5^2 \text{ by Lemmas 2.6 and 2.3.}$$

And also $|Aut(S_4(3))| = 2^7 \cdot 3^4 \cdot 5$, $|Aut(A_5)| = 2^3 \cdot 3 \cdot 5$ by Lemma 2.5. Thus we have that $\pi(S) = \{2, 3, 5\}$ if $S \neq M$.

It means that $\{7, 11, 13\} \subseteq \pi(C_{\bar{G}}(S))$ since $\bar{G}/C_{\bar{G}}(S) \underset{\sim}{<} Aut(S)$. If $C_{\bar{G}}(S)$ is soluble, then $C_{\bar{G}}(S)$ contains a Hall $\{11, 13\}$ -subgroup T_1 . Obviously, T_1 is a cyclic subgroup of order $11 \cdot 13$, this means that $11 \cdot 13 \in \omega(C_{\bar{G}}(S))$. Thus $11 \cdot 13 \in \omega(G)$, a contradiction. \square

Let n be a nature number and p a prime. In the following, $e(n_p)$ denotes a nonnegative integer such that $p^{e(n_p)} \mid n$ but $p^{e(n_p)+1} \nmid n$.

Lemma 3.7 S is not isomorphic to any of the following groups: $S_4(3)$, A_5 , A_6 , $A_5 \times A_5$, $A_5 \times A_5 \times A_5$, $A_6 \times A_6$, $A_5 \times A_6$, $A_5 \times A_5 \times A_6$, $S_4(3) \times A_5$. Therefore, $S \cong M = A_{16}$.

Proof. Suppose $S \cong S_4(3)$, A_5 , A_6 , $A_5 \times A_5$, $A_5 \times A_5 \times A_5$, $A_6 \times A_6$, $A_5 \times A_6$, $A_5 \times A_5 \times A_6$, or $S_4(3) \times A_5$.

Step 1 Let $G_1 := C_{\bar{G}}(S)$, $\bar{G}_1 := G_1 / K_1$ and $S_1 := Soc(\bar{G}_1)$, where K_1 is the maximal normal soluble subgroup of G_1 . Then the followings hold.

(1.1) By Lemma 3.2, $\{7, 11, 13\} \cap \omega(G_1) = \emptyset$, and so $\{7, 11, 13\} \cap \omega(\bar{G}_1) = \emptyset$.

(1.2) By Lemma 3.6(2), $\{7, 11, 13\} \subseteq \pi(G_1)$ and G_1 is nonsoluble. By a similar argument in Lemma 3.4, K_1 is a $\{2, 3, 5\}$ -subgroup, too.

(1.3) Moreover, $\{7, 11, 13\} \subseteq \pi(\bar{G}_1)$ by (1.2) and so \bar{G}_1 is nonsoluble.

(1.4) By Lemma 3.6(1), $e(|G_1|_2) \leq 11$, $e(|G_1|_3) \leq 4$, $e(|G_1|_5) \leq 2$, $e(|G_1|_7) = 2$, and $e(|G_1|_{11}) = e(|G_1|_{13}) = 1$.

(1.5) By the choice of K_1 and (1.3), S_1 is a direct product of some finite nonabelian simple groups as Table 1.

Assume that $\{p, q, r\} = \{7, 11, 13\}$. By Table 1 and (1.4), we have that $\{p, q, r\} \not\subseteq \pi(S_1)$ and so $S_1 \neq M$. Now we assume assert that $\pi(S_1) = \{2, 3, 7, 13\}$. we will show this by three cases.

Case 1: $\{p, q\} \subseteq \pi(S_1)$ but $r \notin \pi(S_1)$.

So we have $r \notin \pi(Aut(S_1))$ by a similar argument as Lemma 3.5(1). Thus $r \in \pi(C_{\bar{G}_1}(S_1))$ since $\bar{G}_1 / C_{\bar{G}_1}(S_1) \underset{\sim}{<} Aut(S_1)$. It follows that $\{p \cdot r, q \cdot r\} \subseteq \omega(G_1)$, which contradicts (1.1).

Case 2: $r \in \pi(S_1)$ but $\{p, q\} \cap \pi(S) = \emptyset$.

It is easy to get that $\{p, q\} \cap \pi(Aut(S_1)) = \emptyset$ by a similar argument as Lemma 3.5(1). So $\{p, q\} \subseteq \pi(C_{\bar{G}_1}(S_1))$ since $\bar{G}_1 / C_{\bar{G}_1}(S_1) \underset{\sim}{<} Aut(S_1)$. Also we have $\{p \cdot r, q \cdot r\} \subseteq \omega(G_1)$, which contradicts (1.1).

Case 3: $\{p, q, r\} \subseteq \pi(S)$.

Since $|G| = |G_1| = |M|$, it follows that $G_1 \cong M$ by Lemma 2.3, which contradicts the hypothesis.

Hence $\pi(S_1) \subseteq \{2, 3, 5\}$ since is onosoluble by (1.5). It follows that $S \cong A_5 \times A_5$, $A_6 \times A_6$, $A_5 \times A_6$, or $S_4(3)$ by (1.4) and (1.5).

Step2. Let $G_2 = C_{\overline{G}_1}(S_1)$, $\overline{G}_2 := G_2 / K_2$ and $S_2 := \text{Soc}(\overline{G}_2)$,

where K_2 is the maximal soluble normal subgroup of G_2 . Then the followings hold:

(2.1) By Lemma 3.2, $\{7 \cdot 13, 11 \cdot 13\} \cap \omega(G_2) = \emptyset$, and so $\{7 \cdot 13, 11 \cdot 13\} \cap \omega(\overline{G}_2) = \emptyset$.

(2.2) By a similar way as Lemma 3.6(2), $\{7, 11, 13\} \subseteq \pi(G_2)$ and G_2 is nonsoluble. By a similar argument in Lemma 3.4, K_2 is a $\{2, 3, 5\}$ -subgroup, too.

(2.3) Moreover, $\{7, 11, 13\} \subseteq \pi(\overline{G}_2)$ by (1.2) and so \overline{G}_2 is nonsoluble.

(2.4) By Lemma 3.6(1), $e(G_2|_2) \leq 8$, $e(G_2|_3) \leq 3$, $e(G_2|_5) = 1$, $e(G_2|_7) = 2$, and $e(G_2|_{11}) = e(G_2|_{13}) = 1$.

(2.5) By the choice of K_2 and (2.4), S_2 is a direct product of some finite nonabelian simple groups as Table 1. In particular, G_2 is nonsoluble.

Assume that $\{p, q, r\} = \{7, 11, 13\}$. By Table 1 and (2.4), we have that $\{p, q, r\} \not\subseteq \pi(S_2)$ and so $S_2 \neq M$. Now we assume assert that $\pi(S_2) = \{2, 3, 5\}$. we will show this by three cases.

Case 1. $\{p, q\} \subseteq \pi(S_2)$ but $r \notin \pi(S_2)$.

So we have $r \notin \pi(\text{Aut}(S_2))$ by a similar argument as Lemma 3.5(1). Thus $r \in \pi(C_{\overline{G}_2}(S_2))$ since $\overline{G}_2 / C_{\overline{G}_2}(S_2) \leq \text{Aut}(S_2)$.

If follows that $\{p \cdot r, q \cdot r\} \in \omega(G_2)$, which contradicts (2.1).

Case 2: $r \in \pi(S_2)$ but $\{p, q\} \cap \pi(S_2) = \emptyset$.

It is easy to get that $\{p, q\} \cap \pi(\text{Aut}(S_2)) = \emptyset$ by a similar argument as Lemma 3.5(1). So $\{p, q\} \subseteq \pi(C_{\overline{G}_2}(S_2))$ since $\overline{G}_2 / C_{\overline{G}_2}(S_2) \leq \text{Aut}(S_2)$. Also we have

$\{p \cdot r, q \cdot r\} \in \omega(G_2)$, which contradicts (2.1).

Case 3: $\{p, q, r\} \subseteq \pi(S_2)$.

Since $|G| = |G_2| = |M|$, it follows that $G = G_2 \cong M$ by Lemma 2.3, which contradicts the hypothesis.

Hence $\pi(S_2) \subseteq \{2, 3, 5\}$. By (2.5), S_2 is nonsoluble. By Lemma 2.3, $S_2 = A_5, A_6$.

Step 3. Let $G_3 := C_{\overline{G}_2}(S_2)$, $\overline{G}_3 := G_3 / K_3$ and $S_3 := \text{Soc}(\overline{G}_3)$, where K_3 is the maximal soluble normal subgroup of G_3 . Then the followings hold:

(3.1) By Lemma 3.2, $\{7 \cdot 13, 11 \cdot 13\} \cap \omega(G_3) = \emptyset$, and so $\{7 \cdot 13, 11 \cdot 13\} \cap \omega(\overline{G}_3) = \emptyset$.

(3.2) By a similar way as Lemma 3.6(2), $\{7, 11, 13\} \subseteq \pi(G_3)$ and G_3 is nonsoluble. By a similar argument in Lemma 3.4, K_2 is a $\{2, 3, 5\}$ -subgroup, too.

(3.3) Moreover, $\{7, 11, 13\} \subseteq \pi(\overline{G}_3)$ by (3.2) and so \overline{G}_3 is nonsoluble.

(3.4) By Lemma 3.6(1), $e(G_3|_2) \leq 5$, $e(G_3|_3) \leq 2$, $e(G_3|_5) = 0$, $e(G_3|_7) = 2$, and $e(G_3|_{13}) = 1$.

(3.5) By the choice of K_3 and (3.4), S_3 is a direct product of some finite nonabelian simple groups as Table 1. In particular, G_3 is nonsoluble.

Hence $\pi(S_3) \subseteq \{2, 3\}$ and so S_3 is soluble. But by (3.5), S_3 is nonsoluble, a contradiction.

By Steps 1 to 3, we have that $S \cong M$. By Lemma 3.6(1), it follows that $|\overline{G}| = |G| = |M|$. Hence $K = 1$ and $G \cong A_{16}$.

By Lemmas 3.1 to 3.7, we complete the proof.

Acknowledgments

The object is partial supported by the NNSF of China (Grant Number: 10871032), the Scientific Research Fund of School of Science of SUSE (Grant Number: 09LXYB02) and NSF of SUSE (Grnt Number: 2010XJKYL017). The authors are very grateful for the helpful suggestions of the referee.

References

- [1] A. Abdollahi, S. Akbari, and H. R. Maimani. Noncommuting graph of a group. *J. of Algebra*, 29(2)(2009), 468-492.
- [2] A. Abdollahi. Characterization of $\text{SL}(2, q)$ by its non-commuting graph. *Beitrage zur Algebra und Geometrie*, 50(2)(2009), 443-448.
- [3] J. H. Conway, R. T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson. ATLAS of finite groups. Clarendon Press, Oxford, 1985
- [4] M. R. Darafsheh, and A. R. Moghaddamfar. Characterization of the groups $PSL_5(2)$, $PSL_6(2)$, and $PSL_7(2)$. *Commutations in Algebra*, 29(1)(2001), 465-475.
- [5] P. Kleidman, and M. Liebeck. The subgroup structure of finite classical groups. Cambridge University Press, Cambridge, 1990.
- [6] B. Khosravi and M. Khatami. A new characterization of $PGL(2, p)$ by its noncommuting graph. In press.
- [7] V. D. Mazurov, and G. Y. Chen. Recognizability of finite simple groups $L_4(2^m)$ and $U_4(2^m)$ by spetrum. *Algebra and Logic*, 47(1)(2008), 49-55.
- [8] A. R. Moghddamfar, W.J. Shi, W. Zhou, and A.R. Zokayi, On the noncommuting graph associated with a finite group. *Siberian Mathematical Journal*, 46(2)(2005), 325-332.
- [9] B. H. Neumann, A problem of Paul Erdős on group. *J. of the Australist Mathematical Society, Series A*, 21(1976), 467-472.
- [10] D. J. S. Robinson. A course in the theory of groups. 2nd, Springer, New York, 2003.
- [11] L. L. Wang, and W. Shi. A new characterization of $L_2(q)$ by its noncommuting graph. *Front. Math. China*, 2(1)(2007), 143-148.
- [12] L. L. Wang, and W. J. Shi. A new characterization of A_{10} by its commuting graph. *Commucations in Algebra*, 36(2)(2008), 523-528.
- [13] L. L. Wang, L. Zhang, and C. Shao. A new characterization by its noncommuting graph. *Journal of Suzhou University (Nature Science Edition)*, 23(2)(2007), 1-5.
- [14] A.V. Zavarnitsine. Exceptional action of the simple

groups $L_4(q)$ in the defining characteristic. Siberian Electronic Mathematical Reports, 5(2008), 68-74.

[15] A.V. Zavarnitsine. Finite groups with narrow prime spectrum. Siberian Electronic Mathematical Reports, 6(2009), 1-12.

[16] A. V. Zavarnitsin, and V. D. Mazurov. Element orders in coverings of symmetric and alternating groups. Algebra and Logic, 38(3)(1999), 159-1.

Table 1 Finite nonabelian simple groups with $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}$

G	$ G $	$O(G)$	G	$ G $	$O(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_6	$2^3 \cdot 3^2 \cdot 5$	2^2
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	A_4
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	6	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	6	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	4
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$O_{10}^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	S_3
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	2
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	A_{11}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	HS	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	${}^2 F_4(2)$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2	${}^3 D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$G_2(3)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2	$L_3(9)$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13$	4
$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6	A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	4

Table 2 Possible values of Z'_x and N'_x

Z_x	Z'_x	N_x	N'_x
$2^2 \cdot 3 \cdot 7$	83	$2^{12} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	124540415999
$2^2 \cdot 3^2 \cdot 7$	251	$2^{12} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$647 \cdot 64163017$
$2^2 \cdot 3^3 \cdot 7$	5·151	$2^{12} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	17·813989647
$2^2 \cdot 3^4 \cdot 7$	2267	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	1297·3556367
$2^2 \cdot 3^5 \cdot 7$	6803	$2^{12} \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	37·79·229·2297
$2^2 \cdot 3^6 \cdot 7$	20411	$2^{12} \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	3·170837333
$2^2 \cdot 3 \cdot 7^2$	587	$2^{12} \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	37·67·7176881
$2^2 \cdot 3^2 \cdot 7^2$	41·43	$2^{12} \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	389·15245491
$2^2 \cdot 3^3 \cdot 7^2$	11·13·17	$2^{12} \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	83·1877·12689
$2^2 \cdot 3^4 \cdot 7^2$	5 ³ ·127	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	2·227·414691
$2^2 \cdot 3^5 \cdot 7^2$	97·491	$2^{12} \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	19·23·43·11689
$2^2 \cdot 3^6 \cdot 7^2$	13·29·379	$2^{12} \cdot 5^3 \cdot 11 \cdot 13$	3 ² ·1117·7283
$2^3 \cdot 3 \cdot 7$	167	$2^{11} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	79·167·4719943
$2^3 \cdot 3^2 \cdot 7$	503	$2^{11} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	31·2237·299317
$2^3 \cdot 3^3 \cdot 7$	1511	$2^{11} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	149·46435651
$2^3 \cdot 3^4 \cdot 7$	5·907	$2^{11} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	19·613·198017
$2^3 \cdot 3^5 \cdot 7$	11·1237	$2^{11} \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	17·45221647
$2^3 \cdot 3^6 \cdot 7$	40823	$2^{11} \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	181·1415779
$2^3 \cdot 3 \cdot 7^2$	5 ² ·47	$2^{11} \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	71·1753·71473
$2^3 \cdot 3^2 \cdot 7^2$	3527	$2^{11} \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	7·29·97·150589
$2^3 \cdot 3^3 \cdot 7^2$	19·557	$2^{11} \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	4603·214733
$2^3 \cdot 3^4 \cdot 7^2$	531751	$2^{11} \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	31·101·105229
$2^3 \cdot 3^5 \cdot 7^2$	5·19051	$2^{11} \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	3623·30313
$2^3 \cdot 3^6 \cdot 7^2$	285767	$2^{11} \cdot 5^3 \cdot 11 \cdot 13$	36607999
$2^4 \cdot 3 \cdot 7$	5·67	$2^{10} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	31135103999
$2^4 \cdot 3^2 \cdot 7$	19·53	$2^{10} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	10378367999
$2^4 \cdot 3^3 \cdot 7$	3023	$2^{10} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	199·17384201
$2^4 \cdot 3^4 \cdot 7$	47·193	$2^{10} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	911·1265809
$2^4 \cdot 3^5 \cdot 7$	5·5443	$2^{10} \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	384383999
$2^4 \cdot 3^6 \cdot 7$	81647	$2^{10} \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	5·59·61·11867
$2^4 \cdot 3 \cdot 7^2$	2351	$2^{10} \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	59·1259·59879
$2^4 \cdot 3^2 \cdot 7^2$	5·17·83	$2^{10} \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	23·64461913
$2^4 \cdot 3^3 \cdot 7^2$	61·347	$2^{10} \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	494207999
$2^4 \cdot 3^4 \cdot 7^2$	11·23·251	$2^{10} \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	2293·71843
$2^4 \cdot 3^5 \cdot 7^2$	59·3229	$2^{10} \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	54911999
$2^4 \cdot 3^6 \cdot 7^2$	5·757	$2^{10} \cdot 5^3 \cdot 11 \cdot 13$	3·7 ² ·124517
$2^5 \cdot 3 \cdot 7$	11·61	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	43·362036093
$2^5 \cdot 3^2 \cdot 7$	5·13·315	$2^9 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	653·7946683

$2^5 \cdot 3^3 \cdot 7$	6047	$2^9 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	1729727999
$2^5 \cdot 3^4 \cdot 7$	18143	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$29 \cdot 19881931$
$2^5 \cdot 3^5 \cdot 7$	$13 \cdot 53 \cdot 79$	$2^9 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$5387 \cdot 35677$
$2^5 \cdot 3^6 \cdot 7$	$5 \cdot 11 \cdot 2969$	$2^9 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	64063999
$2^5 \cdot 3 \cdot 7^2$	4703	$2^9 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	2223925999
$2^5 \cdot 3^2 \cdot 7^2$	103·137	$2^9 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	19·39016421
$2^5 \cdot 3^3 \cdot 7^2$	5·8467	$2^9 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	211·1171109
$2^5 \cdot 3^4 \cdot 7^2$	17·31·241	$2^9 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	7·283·41579
$2^5 \cdot 3^5 \cdot 7^2$	43·8861	$2^9 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	27455999
$2^5 \cdot 3^6 \cdot 7^2$	1143071	$2^9 \cdot 5^3 \cdot 11 \cdot 13$	23·67·5939
$2^6 \cdot 3 \cdot 7$	17·79	$2^8 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	19·5801·70621
$2^6 \cdot 3^2 \cdot 7$	29·139	$2^8 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	673·1277·3019
$2^6 \cdot 3^3 \cdot 7$	5·41·59	$2^8 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	864863999
$2^6 \cdot 3^4 \cdot 7$	131·277	$2^8 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	89·3239191
$2^6 \cdot 3^5 \cdot 7$	108863	$2^9 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	1367·70297
$2^6 \cdot 3^6 \cdot 7$	19·17189	$2^8 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$3^2 \cdot 3559111$
$2^6 \cdot 3 \cdot 7^2$	23·409	$2^8 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	1111967999
$2^6 \cdot 3^2 \cdot 7^2$	13 ² ·167	$2^8 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	7·167·317071
$2^6 \cdot 3^3 \cdot 7^2$	227·373	$2^8 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	71·1740169
$2^6 \cdot 3^4 \cdot 7^2$	5·101·503	$2^8 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	41183999
$2^6 \cdot 3^5 \cdot 7^2$	11·13·73 ²	$2^8 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	37·371027
$2^6 \cdot 3^6 \cdot 7^2$	17·89·1511	$2^8 \cdot 5^3 \cdot 11 \cdot 13$	3·1525333
$2^7 \cdot 3 \cdot 7$	5·67	$2^7 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	3891887999
$2^7 \cdot 3^2 \cdot 7$	11·733	$2^7 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$53 \cdot 24477283$
$2^7 \cdot 3^3 \cdot 7$	17·1423	$2^7 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	432431999
$2^7 \cdot 3^4 \cdot 7$	5 ² ·2903	$2^7 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	337·427727
$2^7 \cdot 3^5 \cdot 7$	217727	$2^7 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	163·294773
$2^7 \cdot 3^6 \cdot 7$	67·9749	$2^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	16015999
$2^7 \cdot 3 \cdot 7^2$	5·53·71	$2^7 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	3557·156307
$2^7 \cdot 3^2 \cdot 7^2$	47·1201	$2^7 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	17·10901647
$2^7 \cdot 3^3 \cdot 7^2$	169343	$2^7 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	23·2685913
$2^7 \cdot 3^4 \cdot 7^2$	41·12391	$2^7 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	4111·5009
$2^7 \cdot 3^5 \cdot 7^2$	5·23·29·457	$2^7 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	971·7069
$2^7 \cdot 3^6 \cdot 7^2$	257·17791	$2^7 \cdot 5^3 \cdot 11 \cdot 13$	7·19·17203
$2^8 \cdot 3 \cdot 7$	5 ³ ·43	$2^6 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	449·541·8011
$2^8 \cdot 3^2 \cdot 7$	16127	$2^6 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	31·37·565517
$2^8 \cdot 3^3 \cdot 7$	49383	$2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	73·2961863
$2^8 \cdot 3^4 \cdot 7$	37·3923	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	43·109·15377

$2^8 \cdot 3^5 \cdot 7$	$5 \cdot 17 \cdot 47 \cdot 109$	$2^6 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$19 \cdot 53 \cdot 23857$
$2^8 \cdot 3^6 \cdot 7$	1306367	$2^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$3 \cdot 691 \cdot 3863$
$2^8 \cdot 3 \cdot 7^2$	$11^2 \cdot 311$	$2^6 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	$29 \cdot 9586931$
$2^8 \cdot 3^2 \cdot 7^2$	$5 \cdot 67 \cdot 337$	$2^6 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	92663999
$2^8 \cdot 3^3 \cdot 7^2$	338687	$2^6 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	30887999
$2^5 \cdot 3^4 \cdot 7^2$	$19 \cdot 53 \cdot 1009$	$2^6 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	$7 \cdot 17 \cdot 31 \cdot 2791$
$2^8 \cdot 3^5 \cdot 7^2$	3048191	$2^6 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	3431999
$2^8 \cdot 3^6 \cdot 7^2$	$5^2 \cdot 11^2 \cdot 3023$	$2^6 \cdot 5^3 \cdot 11 \cdot 13$	$3^2 \cdot 79 \cdot 1609$
$2^9 \cdot 3 \cdot 7$	13·827	$2^5 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$17 \cdot 3217 \cdot 17791$
$2^9 \cdot 3^2 \cdot 7$	5·6451	$2^5 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	8713·37223
$2^9 \cdot 3^3 \cdot 7$	11·19·463	$2^5 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	108107999
$2^9 \cdot 3^4 \cdot 7$	13·137·163	$2^5 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	151·238649
$2^9 \cdot 3^5 \cdot 7$	870911	$2^5 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	263·45673
$2^9 \cdot 3^6 \cdot 7$	5·181	$2^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	4003999
$2^9 \cdot 3 \cdot 7^2$	73·1031	$2^5 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	138995999
$2^9 \cdot 3 \cdot 7^2$	23·9817	$2^5 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	$7^2 \cdot 79 \cdot 11969$
$2^9 \cdot 3 \cdot 7^2$	5 ³ ·5419	$2^5 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	149·103651
$2^9 \cdot 3 \cdot 7^2$	127·16001	$2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	857·6007
$2^9 \cdot 3 \cdot 7^2$	6096383	$2^5 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	71·24169
$2^9 \cdot 3 \cdot 7^2$	18289151	$2^5 \cdot 5^3 \cdot 11 \cdot 13$	17·33647
$2^{10} \cdot 3 \cdot 7$	21503	$2^4 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	823·591113
$2^{10} \cdot 3^2 \cdot 7$	64511	$2^4 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	229·708131
$2^{10} \cdot 3^3 \cdot 7$	5·38707	$2^4 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	17·29·83·1321
$2^{10} \cdot 3^4 \cdot 7$	580607	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	3467·5197
$2^{10} \cdot 3^5 \cdot 7$	739·2357	$2^4 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	61·98459
$2^{10} \cdot 3^6 \cdot 7$	5225471	$2^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	3·667333
$2^{10} \cdot 3 \cdot 7^2$	13·11579	$2^4 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	53·101·12983
$2^{10} \cdot 3^2 \cdot 7^2$	11·61·673	$2^4 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	23165999
$2^{10} \cdot 3^3 \cdot 7^2$	71·19081	$2^4 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	19·547·743
$2^{10} \cdot 3^4 \cdot 7^2$	5·13·31·2017	$2^4 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	23·111913
$2^{10} \cdot 3^5 \cdot 7^2$	2053·5939	$2^4 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	577·1487
$2^{10} \cdot 3^6 \cdot 7^2$	23·263·6047	$2^4 \cdot 5^3 \cdot 11 \cdot 13$	3·7·13619
$2^{11} \cdot 3 \cdot 7$	29·1483	$2^3 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	243242999
$2^{11} \cdot 3^2 \cdot 7$	129023	$2^3 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	19·4267421
$2^{11} \cdot 3^3 \cdot 7$	387071	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	401·67399
$2^{11} \cdot 3^4 \cdot 7$	5·11·43·491	$2^3 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	127·70937
$2^{11} \cdot 3^5 \cdot 7$	41·84967	$2^3 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$17^2 \cdot 10391$
$2^{11} \cdot 3^6 \cdot 7$	10450943	$2^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	1000999

$2^{11} \cdot 3 \cdot 7^2$	$5 \cdot 19 \cdot 3169$	$2^3 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	$107 \cdot 324757$
$2^{11} \cdot 3^2 \cdot 7^2$	$97 \cdot 9311$	$2^3 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	11582999
$2^{11} \cdot 3^3 \cdot 7^2$	$47 \cdot 57649$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	$683 \cdot 5653$
$2^{11} \cdot 3^4 \cdot 7^2$	8128511	$2^3 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	$7 \cdot 53 \cdot 3469$
$2^{11} \cdot 3^5 \cdot 7^2$	$5 \cdot 839 \cdot 5813$	$2^3 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	$421 \cdot 1019$
$2^{11} \cdot 3^6 \cdot 7^2$	$79 \cdot 926033$	$2^3 \cdot 5^3 \cdot 11 \cdot 13$	$29 \cdot 4931$
$2^{12} \cdot 3 \cdot 7$	$5 \cdot 17203$	$2^2 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$373 \cdot 326063$
$2^{12} \cdot 3^2 \cdot 7$	$83 \cdot 3109$	$2^2 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	40540499
$2^{12} \cdot 3^3 \cdot 7$	774143	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	13513499
$2^{12} \cdot 3^4 \cdot 7$	2322431	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$101 \cdot 103 \cdot 433$
$2^{12} \cdot 3^5 \cdot 7$	$5 \cdot 1393459$	$2^2 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	1501499
$2^{12} \cdot 3^6 \cdot 7$	20901887	$2^9 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$3^4 \cdot 37 \cdot 167$
$2^{12} \cdot 3 \cdot 7^2$	60211	$2^2 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	$23 \cdot 755413$
$2^{12} \cdot 3^2 \cdot 7^2$	$5 \cdot 17 \cdot 79 \cdot 269$	$2^2 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	$7 \cdot 37 \cdot 59 \cdot 379$
$2^{12} \cdot 3^3 \cdot 7^2$	$11 \cdot 23 \cdot 21419$	$2^2 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	$89 \cdot 109 \cdot 199$
$2^{12} \cdot 3^4 \cdot 7^2$	$29 \cdot 37 \cdot 103 \cdot 139$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	$83 \cdot 7753$
$2^{12} \cdot 3^5 \cdot 7^2$	48771071	$2^2 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	214499
$2^{12} \cdot 3^6 \cdot 7^2$	$5 \cdot 41 \cdot 59 \cdot 12097$	$2^2 \cdot 5^3 \cdot 11 \cdot 13$	$3 \cdot 23833$
$2^{13} \cdot 3 \cdot 7$	172031	$2 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$41 \cdot 761 \cdot 1949$
$2^{13} \cdot 3^2 \cdot 7$	$5 \cdot 233 \cdot 443$	$2 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$31 \cdot 653879$
$2^{13} \cdot 3^3 \cdot 7$	$13 \cdot 119099$	$2 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$67 \cdot 100847$
$2^{13} \cdot 3^4 \cdot 7$	$479 \cdot 9697$	$2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$751 \cdot 2999$
$2^{13} \cdot 3^5 \cdot 7$	$11 \cdot 1266781$	$2 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	750749
$2^{13} \cdot 3^6 \cdot 7$	$5^2 \cdot 211 \cdot 10273$	$2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$19 \cdot 13171$
$2^{13} \cdot 3 \cdot 7^2$	$101 \cdot 11923$	$2 \cdot 3^5 \cdot 5^3 \cdot 11 \cdot 13$	8687249
$2^{13} \cdot 3^2 \cdot 7^2$	3612671	$2 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13$	$43 \cdot 67343$
$2^{13} \cdot 3^3 \cdot 7^2$	$5 \cdot 211 \cdot 10273$	$2 \cdot 3^3 \cdot 5^3 \cdot 11 \cdot 13$	965249
$2^{13} \cdot 3^4 \cdot 7^2$	$17 \cdot 521 \cdot 3671$	$2 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13$	$31 \cdot 97 \cdot 107$
$2^{13} \cdot 3^5 \cdot 7^2$	$19 \cdot 71 \cdot 72307$	$2 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13$	$23 \cdot 4663$
$2^{13} \cdot 3^6 \cdot 7^2$	$311 \cdot 940921$	$2 \cdot 5^3 \cdot 11 \cdot 13$	$7 \cdot 5107$
$2^{14} \cdot 3 \cdot 7$	$17 \cdot 37 \cdot 547$	$3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$2 \cdot 15202687$
$2^{14} \cdot 3^2 \cdot 7$	1032191	$3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$2^2 \cdot 317 \cdot 7993$
$2^{14} \cdot 3^3 \cdot 7$	$5^2 \cdot 123863$	$3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$2 \cdot 631 \cdot 2667$
$2^{14} \cdot 3^4 \cdot 7$	$19 \cdot 59 \cdot 8287$	$3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$2^2 \cdot 281531$
$2^{14} \cdot 3^5 \cdot 7$	$2399 \cdot 11617$	$3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$2 \cdot 1876871$
$2^{14} \cdot 3^6 \cdot 7$	$29 \cdot 2883019$	$5^3 \cdot 7 \cdot 11 \cdot 13$	$2^2 \cdot 3 \cdot 10427$
$2^{14} \cdot 3 \cdot 7^2$	$193 \cdot 12479$	$3^5 \cdot 5^3 \cdot 11 \cdot 13$	$2^3 \cdot 163 \cdot 3331$
$2^{14} \cdot 3^2 \cdot 7^2$	$2687 \cdot 2689$	$3^4 \cdot 5^3 \cdot 11 \cdot 13$	$2 \cdot 41 \cdot 17657$
$2^{14} \cdot 3^3 \cdot 7^2$	$13 \cdot 83 \cdot 20089$	$3^3 \cdot 5^3 \cdot 11 \cdot 13$	$2^6 \cdot 7541$
$2^{14} \cdot 3^4 \cdot 7^2$	$5 \cdot 11 \cdot 733 \cdot 1613$	$3^2 \cdot 5^3 \cdot 11 \cdot 13$	$2 \cdot 7 \cdot 11491$
$2^{14} \cdot 3^5 \cdot 7^2$	$2437 \cdot 80051$	$3 \cdot 5^3 \cdot 11 \cdot 13$	$2^3 \cdot 6703$
$2^{14} \cdot 3^6 \cdot 7^2$	$13 \cdot 17 \cdot 1423 \cdot 1861$	$5^3 \cdot 11 \cdot 13$	$2 \cdot 3^3 \cdot 331$

Table 3 Possible values of Z'_y and N'_y

Z_y	Z'_y	N_y	N'_y
$5 \cdot 11$	$2 \cdot 3^3$	$2^{14} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13$	$367 \cdot 518275697$
$5^2 \cdot 11$	$2 \cdot 137$	$2^{14} \cdot 3^6 \cdot 5 \cdot 7^2 \cdot 13$	38041436159
$5^3 \cdot 11$	$2 \cdot 3 \cdot 229$	$2^{14} \cdot 3^6 \cdot 7^2 \cdot 13$	$23 \cdot 389 \cdot 850373$

Table 4 Possible values of Z'_z and N'_z

Z_z	Z'_z	N_z	N'_z
$3 \cdot 13$	$2 \cdot 19$	$2^{14} \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11$	$31 \cdot 9187 \cdot 941867$
$3^2 \cdot 13$	$2^2 \cdot 19$	$2^{14} \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11$	89413631999
$3^3 \cdot 13$	$2 \cdot 5^2 \cdot 7$	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11$	29804543999
$3^4 \cdot 13$	$2^2 \cdot 263$	$2^{14} \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11$	$23^2 \cdot 18780431$
$3^5 \cdot 13$	$2 \cdot 1579$	$2^{14} \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 13$	3311615999