Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 36 (2011) 3062-3066

ISSN: 2229-712X

Directed edge - graceful labeling of trees

B.Gayathri[1] and V.Vanitha[2]

[1]Department of Mathematics, Periyar E.V.R. College, Trichy – 620 023.

[2]Mother Teresa Women's University, Kodaikanal – 624 102.

ARTICLE INFO

Article history:

Received: 19 April 2011;

Received in revised form: 15 June 2011;

Accepted: 24 June 2011;

Keywords

Festoon,

Twig,

AMS (MOS) Subject Classification: 05C78.

ABSTRACT

Rosa [13] introduced the notion of graceful labelings. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [11]. Bloom and Hsu [3, 4, 5] extended the notion of graceful labeling to directed graphs. In 1985, Lo [12] introduced the notion of edge – graceful graphs. We introduced [8] the concept of edge – graceful labelings to directed graphs and further studied in [9]. In this paper we investigate directed edge – graceful labeling of trees.

Introduction

All graphs in this paper are finite and directed. Terms not defined here are used in the sense of Harary [10]. The symbols V(G) and E(G) will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X–ray crystallography, radar, astronomy, circuit design, communication network addressing, database management etc. [1, 2]. A good account on graceful labeling problems and other types of graph labeling problems can be found in the dynamic survey of J.A. Gallian [6].

A graph G is called a graceful labeling if f is an injection from the vertices of G to the set {0, 1, 2, ..., q} such that, when each edge xy is assigned the label | f(x) – f(y) |, the resulting edge labels are distinct.

A graph G(V, E) is said to be edge–graceful if there exists a bijection f from E to {1, 2, ..., |E|} such that the induced mapping from V to {0, 1, ..., |V| – 1} given by, = (□f(xy)) mod(|V|) taken overall edges xy incident at x is a bijection.

A necessary condition for a graph G with p vertices and q edges to be edge–graceful is q(q + 1) □ . Bloom and Hsu [3, 4, 5] extended the notion of graceful labelling to directed graph. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [11]. In [8] we extended the concept of edge–graceful labelings to directed graphs and further studied in [9]. In this paper we investigate directed edge – graceful labeling of trees.

A (p, q) graph G is said to be directed edge – graceful if there exists an orientation of G and a labeling f of the arcs A of G with {1, 2, ..., q} such that induced mapping g on V defined by, g(v) = (mod p) is a bijection where, = the sum of the labels of all arcs with head v and = the sum of the labels of all arcs with v as tail.

A graph G is said to be directed edge–graceful graph if it has directed edge–graceful labelings. Here, we investigate directed edge – graceful labeling of trees.

Prior Results

Theorem 2.1: [8] The path P2n+1 is directed edge-graceful for all n ≥1.

Theorem 2.2: [8] The cycle graph C2n+1 is directed edge-graceful for all n ≥1.

Theorem 2.3 : [8] The Butterfly graph Bn is directed edge-graceful if n is odd.

Theorem 2.4 : [8] The Butterfly graph Bn is directed edge – graceful if n is even and n ≥ 4.

Theorem 2.5 : [8] The snail graph SN(2n + 1) is directed edge-graceful for all n ≥ 1.

Theorem 2.6 : [8] is directed edge-graceful if n is even and n ≥ 4.

Theorem 2.7: [9] The graph P3 ≥ K1,2n+1 is directed edge-graceful for all n ≥ 1.

Theorem 2.8 : [9] The graph P2m @ K1,2n+1 is directed edge-graceful for all m ≥ 2 and n ≥ 1.

Theorem 2.9 : [9] The graph P2m+1 @ K1,2n is directed edge-graceful for all m ≥ 1 and n ≥ 1.

Main results

Theorem

The festoon graph F(n, m) of odd order is directed edge - graceful graph.

Proof

The festoon graph F(n, m) is of odd order if n is odd and m is even. Let G = F(n, m) and V[F(n, m)] = {w1, w2, ..., wn, v11, v12, ..., v1m, v21, v22, ..., v2m, ..., vn1, vn2, ..., vnm} be the set of vertices. Now we orient the edges of F(n, m) such that the arc set A is given by,

$$A = \left\{(w_{2i-1}, w_{2i}), 1 \le i \le \frac{n-1}{2}\right\} \cup \left\{(w_{2i+1}, w_{2i}), 1 \le i \le \frac{n-1}{2}\right\}$$

Tele:

E-mail addresses: rajarajanvani@gmail.com, maduraigayathri@gmail.com

u {(wi, vij), 1 ≤ i≤n, 1 ≤ j ≤ m}

The edges and their orientation of F(n, m) are as in Fig. (1).

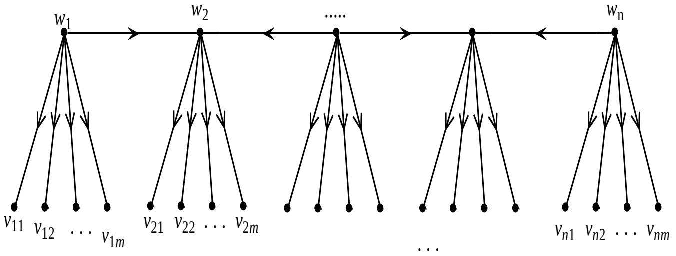

Fig. 1: F(n, m) with orientation

We now label the arcs of A as follows.

$$f\left(\left(w_{2i+1}, w_{2i}\right)\right) = i,\ 1 \le i \le \frac{n-1}{2}$$

$$f\left(\left(w_{2i-1}, w_{2i}\right)\right) = nm + \frac{n-1}{2} + i,\ 1 \le i \le \frac{n-1}{2}$$

$$f\left(\left(w_i, v_{i(2j-1)}\right)\right) = \left(\frac{n-1}{2}\right) + \frac{m}{2}(i-1) + j,\ 1 \le i \le n,\ 1 \le j\ \frac{m}{2}$$

$$f\left(\left(w_i, v_{i(2j)}\right)\right) = n(m+1) - \left(\frac{n-1}{2}\right) - \frac{m}{2}(i-1) - j,\ 1 \le i \le n,\ 1 \le j\ \frac{m}{2}$$

The values of $f^+(w_i)$, $f^+(v_{ij})$ and $f^-(w_i)$, $f^-(v_{ij})$ are computed as under.

$$f^+(w_1) = 0$$

$$f^-(w_1) = -\left[n(m+1)\left(\frac{m}{2}+1\right) - \left(\frac{n-1}{2}\right)\right]$$

$$f^+(w_{2i}) = (nm+n) - \left(\frac{n-1}{2}\right) - 1 + 2i,\ 1 \le i \le \frac{n-1}{2}$$

$$f^-(w_{2i}) = -\left[\frac{m}{2}(nm+n)\right]$$

$$f^+(w_{2i+1}) = 0$$

$$f^-(w_{2i+1}) = -\left[n(m+1)\left(\frac{m}{2}+1\right) - \left(\frac{n-1}{2}\right) + 2i\right],\ 1 \le i \le \frac{n-3}{2}$$

$$f^+(w_n) = 0$$

$$f^-(w_n) = -\left[\frac{m}{2}(nm+n) + \left(\frac{n-1}{2}\right)\right]$$

$$f^+(v_{i(2j-1)}) = \left(\frac{n-1}{2}\right) + \frac{m}{2}(i-1) + j,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

$$f^-(v_{i(2j-1)}) = 0,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

$$f^+(v_{i(2j)}) = 0,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

$$n(m+1) - \left(\frac{n-1}{2}\right) - \frac{m}{2}(i-1) - j,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

$$f^-(v_{i(2j)}) = 0,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

Then the induced vertex labels are,

$$g(v_{i(2j-1)}) = \left(\frac{n-1}{2}\right) + \frac{m}{2}(i-1) + j,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

$$g(v_{i(2j)}) = n(m+1) - \left(\frac{n-1}{2}\right) - \frac{m}{2}(i-1) - j,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

Case (i): $\left(\frac{n-1}{2}\right)$ is even

$$g(w_{2i-1}) = \frac{n-1}{2} + 2 - 2i,\ 1 \le i \le \frac{n+3}{4}$$

$$g(w_{2i}) = n(m+1) - \left(\frac{n-1}{2}\right) - 1 + 2i,\ 1 \le i \le \frac{n-1}{4}$$

$$g\left(w_{\frac{n-1}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{n-1}{4}$$

$$g\left(w_{\frac{n+1}{2}+2i}\right) = nm + n - 2i,\ 1 \le i \le \frac{n-1}{4}$$

Case (ii): $\left(\frac{n-1}{2}\right)$ is odd

$$g(w_{2i-1}) = \left(\frac{n+3}{2}\right) - 2i,\ 1 \le i \le \frac{n+1}{4}$$

$$g(w_{2i}) = nm + \left(\frac{n-1}{2}\right) + 2i,\ 1 \le i \le \frac{n-3}{4}$$

$$g\left(w_{\frac{n-3}{2}+2i}\right) = 2i - 2,\ 1 \le i \le \frac{n+1}{4}$$

$$g\left(w_{\frac{n-1}{2}+2i}\right) = nm + n + 1 - 2i,\ 1 \le i \le \frac{n+1}{4}$$

Clearly, $g(V) = \{0, 1, ..., n(m+1) - 1\} = \{0, 1, ..., p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $F(n, m)$ of odd order is a directed edge-graceful graph. The directed edge-graceful labeling of $F(5, 10)$ and $F(7, 6)$ are given in Fig. (2) and Fig. (3) respectively.

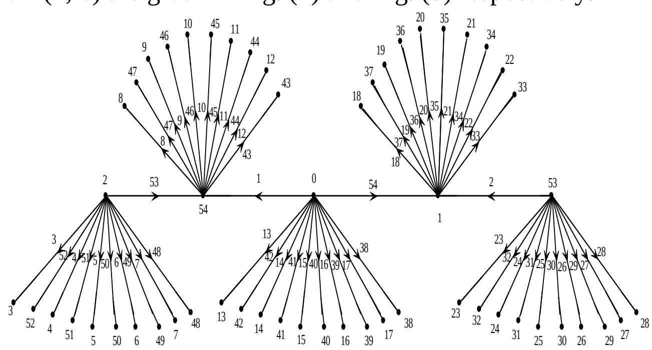

Fig. 2: F(5, 10) with directed edge-graceful labeling

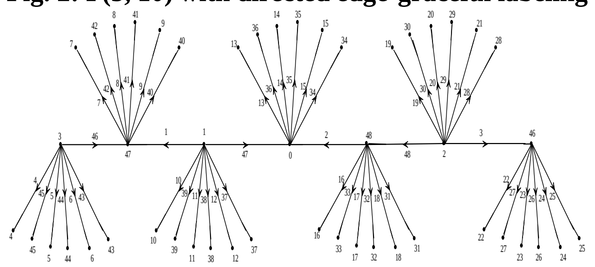

Fig. 3: F(7, 6) with directed edge-graceful labeling

Theorem

The graph $\left\langle K_{1,n} : m \right\rangle$, n is even and m, $n \geq 2$ is directed edge-graceful graph.

Proof

Let $G = \left\langle K_{1,n} : m \right\rangle$ n is even and $m, n \geq 2$ and

$V\left[\left\langle K_{1,n} : m \right\rangle\right] = \{u_1, u_2, ..., u_m, w_1, w_2, ..., w_{m-1}, u_{11}, u_{12}, ..., u_{1n}, u_{21}, u_{22}, ..., u_{2n}, ..., u_{m1}, u_{m2}, ..., u_{mn}\}$ be the vertices. Now we orient the edges of $\left\langle K_{1,n} : m \right\rangle$ such that the arc set A is given by,

$A = \{(u_i, w_i), 1 \leq i \leq m-1\} \cup \{(u_{i+1}, w_i), 1 \leq i \leq m-1\}$
$\cup \{(u_i, u_{ij}), 1 \leq i \leq m, 1 \leq j \leq n\}$

The edges and their orientation of $\left\langle K_{1,n} : m \right\rangle$ are as in Fig. (4).

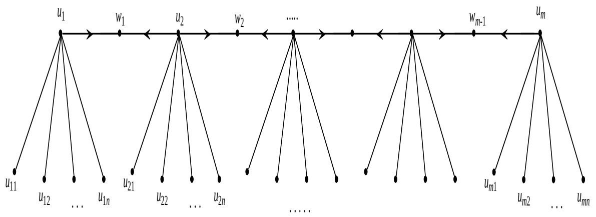

Fig. 4: $\left\langle K_{1,n} : m \right\rangle$ with orientation

We now label the arcs of A as follows.

$$f\left(\left(u_{i+1}, w_i\right)\right) = i, \ 1 \leq i \leq m-1$$

$$f\left(\left(u_i, w_i\right)\right) = m(n+1) - 1 + i, \ 1 \leq i \leq m-1$$

$$f\left(\left(u_i, u_{i(2j-1)}\right)\right) = m - 1 + \frac{n}{2}(i-1) + j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

$$f\left(\left(u_i, u_{i(2j)}\right)\right) = nm + m - j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

The values of $f^+(u_i)$, $f^+(w_i)$, $f^+(u_{ij})$ and $f^-(u_i)$, $f^-(w_i)$, $f^-(u_{ij})$ are computed as under.

$$f^+(u_1) = 0$$

$$f^-(u_1) = -\left[\frac{n}{2}(nm + 2m - 1) + m(n+1)\right]$$

$$f^+(w_i) = m(n+1) - 1 + 2i, \ 1 \leq i \leq m-1$$

$$f^-(w_i) = 0, \ 1 \leq i \leq m-1$$

$$f^+(u_{i+1}) = 0, \ 1 \leq i \leq m-2$$

$$f^-(u_{i+1}) = -\left[\frac{n}{2}(nm + 2m - 1) + m(n+1) + 2i\right], \ 1 \leq i \leq m-2$$

$$f^+(u_m) = 0$$

$$f^-(u_m) = -\left[\frac{n}{2}(nm + 2m - 1) + m - 1\right]$$

$$f^+(u_{i(2j-1)}) = m - 1 + \frac{n}{2}(i-1) + j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

$$f^-(u_{i(2j-1)}) = 0, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

$$f^+(u_{i(2j)}) = nm + m - j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

$$f^-(u_{i(2j)}) = nm + m - j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

Then the induced vertex labels are

$$g(u_{i2j-1}) = m - 1 + \frac{n}{2}(i-1) + j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

$$g(u_{i2j}) = nm + m - j, \ 1 \leq i \leq m, \ 1 \leq j \leq \frac{n}{2}$$

Case (i): m is even

$$g(u_i) = m + 1 - 2i, \ 1 \leq i \leq \frac{m}{2}$$

$$g(w_i) = m(n+1) - 1 + 2i, \ 1 \leq i \leq \frac{m-2}{2}$$

$$g\left(w_{\frac{m-2}{2}+i}\right) = 2i - 2, \ 1 \leq i \leq \frac{m}{2}$$

$$g\left(u_{\frac{m}{2}+i}\right) = m(n+2) - 2i, \ 1 \leq i \leq \frac{m}{2}$$

Case (ii): m is odd

$$g(u_i) = m + 1 - 2i, \ 1 \leq i \leq \frac{m+1}{2}$$

$$g(w_i) = m(n+1) - 1 + 2i, \ 1 \leq i \leq \frac{m-1}{2}$$

$$g\left(w_{\frac{m-1}{2}+i}\right) = 2i - 1, \ 1 \leq i \leq \frac{m-1}{2}$$

$$g\left(u_{\frac{m+1}{2}+i}\right) = nm + 2m - 1 - 2i, \ 1 \leq i \leq \frac{m-1}{2}$$

Clearly, $g(V) = \{0, 1, ..., nm + 2m - 2\} = \{0, 1, ..., p-1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $\left\langle K_{1,n} : m \right\rangle$ is a directed edge-graceful graph.

The directed edge-graceful labeling of $\left\langle K_{1,6} : 5 \right\rangle$ and $\left\langle K_{1,4} : 7 \right\rangle$ are given in Fig. (5) and Fig. (6) respectively.

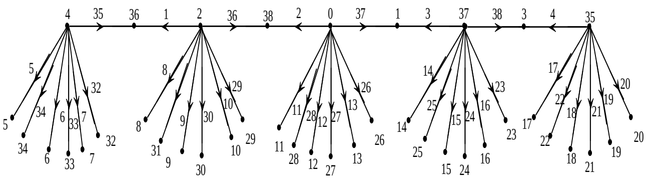

Fig. 5: $\left\langle K_{1,6} : 5 \right\rangle$ with directed edge-graceful labeling

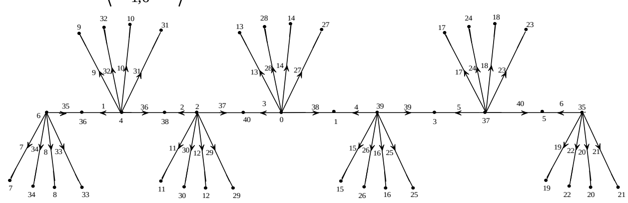

Fig. 6: $\left\langle K_{1,4} : 7 \right\rangle$ with directed edge-graceful labeling

Theorem

The twig graph T_n is directed edge-graceful if n is odd and $n \geq 5$.

Proof

Let $G = T_n$ and $V[T_n] = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-2}, v_1', v_2', \ldots, v_{n-2}'\}$ be the set of vertices. Now we orient the edges of T_n such that, the arc set A is given by,

$$A = \left\{(u_{2i-1}, u_{2i}),\ 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{(u_{2i+1}, u_{2i}),\ 1 \leq i \leq \frac{n-1}{2}\right\}$$

$$\cup \left\{(v_i, u_{i+1}),\ 1 \leq i \leq n-2\right\} \cup \left\{(v_i', u_{i+1}),\ 1 \leq i \leq n-2\right\}$$

The edges and their orientation of T_n are as in Fig (7).

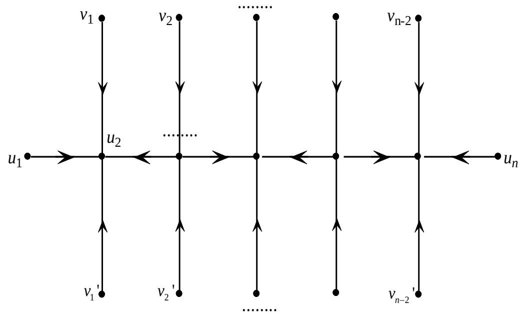

Fig. 7: T_n with orientation

We now label the arcs of A as follows.

$$f((u_{2i-1}, u_{2i})) = 2(n-2) + \left(\frac{n-1}{2}\right) + i,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f((u_{2i+1}, u_{2i})) = i,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f((v_i, u_{i+1})) = \left(\frac{n-1}{2}\right) + i,\ 1 \leq i \leq n-2$$

$$f((v_i', u_{i+1})) = 2(n-2) + \left(\frac{n-1}{2}\right) + 1 - i,\ 1 \leq i \leq n-2$$

The values of $f^+(u_i)$, $f^+(v_i)$, $f^+(v_i')$ and $f^-(u_i)$, $f^-(v_i)$, $f^-(v_i')$ are computed as under.

$$f^+(u_1) = 0$$

$$f^-(u_1) = -\left[2(n-2) + \left(\frac{n+1}{2}\right)\right]$$

$$f^+(u_{2i}) = 4(n-2) + 3\left(\frac{n+1}{2}\right) - 2 + 2i,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f^-(u_{2i}) = 0,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f^+(u_{2i+1}) = 2(n-2) + n,\ 1 \leq i \leq \frac{n-3}{2}$$

$$f^-(u_{2i+1}) = -\left[2(n-2) + \left(\frac{n+1}{2}\right) + 2i\right],\ 1 \leq i \leq \frac{n-3}{2}$$

$$f^+(u_n) = 0$$

$$f^-(u_n) = -\left(\frac{n-1}{2}\right)$$

$$f^+(v_i) = 0,\ 1 \leq i \leq n-2$$

$$f^-(v_i) = -\left[\left(\frac{n-1}{2}\right) + i\right],\ 1 \leq i \leq n-2$$

$$f^+(v_i') = 0,\ 1 \leq i \leq n-2$$

$$f^-(v_i') = -\left[2(n-2) + \left(\frac{n-1}{2}\right) + 1 - i\right],\ 1 \leq i \leq n-2$$

Then the induced vertex labels are,

$$g(v_i) = 2(n-2) + \left(\frac{n-1}{2}\right) + 1 - i,\ 1 \leq i \leq n-2$$

$$g(v_i') = \left(\frac{n-1}{2}\right) + i,\ 1 \leq i \leq n-2$$

Case (i) $\left(\frac{n-1}{2}\right)$ is odd

$$g(u_{2i-1}) = \frac{n+3}{2} - 2i,\ 1 \leq i \leq \frac{n+1}{4}$$

$$g(u_{2i}) = 2(n-2) + \left(\frac{n-1}{2}\right) + 2i,\ 1 \leq i \leq \frac{n-3}{4}$$

$$g\left(u_{\frac{n-3}{2}+2i}\right) = 2i - 2,\ 1 \leq i \leq \frac{n+1}{4}$$

$$g\left(u_{\frac{n-1}{2}+2i}\right) = 2(n-2) + n + 1 - 2i,\ 1 \leq i \leq \frac{n+1}{4}$$

Case (ii) $\left(\frac{n-1}{2}\right)$ is even

$$g(u_{2i-1}) = \frac{n+3}{2} - 2i,\ 1 \leq i \leq \frac{n+3}{4}$$

$$g(u_{2i}) = 2(n-2) + \left(\frac{n-1}{2}\right) + 2i,\ 1 \leq i \leq \frac{n-1}{4}$$

$$g\left(u_{\frac{n-1}{2}+2i}\right) = 2i - 1,\ 1 \leq i \leq \frac{n-1}{4}$$

$$g\left(u_{\frac{n+1}{2}+2i}\right) = 2(n-2) + n - 2i,\ 1 \leq i \leq \frac{n-1}{4}$$

Clearly, $g(V) = \{0, 1, \ldots, 3n-5\} = \{0, 1, \ldots, P-1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, T_n is a directed edge-graceful graph if n is odd. The directed edge-graceful labeling of T_{13} and T_{15} are given in Fig. (8) and Fig. (9) respectively.

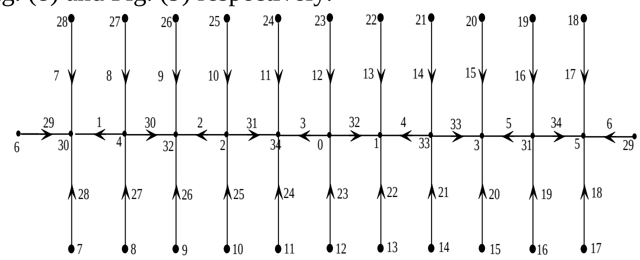

Fig. 8: T13 with directed edge-graceful labeling

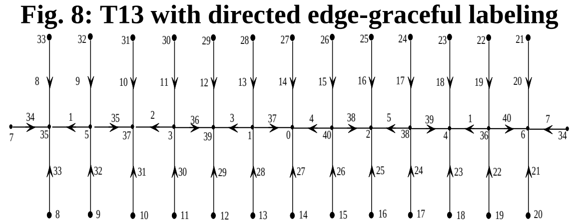

Fig. 9: T15 with directed edge-graceful labeling

References

[1] G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs, Proc–IEEE, 65 (1977) 562–570.

[2] G. S. Bloom and S. W. Golomb, Numbered complete graphs, Unusual rulers, and assorted applications, in Theory and Applications of Graphs, Lecture Notes in Math., 642, Springer – Verlag, New York (1978) 53–65.

[3] G. S. Bloom and D. F. Hsu, On graceful digraphs and a problem in network addressing, Congr. Numer., 35 (1982) 91–103.

[4] G. S. Bloom and D. F. Hsu, On graceful directed graphs that are computational models of some algebraic systems, Graph Theory with Applications to Algorithms and Computers, Ed. Y. Alavi, Wiley, New York (1985).

[5] G. S. Bloom and D. F. Hsu, On graceful directed graphs, SIAM J. Alg. Discrete Meth., 6 (1985) 519–536.

[6] J. A. Gallian – A dynamic survey of graph labeling – The electronic journal of combinatorics (2010).

[7] B. Gayathri and M. Duraisamy, Even edge – graceful labeling of fan graphs, Bulletin of Pure and Applied Sciences, 26E (No.2) (2007) 285–293.

[8] B. Gayathri and V. Vanitha, Directed edge – graceful labeling of some trees and cycle related graphs, Bharathidasan University journal of Science and Technology, (in press).

[9] B. Gayathri and V. Vanitha, Directed edge – graceful labeling of path and star related trees, Presented in the International Conference on Mathematics and Computer Science (ICMCS-2011), Loyola College Chennai.

[10] F. Harary – Graph theory – Addison Wesley, Reading Mass (1972).

[11] S. Josephine Jeyarani, Studies in graph theory labelings of graphs, Ph.D. thesis, Manonmaniam Sundaranar University, (1996).

[12] S. Lo – On edge – graceful labelings of graphs, Congr. Numer., 50(1985) 231–241.

[13] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breac, N. Y. and Dunod Paris (1967) 349–355.