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Bifurcations for a tuberculosis disease population model with delay

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ABSTRACT
In this paper, a tuberculosis disease population model with time delays is investigated. The linear stability of the equilibrium and the existence of Hopf bifurcation with delay is investigated. Numerical simulations for justifying the theoretical results are illustrated. Finally, main conclusions are given.

Introduction

Recently substantial efforts have been made in tuberculosis disease population model which is frequently used[1,3]. In 2008, Koriko and Yusuf [2] studied the stability of the following tuberculosis disease population model:

$$\begin{cases} \dot{S} = vfN - \alpha I_A S - \delta S + T_A I_A + T_L I_L, \\ \dot{I}_L = (1-p)\alpha I_A S - \beta_A I_L - T_L I_L - \delta I_L, \\ \dot{I}_A = P\alpha I_A S + \beta_A I_L - T_A I_A - \delta I_A - \varepsilon I_A, \end{cases} \quad (1)$$

where S, I_L, I_A denote the susceptible population, the latently infected population and the actively infected population, respectively.

The model parameters and their respective descriptions are given as follows. N: total number of new people into the location of interest,

S: Number of susceptible people in the location, I_L: number of TB latently infected people, I_A: number of TB actively infected people,

δ: probability that a susceptible person is not vaccinated, f: efficacy rate of latent TB therapy, T_L:success rate of latent TB therapy, T_A: active TB treatment cure rate, a: TB instantaneous incidence rate per susceptible, d :humans natural death rate, p: proportion of infection instantaneously degenerating into active TB, e:TB-induced death rate, β_A:breakdown rate from latent to active TB. In details, one can see [2].

Taking into account that there is a certain time delay during the process that the model population become actively infected population (i.e., the population will take τ units of time to become actively infected population), the dynamic behavior of the system not only is affected by the current state of the system, but also the past state of the system, i.e., there exists inherent lag in the system. Based on this view of point, in this paper, we will revise system (1) as follows:

$$\begin{cases} \dot{S} = vfN - \alpha I_A(t-\tau)S - \delta S + T_A I_A(t-\tau) + T_L I_L, \\ \dot{I}_L = (1-p)\alpha I_A(t-\tau)S - \beta_A I_L - T_L I_L - \delta I_L, \\ \dot{I}_A = p\alpha I_A(t-\tau)S + \beta_A I_L - T_A I_A(t-\tau) - \delta I_A(t-\tau) - \varepsilon I_A(t-\tau), \end{cases} \quad (2)$$

For convenience, we denote S, I_L, I_A as x, y, z, respectively and set $a = vfN, b = \alpha, c = T_A, d = T_L, \beta = \beta_A$, then system (2) can be taken the form

$$\begin{cases} \dot{x} = a - bxz(t-\tau) - \delta x + cz(t-\tau) + dy, \\ \dot{y} = (1-p)bxz(t-\tau) - \beta y - dy - \delta y, \\ \dot{z} = pbxz(t-\tau) + \beta y - cz(t-\tau) - \delta z(t-\tau) - \varepsilon z(t-\tau), \end{cases} \quad (3)$$

It is well known that delay may have very complicated impact on the dynamics of a system. To obtain a deep and clear understanding of dynamics of tuberculosis disease population system with delay, in this paper, we will study the Hopf bifurcation of model (3). Choosing the delay τ as the bifurcation parameter, we shall investigated the effect of the delay τ on the dynamics of system (3).

The remainder of the paper is organized as follows. In Section 2, we discuss the stability of the equilibrium and the existence of Hopf bifurcations occurring at the equilibrium. In Section 3, numerical simulations are carried out to validate our main results. Some main conclusions are drawn in Section 4.

Stability of the equilibrium and local Hopf bifurcations

It is easy to see that system (2) has a disease-free equilibrium $E_0(x^*, 0, 0)$, where $x^* = a/\delta$, and the endemic equilibrium satisfies the following equations

$$\begin{cases} x = \dfrac{a + cz + dy}{\delta + \alpha z}, \\ y = \dfrac{(1-p)\alpha xz}{\beta + y + \delta}, \\ z = \dfrac{\beta y}{c + \varepsilon + \delta - p\alpha x}. \end{cases}$$

For simplicity, we only consider the stability and local Hopf bifurcation near the disease-free equilibrium $E_0(x^*, 0, 0)$. The linearization of Eq. (3) at $E_0(x^*, 0, 0)$ takes the form

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$$\begin{cases} \dot{x} = -\delta x + dy + (c - bx^*)z(t-\tau), \\ \dot{y} = -(\beta + d + \delta)y + (1-p)bx^* z(t-\tau), \\ \dot{z} = \beta y + (pbx^* - c - \delta - \varepsilon)z(t-\tau). \end{cases} \quad (4)$$

The characteristic equation of system (4) takes the form

$$(\lambda + \delta)(\lambda^2 + m_1\lambda + m_0 + n_0 e^{-\lambda\tau}) = 0, \quad (5)$$

where $m_0 = -(\beta + d + \delta)(pbx^* - c - \delta - \varepsilon)$,

$m_1 = \beta + d + \delta - pbx^* + c + \delta + \varepsilon, n_0 = \beta(p-1)bx^*.$

The stability of the positive equilibrium of system (3) depends on the locations of the roots of the characteristic equation (5), When all roots of Eq.(5) locate in the left half of complex plane, the trivial solution is stable, otherwise, it is instable.

It follows from (5) that $\lambda_1 = -\delta < 0$, then we only need to investigate the distribution of roots of the following equation

$$\lambda^2 + m_1\lambda + m_0 + n_0 e^{-\lambda\tau} = 0. \quad (6)$$

For $\tau = 0$, (6) becomes

$$\lambda^2 + m_1\lambda + m_0 + n_0 = 0. \quad (7)$$

It is easy to see that a set of necessary and sufficient conditions that all roots of (7) have a negative real part is given in the following form:

(H1) $m_1 > 0, m_0 + n_0 > 0.$

For $\omega > 0$, $i\omega$ is a root of (6), then

$$-\omega^2 + im_1\omega + m_0 + +n_0(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} n_0\cos\omega\tau = \omega^2 - m_0, \\ n_0\sin\omega\tau = -m_1\omega. \end{cases} \quad (8)$$

which leads to

$$\omega^4 + (m_1^2 - 2m_0)\omega^2 + m_0^2 - n_0^2 = 0. \quad (9)$$

In the sequel, we consider three cases.

(a) If the condition

(H2) $m_1^2 - 2m_0 > 0, m_0^2 - n_0^2 > 0$ or

$\Delta = (m_1^2 - 2m_0)^2 - 4(m_0^2 - n_0^2) < 0$

holds, then Eq.(9) has no positive root.

(b) If the condition

(H3) $m_1^2 - 2m_0 < 0, m_0^2 - n_0^2 > 0$ or

$\Delta = (m_1^2 - 2m_0)^2 - 4(m_0^2 - n_0^2) > 0$

holds, then Eq.(9) has two positive roots.

$$\omega_\pm = \frac{\sqrt{2}}{2}[-m_1^2 + 2m_0 \pm \sqrt{\Delta}]^{\frac{1}{2}}. \quad (10)$$

(c) If the condition

(H4) $m_1^2 - 2m_0 < 0, m_0^2 - n_0^2 < 0$ or

$\Delta = (m_1^2 - 2m_0)^2 - 4(m_0^2 - n_0^2) = 0$

holds, then Eq.(8) has only one positive root $\omega_\pm$.

Without loss of generality, we assume that (9) has two positive roots denoted by $\omega_\pm$. It follows from (8) that

$$\tau_j^\pm = \frac{1}{\omega_\pm}\left[\arccos\frac{\omega_\pm^2 - m_0}{n_0^2} + 2j\pi\right]. \quad (11)$$

at which Eq.(6) has a pair of purely imaginary roots $\pm i\omega_\pm$, where $j = 0,1,2,\ldots$. Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq.(6) such that $\alpha(\tau_j^\pm) = 0, \omega(\tau_j^\pm) = \omega_\pm$. Due to functional differential equation theory, for every $\tau_j^\pm, j = 0,1,2,\ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_j^\pm| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (6) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda + m_1)e^{\lambda\tau}}{n_0\lambda} - \frac{\tau}{\lambda},$$

which, together with (7), leads to

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^\pm} = \mathrm{Re}\left\{\frac{(2\lambda + m_1)e^{\lambda\tau}}{n_0\lambda}\right\}_{\tau=\tau_j^\pm}$$

$$= \frac{2\omega_\pm\cos\omega_\pm\tau_j^\pm + m_1\sin\omega_\pm\tau_j^\pm}{n_0\omega_\pm}$$

$$= \frac{2\omega_\pm^2 - m_1^2 - 2m_0}{n_0^2} = \frac{\pm\sqrt{\Delta}}{n_0^2},$$

where $\Delta = (m_1^2 - 2m_0)^2 - 4(m_0^2 - n_0^2)$.

Thus, if $\Delta \neq 0$, we obtain

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j^+}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^+}\right\}$$

$$\mathrm{sign}\left\{\frac{1}{n_0^2}\left\{+\sqrt{\Delta}\right\}\right\} = 1 > 0 \quad \text{and}$$

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j^-}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^-}\right\}$$

$$\mathrm{sign}\left\{\frac{1}{n_0^2}\left\{-\sqrt{\Delta}\right\}\right\} = 1 < 0.$$

According to above analysis and the results of Ruan and Wei[6],Yang[5] and Hale[3], we have the following result.

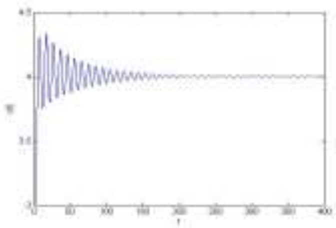
Theorem2.1. Let $\tau_j^\pm (j = 0,1,2,\ldots)$ be defined by (11) and $\tau = \min\{\tau_0^+, \tau_0^-\}$. If (H1)-(H3) hold, then the equilibrium $E_0(x^*,0,0)$ of system (3) is asymptotically stable for $\tau \in [0,\tau_0)$. If (H1), (H2) and (H4) hold, system (3) undergoes a Hopf bifurcation at the equilibrium $E_0(x^*,0,0)$ when $\tau = \tau_j^\pm (j = 0,1,2,\ldots)$

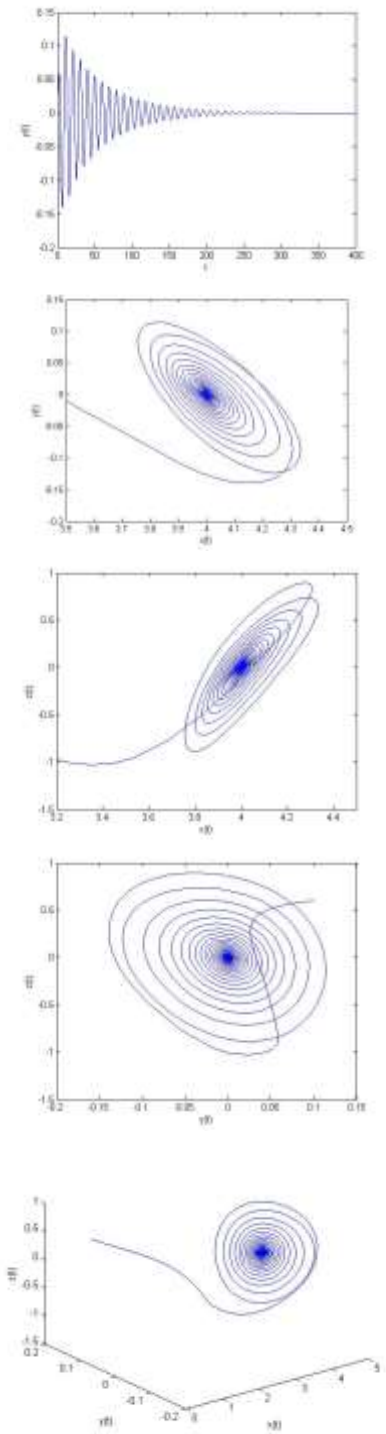
Numerical Examples

In this section, we will prove some numerical results of system (3) to illustrate our results obtained in Section 2. We consider the system

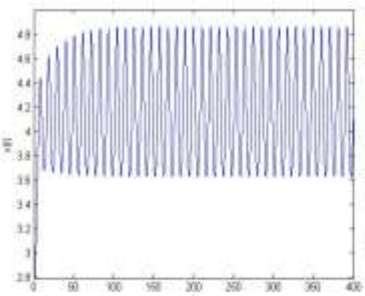
$$\begin{cases} \dot{x} = 2 - 0.2xz(t-\tau) - 0.5x + 0.5z(t-\tau) + 0.3y, \\ \dot{y} = 0.16xz(t-\tau) - 4y - 0.3y - 0.5y, \\ \dot{z} = 0.4xz(t-\tau) + 4y - 0.5z(t-\tau) - 0.5z(t-\tau) - 0.3z(t-\tau), \end{cases} \quad (12)$$

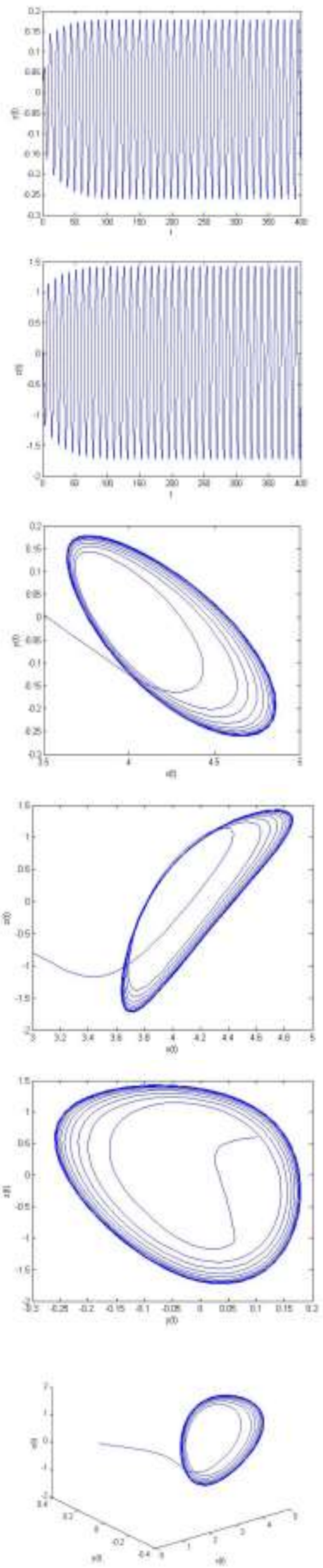
which has a disease-free equilibrium $E_0(4,0,0)$ and satisfies the conditions indicated in Theorem 2.1. When $\tau = 0$, the disease-free equilibrium $E_0(4,0,0)$ is asymptotically stable. The disease-free equilibrium $E_0(4,0,0)$ is stable when $\tau < \tau_0 \approx 1.66$ which is illustrated by the computer simulations (see Fig.1). When τ passes through the critical value τ_0, the disease-free equilibrium $E_0(4,0,0)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the disease-free equilibrium $E_0(4,0,0)$ (see Fig.2).





Figs.1-7 When $\tau = 2.57 < \tau_0 \approx 2.6$. The disease-free equilibrium $E_0(4,0,0)$ of system (12) is asym pt otically stable. The initial value is (0.3,0.1,0.6).





Figs.8-14 When Hopf bifurcation of system (12) occurs from the disease-free equilibrium. The initial value is (0.3,0.1,0.6)

Conclusions

In this paper, the local stability of the equilibrium and local Hopf bifurcation in a FitzHugh-Nagumo model with time delay are investigated. It is showed that if the conditions (H1)-(H3) hold, the equilibrium of system (1.2) is asymptotically stable for all. If the conditions (H1),(H2) and (H4) hold, as the delay increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur at the equilibrium , i.e., a family of periodic orbits bifurcates from the the disease-free equilibrium .Moreover, the length of delay preserving the stability of the equilibrium is estimated. Some numerical simulations are performed to verify our theoretical results found.

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