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On an iterative algorithm for sharpening Sahai-Sathe's upper-lower bounds on the variance of UMVU estimator in inverse sampling

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ABSTRACT

Sathe, Y. S. (1977) found a new set of sharp upper and lower bounds on the variance of the UMVUE in the case of inverse sampling motivated by the fact that a close-form expression to capture the actual variance is unavailable. Sahai, Ajit (1985) improved the variance upper bound in Sathe, Y. S. (1977). This note is motivated by their papers and by their inherent desire to capture that variance possibly more closely using their set of the sharp Sahai-Sathe's upper-and-lower bounds. It was very heartening for the authors to realize that the same is possible. In fact the seminal result achieved by the authors in this note could be used for improving the sharpness of these bounds iteratively till it pleases the one using it for the purpose of being close to the actual variance of the UMVUE in the absence of its capture in the closed-form. The achievement is briefly illustrated through a modest empirical study to bring forth the power of the proposed iterative algorithm for the aforesaid purpose.

Introduction

It is not uncommon to be presented with a scenario in biomedical research wherein the success probability is challenging, and the randomized clinical trial requires the application of inverse sampling to have a statistically sound estimate of that probability.

As an illustration, consider the case of the object population of HIV+ patients undergoing treatment to delay the onset of AIDS status as far as plausible. Now, suppose that one of the aims of the currently ongoing research investigations is to estimate the probability p of the success of these treatments with success defined as the deferment of AIDS' onset by a decade or more. Statistically, we may have to employ inverse sampling to have a good estimate of p, based on the number of patients X_r preceding the r successes, arriving at r using well-known statistical criteria.

Sathe (1977) found a set of sharp upper and lower bounds on the variance of the UMVUE (uniformly minimum-variance unbiased estimator) in the case of inverse sampling, motivated by the fact that a closed-form expression to capture the actual variance is unavailable. Sahai (1985) subsequently improved the variance upper bound. The current note is motivated by this prior work paper and an analytic need to capture that variance more closely using prior sets of sharp upper and lower bounds.

Let $U = (u_1, u_2, u_3, \ldots)$ denote a sequence of Bernoulli trials with an unknown probability of success p, where $P(u_i = 1) = p$. Consider the problem of estimating the probability p by inverse sampling. Therefore, the outcomes u_i's are observed sequentially until r successes occur. Let X_r stand for the number of trials required for that. Then, it is well established that the MVUE (minimum variance unbiased estimator) of p is $Tr = (r-1)/(X_r-1)$. (1.1)

As observed in Best (1974) and Mikulski and Smith (1976), in the absence of a close-form expression for its variance $V(T_r)$, and therefore in the absence of any simple tractable expression, Sahai (1985) and Sathe (1977), respectively, found a new set of sharp upper and lower bounds on $V(T_r)$, which is the variance of the MVUE T_r [where, $r > 2$], as below.

VUBS {Variance Upper Bound by Sahai (1985)} $\equiv 2{*}p^2q/D1$. (1.2)

VLBS {Variance Lower Bound by Sathe (1977)} $\equiv 2{*}p^2q/D2$. (1.3)

Wherein, $E = (r-1){*}(r-2+p)$; $F = E^2 + 8pq + (r-1){*}\,8p^2q^2/(r-2)$; $G = E + 4pq$; $H = F^{1/2} - G$; $D1 = H{*}(p/4q)$. (1.4)

$D2 = (r-q) + [(r-3q)^2 + 8pq]^{1/2}$; Where, $r > 2$. (1.5)

Main Results

The main findings achieved by the authors in this note, which could be used for improving the sharpness of these bounds iteratively till it pleases the one using it for the purpose of being close to the actual variance of the UMVUE in the absence of its capture in the closed-form, is stated in the following lemma.

Lemma 2.1. Given that: $L \le Q \le U$. (2.1)

We have: $L + DUL/4 \le Q \le U - DUL/4$ (2.2)

Wherein: $DUL = U - L$. (2.3)

Proof: Consider $U(\lambda) = U(1 + \lambda + \lambda^2) - L.(\lambda + \lambda^2) = U + (\lambda + \lambda^2)\,DUL$; $\lambda \ge 0$. (2.4)

Then, $\delta U(\lambda)/\delta\lambda = 0$, $\lambda_0 = -1/2$ is the stationary point corresponding to minimum as $\delta^2 U(\lambda)/\delta\lambda^2 = 2.\,DUL \ge 0$ (2.5)

Similarly, Consider $L(\mu) = L(1 + \mu + \mu^2) - U.(\mu + \mu^2) = L - (\mu + \mu^2)\,DUL$; $\mu \ge 0$. (2.6)

Then, $\delta L(\mu)/\delta\mu = 0$, $\mu_0 = -1/2$ is the stationary point corresponding to maximum as $\delta^2 L(\mu)/\delta\mu^2 = -2.\,DUL \le 0$ (2.7)

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This, therefore, implies that: L (μ_0) ≤ Q ≤ U (λ_0); wherein $\lambda_0 = -1/2 = \mu_0$. This is the same as (2.2) with (2.3). Q.E.D.

The proposed iterative algorithm for sharper lower and upper bounds

With reference to Lemma 2.1 in the preceding section and in the context of equations (2.1), (2.2), and (2.3), the lower and the upper bounds VUBS and VLBS achieved by Sahai (1985) and Sathe (1977) may be addressed. Applying (2.1) to (2.3) to these bounds, the first iteration of the proposed algorithm yields:

VUBSS {Variance Upper Bound by Skrepnek and Sahai} [1] ≡ VUBS – DUL/4, (3.1)

VLBSS {Variance Lower Bound by Skrepnek and Sahai} [1] ≡ VLBS + DUL/4, (3.2)

Wherein, DUL = VUBS – VLBS. (3.3)

Similarly, applying (2.1) to (2.3) to the aforesaid bounds, the second iteration of the proposed algorithm yields:

VUBSS {Variance Upper Bound by Skrepnek and Sahai} [2] ≡ VUBSS [1] – DUL [1]/4, (3.4)

VLBLS {Variance Lower Bound by Skrepnek and Sahai} [2] ≡ VLBSS [1] + DUL [1]/4, (3.5)

Wherein, DUL [1] = VUBSS [1] – VLBSS [1]. (3.6)

Applying (2.1) to (2.3) to the bounds to the 'I + 1st.' {I = 1 (1)…} iteration of the proposed algorithm, we have:

VUBSS {Variance Upper Bound by Skrepnek and Sahai} [I+1] ≡ VUBSS [I] – DUL [I]/4, (3.7)

VLBSS {Variance Lower Bound by Skrepnek and Sahai} [I+1] ≡ VLBSS [I] + DUL [I]/4, (3.8)

Wherein, DUL [I] = VUBSS [I] – VLBSS [I]. (3.9)

The above details complete a comprehensible definition of our proposed iterative algorithm for the derivation of sharper upper and lower bounds on the variance of the MVUE of p in inverse sampling.

Empirical Study Illustrating the Application of the Proposed Iterative Algorithm

To illustrate the application of the proposed iterative algorithm for sharpening the upper and lower bounds, the respective bounds VUBS and VLBS of Sahai (1985) and Sathe (1977) are employed to generate the iteratively sharper VUBSS [I] and VLBSS [I], as described in the preceding sections for I = 1, 2, 3, 4, and 5. Three values of r had been considered: 3, 6, and 10, combined with the four illustrative values of p as 0.1, 0.2, 0.3, and 0.4. The resultant values, calculated using MAPLE 13 (Maplesoft, Waterloo, Ontario), appear in the Appendix as Table A.I, Table A.II, and Table A.III.

Conclusion

Results of the current investigation illustrate the gainfulness of the proposed iterative algorithm sharpening upper and lower bounds. The values VUBSS [I] and VLBSS [I] represent the respectively magnified values of the sharpened upper and lower bounds by a factor of 10^2, achieved at the I^{th} iteration of this proposed Skrepnek-Sahai Sharpening Algorithm. The values (i.e., 10^2 times) of VUBSS [5] and VLBSS [5], at the fifth iteration of the algorithm are quite close, as presented.

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Appendix

		TABLE A.I		
Number of Bounds	p = 0.1	p = 0.2	p = 0.3	p = 0.4
VUBS	0.6714271140	2.000000000	3.496902535	4.913738118
VUBSS[1]	0.6535703355	1.959687576	3.434956844	4.819243032
VUBSS[2]	0.6446419462	1.939531364	3.403983999	4.771995490
VUBSS[3]	0.6401777516	1.929453258	3.388497576	4.748371718
VUBSS[4]	0.6379456543	1.924414205	3.380754365	4.736559832
VUBSS[5]	0.6368296056	1.921894679	3.376882759	4.730653890
r = 3				
VLBSS[5]	0.6345975084	1.916855626	3.369139548	4.718842004
VLBSS[5]	0.6334814597	1.914336100	3.365267942	4.712936062
VLBSS[5]	0.6312493624	1.909297047	3.357552473	4.701124176
VLBSS[5]	0.6267851678	1.899218941	3.342038308	4.677500404
VLBSS[5]	0.6178567785	1.879062729	3.311065463	4.630252862
VLBS	0.6000000000	1.838750305	3.249119772	4.535757776
		TABLE A.II		
VUBS	0.2120998333	0.7152988750	1.342809000	1.960136500
VUBSS[1]	0.2119703435	0.7147750996	1.341866104	1.958767936
VUBSS[2]	0.2119055986	0.7145132120	1.341394656	1.958083654
VUBSS[3]	0.2118732261	0.7143822682	1.341158932	1.957741513
VUBSS[4]	0.2118570399	0.7143167962	1.341041070	1.957570443
VUBSS[5]	0.2118489468	0.7142184060	1.340982140	1.957484908
r = 6				
VLBSS[5]	0.2118327605	0.7142185884	1.340864278	1.957313837
VLBSS[5]	0.2118246674	0.7141858524	1.340805348	1.957228302
VLBSS[5]	0.2118084812	0.7141203804	1.340687486	1.957057232
VLBSS[5]	0.2117761087	0.7139894366	1.340451762	1.956715091
VLBSS[5]	0.2117113638	0.7137275490	1.339980314	1.956030809
VLBS	0.2115818740	0.7132037736	1.339037418	1.954662245
		TABLE A.III		
VUBS	0.1094607222	0.3792113750	0.7283755715	1.084178833
VUBSS[1]	0.1094525790	0.3791742134	0.7283034409	1.084071836
VUBSS[2]	0.1094485074	0.3791556326	0.7282673756	1.084018338
VUBSS[3]	0.1094464716	0.3791463422	0.7282493430	1.083991588
VUBSS[4]	0.1094454537	0.3791416970	0.7282403267	1.083978214
VUBSS[5]	0.1094449447	0.3791393744	0.7282358185	1.083971526
r = 10				
VLBSS[5]	0.1094439268	0.3791347294	0.7282268022	1.083958152
VLBSS[5]	0.1094434178	0.3791324068	0.7282222940	1.083951464
VLBSS[5]	0.1094423999	0.3791277616	0.7282132777	1.083938090
VLBSS[5]	0.1094403641	0.3791184712	0.7281952451	1.083911340
VLBSS[5]	0.1094362925	0.3790998904	0.7281591798	1.083857842
VLBS	0.1094281493	0.3790627288	0.7280870492	1.083750845

Abbreviations: VUBS - Variance Upper Bound by Sahai (1985); VLBS - Variance Lower Bound by Sathe (1977); VUBSS - Variance Upper Bound by Skrepnek and Sahai; VLBSS - Variance Lower Bound by Skrepnek and Sahai.