# Bi-Edge - graceful and global edge - graceful labeling of some graphs 

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## Introduction

Graphs have many applications in various areas of Computer science. Parallel computers can be modeled by a graph, in which the vertices of the graph represent the processors and the edges represent the communications among processors.

The main objective is to derive a graph from the set of objects and to order(ranking) the nodes according to their relations with others in the graph. The basic idea of ranking approach is to share the total processes among p nodes and to assign the tasks according to their modulo ranking by finding the strength of the node as the sum of the weights of the edges with which it is incident.

The mathematical idea for performing this task in the network topology is to check whether the topology is edgegraceful or not. Here, the weights of the edges represent the labels of the edges.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices or edges then the labeling is called vertex or edge labeling. Graph labelings were first introduced in the late 1960's. In the recent years, dozens of graph labeling techniques have been studied in over 1000 papers.

Labeled graphs serve as useful models for a broad range of applications such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, and models for constraint programming over finite domains [1, 2].

By a $(p, q)$ graph $G$, we mean a graph $G=(V, E)$ with $|V|$ $=p$ and $|E|=q$. Most graph labeling methods trace their origin from the definition introduced by Rosa in 1967. Golomb [8] subsequently called such labeling as graceful.

In 1985, Lo [14] introduced the edge - graceful graph which is a dual notion of graceful labeling.

A $(\mathrm{p}, \mathrm{q})$ graph G is said to have an edge-graceful labeling if there exists an injection f from the edge set to $\{1,2,3, \ldots, \mathrm{q}\}$ so

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that the induced mapping $\mathrm{f}^{+}$defined on the vertex set defined by $\mathrm{f}^{+}(\mathrm{x})=\Sigma\{\mathrm{f}(\mathrm{xy})$ : xy is an edge $\}(\bmod \mathrm{p})$ are distinct.

A graph $G$ is said to be edge - graceful if it admits an edge graceful labeling. Lo[14], proved that G is edge - graceful then $q(q+1) \equiv \frac{(p+1) p}{2}(\bmod p)$.

In this paper, we introduce the bi-edge graceful and global edge-graceful labeling of graphs which are extension of edgegraceful labeling. Here we investigated these types of graphs and we have completely characterized the global edge-graceful graphs.

For global edge - gracefulness of a $(p, q)$ graph $G$, we assume that both $G$ and $G^{c}$ are connected. In other words, $q$ takes the value between $p-1$ and $\frac{(p-1)(p-2)}{2}$. Throughout this paper, we assume that $G$ is a connected simple undirected graph.

## Bi-edge-graceful

## Definition

A graph $G$ is said to be bi-edge - graceful if both $G$ and its line graph $L(G)$ are edge - graceful.

## Theorem

The path $P_{n}$ is not bi-edge - graceful.
Proof
Let $P_{n}$ be the path of $n$ vertices and $n-1$ edges.
Case (1) : $n$ is odd
$\mathrm{L}\left(\mathrm{P}_{n}\right)=\mathrm{P}_{n-1}$ which is an even order tree and is not edge graceful by the necessary conditions of Lo and hence $p_{n}$ is not bi-edge - graceful.
Case (2): $n$ is even
In this case, $\mathrm{P}_{n}$ is of even order and again by the same arguments it is not bi-edge - graceful.

## Theorem

The graph $C_{2 n+1}$ is bi-edge - graceful.
Proof
Let $C_{2 n+1}$ be the cycle with $2 n+1$ vertices and $2 n+1$ edges.

Case (1): $C_{2 n+1}$ is edge - graceful.
Let $v_{1}, v_{2}, v_{3}, \ldots, v_{2 n+1}$ be the vertices of $C_{2 n+1}$ and the edges $e_{i}$ are defined as follows (see fig.2.1).


Fig. 2.1 $\mathbf{C}_{\mathbf{2} \mathbf{n + 1}}$ with ordinary labeling
$e_{i}=\left(v_{i}, v_{i+1}\right)$ for $i=1 \leq i \leq 2 n$
$e_{2 n+1}=\left(v_{2 n+1}, v_{1}\right)$
First we label the edges of $C_{2 n+1}$
$f\left(e_{i}\right)=i$ for $1 \leq i \leq 2 n+1$
Then the induced vertex labels are
$f^{+}\left(v_{i}\right)=2 i-1 \quad$ for $\quad 1 \leq i \leq n$
$f^{+}\left(v_{i}\right)=2(i-n-1)$ for $n+1 \leq 2 n+1$
These labels are arranged in order. Let
$A=\left\{f^{+}\left(v_{i}\right) / 1 \leq i \leq n\right\}=\{1,3, \ldots, 2 n-1\}$
$B=\left\{f^{+}\left(v_{i}\right) / n+1 \leq i \leq 2 n+1\right\}=\{0,2, \ldots, 2 n\}$
$\therefore$
$f^{+}(V)=A \cup B=\{1,3, \ldots, 2 n-1,0,2, \ldots, 2 n\} \subseteq\{0,1,2, \ldots, p-1\}$
where $p=2 n+1$. Hence $C_{2 n+1}$ is edge - graceful.
Case (2): $L\left(C_{2 n+1}\right)$ is edge - graceful.
$L\left(C_{2 n+1}\right)=C_{2 n+1}$ which contains $2 n+1$ vertices and $2 n+1$ edges. Therefore by case (1) the graph $L\left(C_{2 n+1}\right)$ is edge graceful. Hence the cycle $C_{2 n+1}$ is bi-edge - graceful. The biedge - graceful labeling of $\mathrm{C}_{7}$ is given in fig. 2.2.


Fig. 2.2.Biedge - graceful labeling of $\mathbf{C}_{7}$

## Theorem

The graph $\mathrm{C}_{2 n}$ is not bi-edge - graceful.

## Proof

Here the cycle $\mathrm{C}_{2 n}$ contains $2 n$ vertices and $2 n$ edges. Clearly $p=2 n$ and $q=2 n$.Therefore, it does not satisfy the Lo's necessary condition. Hence $C_{2 n}$ is not edge - graceful. Thus $C_{2 n}$ is not bi-edge - graceful.

## Theorem

The graph $K_{1, n}$ is bi-edge - graceful if $n=4 t, t \geq 1$.
Proof
First we present the proof of $\mathrm{K}_{1, n}$ is edge - graceful.
Case (1): $K_{l, n}$ is edge - graceful if $n=4 t, t \geq 1$.
Let $\mathrm{v}, v_{1}, v_{2}, v_{3}, \ldots, v_{n}$ be the vertices of $K_{l, n}$ and the edges $e_{i}$ are defined as follows( see fig.2.3):
$e_{i}=\left(v, v_{i}\right)$ for $1 \leq i \leq n$


Fig. 2.3. $\mathrm{K}_{1, \mathrm{n}}$ with ordinary labeling
We now label the edges as follows:
$f\left(e_{\mathrm{i}}\right)=\mathrm{i}$ for $1 \leq i \leq n$
Then the induced vertex labels are
$f^{+}(v)=0$
$f^{+}\left(v_{i}\right)=i$ for $1 \leq i \leq n$
These labels are arranged in order. Let
$A=\left\{f^{+}(v)\right\}=\{0\}$
$B=\left\{f^{+}\left(v_{i}\right) \mid l \leq i \leq n\right\}=\{1,2,3, \ldots n\}$
$\therefore f^{+}(V)=A \cup B \subseteq\{0,1,2,3, \ldots, n\}$
Hence $K_{l, n}$ is edge - graceful if $n=4 t, t \geq 1$.
Case (2): $L\left(K_{1, n}\right)$ is edge - graceful if $n=4 t, t \geq 1$.
Let $v_{1}, v_{2}, v_{3}, \ldots, v_{n}$ be the vertices of $L\left(K_{l, n}\right)$ and the edges are (See fig. 2.4)
$e_{i(i+1)}=\left(v_{i} v_{i+1}\right)$ for $i=1,2,3, \ldots, n-1$
$e_{i(i+2)}=\left(v_{i} v_{i+2}\right)$ for $i=1,2,3, \ldots, n-2$
$e_{i(i+3)}=\left(v_{i} v_{i+3}\right)$ for $i=1,2,3, \ldots, n-3$
$e_{1 n}=\left(v_{1} v_{n}\right)$


Fig. 2.4. $L\left(k_{1, n}\right)$ with ordinary labeling
We now label the edges as follows:
For $\quad 1 \leq i \leq n-1, \quad i+1 \leq j \leq n$

$$
f\left(e_{i j}\right)=i+\frac{(j-i-1)(2 n-j+i)}{2}
$$

Then the induced vertex labels are
$f^{+}\left(v_{i}\right)=n-i \quad$ for $\quad 1 \leq i \leq n-1 \quad$ iodd
$f^{+}\left(v_{i}\right)=\frac{n}{2}-i \quad$ for $\quad 2 \leq i \leq \frac{n}{2} \quad$ ieven
$f^{+}\left(v_{i}\right)=\frac{3 n}{2}-i \quad$ for $\quad \frac{n}{2}+2 \leq i \leq n \quad$ ieven
These labels are distinct. Hence $\mathrm{L}\left(K_{1, n}\right)$ is edge - graceful if $n=$ $4 t, t \geq 1$.
By case (1) and case (2) the graph $K_{l, n}$ is biedge - graceful for $n=4 t, t \geq 1$. The edge - graceful labeling of $\mathrm{L}\left(K_{1,8}\right)$ is given in fig.2.5.


Fig. 2.5. Edge - graceful labeling of $\mathbf{L}\left(\mathbf{K}_{1,8}\right)$
Global edge-graceful

### 3.1 Definition

A graph $G$ is called global edge - graceful if both G and its complement $G^{c}$ are edge - graceful.
3.2. Theorem

The graph $C_{n}(n \geq 3)$ is global edge - graceful if $n$ is prime.
Proof
Case 1: $C_{n}$ is edge - graceful.
Let $\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots, \mathrm{v}_{\mathrm{n}}$ be the vertices of $C_{n}$ and the edges $e_{i}$ are defined as follows
$\mathrm{e}_{\mathrm{i}}=\left(\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i}+1}\right) \quad$ for $1 \leq \mathrm{i} \leq \mathrm{n}-1$
$\mathrm{e}_{\mathrm{n}}=\left(\mathrm{v}_{\mathrm{n}}, \mathrm{v}_{1}\right) \quad$ (see fig. 3.1)
By Theorem 2.3, $C_{n}$ is edge - graceful


Fig. 3.1 $C_{n}$ with ordinary labeling
Case 2: $C_{n}^{c}$ is edge - graceful.
Let the vertices are defined as in case 1 and the edges are

$$
e_{1 j}=\left(v_{l}, v_{j}\right), \quad i+2 \leq j \leq n-1
$$

For $2 \leq \mathrm{i} \leq \mathrm{n}-2$
$e_{i j}=\left(v_{i}, v_{j}\right), i+2 \leq j \leq n$.


Fig.3.2 $C_{n}^{c}$ with ordinary labeling
We now label the edges as follows (see fig. 3.2)
For $i=1 \& i+2 \leq j \leq n-1$
$f\left(e_{1 j}\right)=1+\frac{(j-3)(2 n-j)}{2}$
For $2 \leq i \leq n-2, i+2 \leq j \leq n$
$f\left(e_{i j}\right)=i+\frac{(j-i-2)(2 n-j+i-1)}{2}$
Then the induced vertex labels are
For $\mathrm{n}=5,11,17,23,29, \ldots$
for $\quad 1 \leq i \leq \frac{n-2}{3}, \quad f^{+}\left(v_{i}\right)=n-2-3 i$
for $\quad \frac{n+1}{3} \leq i \leq \frac{2 n-4}{3}, \quad f^{+}\left(v_{i}\right)=2 n-2-3 i$
for $\quad \frac{2 n-1}{3} \leq i \leq n-1, \quad f^{+}\left(v_{i}\right)=3 n-2-3 i$
for $\quad i=n, \quad f^{+}\left(v_{i}\right)=n-2$
For $\mathrm{n}=7,13,19,31, \ldots$
for $1 \leq i \leq \frac{n-4}{3}, \quad f^{+}\left(v_{i}\right)=n-2-3 i$
for $\quad \frac{n-1}{3} \leq i \leq \frac{2 n-2}{3}, \quad f^{+}\left(v_{i}\right)=2 n-2-3 i$
for $\quad \frac{2 n+1}{3} \leq i \leq n-1, \quad f^{+}\left(v_{i}\right)=3 n-2-3 i$
for $\quad i=n, \quad f^{+}\left(v_{i}\right)=n-2$
These labels are distinct. Hence the graph $C_{n}^{C}$ is edge graceful. Then by case 1 and case $2 C_{n}$ is global edge - graceful graph.
The edge - graceful labeling of $C_{11}^{C}$ is given in fig. 3.3.


Fig.3.3 Edge - graceful labeling of $C_{11}^{c}$

## Theorem

If a $(p, q)$ graph G is global edge - graceful of odd order then $\quad \mathrm{q} \equiv 0(\bmod \mathrm{p})$
Proof
If G is a $(p, q)$ graph then $G^{c}$ is a $\left(p, q_{1}=\frac{p(p-1)}{2}-q\right)$
graph. Let $G$ be a global edge - graceful of odd order. Applying Lo's condition we have

$$
\begin{equation*}
q(q+1) \equiv \frac{p(p+1)}{2}(\bmod p) \tag{1}
\end{equation*}
$$

Applying Lo's condition for $G^{c}\left(p, q_{1}\right)$ we have

$$
q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)
$$

From equation (1) \& (2) we have, $q(q+1) \equiv q_{1}\left(q_{1}+1\right)(\bmod p)$
$\Rightarrow q(q+1) \equiv\left[\frac{p(p-1)}{2}-q\right]\left[\frac{p(p-1)}{2}-q+1\right](\bmod p)$
Since $p$ is odd, equation (3) reduces to

$$
q(q+1) \equiv q^{2}-q(\bmod p) \Rightarrow 2 q \equiv 0(\bmod p)
$$

which implies $q \equiv 0(\bmod p)$ since $p$ is odd.

## Theorem

The odd order global edge - graceful graphs are precisely ( $p, l p$ ) graphs, where $l \leq l \leq \frac{p-1}{2}$.

## Proof

If a $(p, q)$ graph $G$ is odd order global edge - graceful for some $\quad q \neq l p, l \leq l \leq \frac{p-1}{2}$. Then one can easily verify that $q \equiv$ $i(\bmod p)$ for some $i, l \leq i \leq p-1$. This implies that $q \neq 0(\bmod$ $p$ ), a contradiction to the statement of Theorem 3.3.

## Theorem

If a $(p, q)$ graph $G$ is an odd order tree then $G$ is not a global edge - graceful graph.

## Proof

Suppose $G$ is a global edge - graceful tree then by Theorem 3.3, $q \equiv 0(\bmod p)$, a contradiction to $q=p-1$.

## Theorem

If a $(p, q)$ graph $G$ with $p=4 t-2, t \geq 1$ is global edge graceful then $q \equiv 0(\bmod p)$ or $q \equiv \frac{p}{2}(\bmod p)$.

## Proof

By the proof of Theorem 3.3, from equation (3) we get
$q(q+1) \equiv \frac{p^{2}(p-1)^{2}}{4}-q p(p-1)+\frac{p(p-1)}{2}+q^{2}-q(\bmod p) \ldots$
We have,
$q(q+1) \equiv \frac{p^{2}(p-1)^{2}}{4}+\frac{p(p-1)}{2}+q^{2}-q(\bmod p)$
We compute
$\frac{p^{2}(p-1)^{2}}{4}+\frac{p(p-1)}{2}$ with $p=4 t-2$
Then $\frac{p^{2}(p-1)^{2}}{4}+\frac{p(p-1)}{2}=\frac{p(p-1)}{4}[p(p-1)+2]$

$$
=\frac{(4 t-2)(4 t-3)}{4}[(4 t-2)(4 t-3)+2]
$$

$=\frac{(4 t-2)(4 t-3)}{4}\left[16 t^{2}-20 t+8\right]$
$=(4 t-2)(4 t-3)\left(4 t^{2}-5 t+2\right)$
$\equiv 0(\bmod p)$
From (5) and (6)

$$
\begin{equation*}
q(q+1) \equiv q^{2}-q(\bmod p) \tag{6}
\end{equation*}
$$

which implies $2 q \equiv 0(\bmod p)$.
Therefore, $2 q=x p$ for some x .
If $x$ is even then $q \equiv 0(\bmod p)$.
If $x$ is odd then
$q-\frac{p}{2}=\frac{x p}{2}-\frac{p}{2}=\frac{(x-1) p}{2} \equiv 0(\bmod p)$.
$\therefore q \equiv \frac{p}{2}(\bmod p)$.

## Remark

$q \equiv 0(\bmod p)$ or $q \equiv \frac{p}{2}(\bmod p)$ is not a sufficient condition for a $(p, q)$ graph $G$ with $p=4 t-2$ to be global edge graceful can be seen by the examples presented below.

## Example 1

Consider a $(6,9)$ graph. Clearly, $q \equiv \frac{p}{2}(\bmod p)$. But $q(q$ $+1)=90 \equiv 0(\bmod 6)$ and $\frac{p(p+1)}{2}=21 \equiv 3(\bmod 6)$. So, by Lo's necessary condition, $(6,9)$ graph is not edge - graceful. Hence it is not a global edge - graceful graph.

## Example 2

Consider a $(6,12)$ graph. Clearly, $q \equiv 0(\bmod p)$. But $q(q+$ 1) $=156 \equiv 0(\bmod 6)$ and $\frac{p(p+1)}{2}=21 \equiv 3(\bmod 6)$. So, by Lo's necessary condition, $(6,12)$ graph is not edge - graceful. Hence it is not a global edge - graceful graph.

## Theorem

Any $(p, q)$ graph $G$ where $p$ is of the form $4 t-2, t \geq 1$ is not global edge - graceful.

## Proof

Suppose G (p, q) with $\mathrm{p}=4 \mathrm{t}-2, t \geq 1$ is global edge graceful then by Theorem 3.6, we have $q \equiv 0(\bmod p)$ or $\frac{p}{2}(\bmod p)$.
Case 1: $q \equiv 0(\bmod p)$
Then, $q=s p$ for some $s$.
So, $q(q+1)=s p(s p+1) \equiv 0(\bmod p)$
and $\frac{p(p+1)}{2}=\frac{(4 t-2)(4 t-1)}{2}=(2 t-1)(4 t-1)$
$=8 \mathrm{t}^{2}-6 \mathrm{t}+1=(4 \mathrm{t}-2) 2 \mathrm{t}+(1-2 \mathrm{t}) \equiv-\frac{p}{2}(\bmod p)$.
Therefore, $q(q+1) \neq \frac{p(p+1)}{2}(\bmod p)$. So, by Lo's condition $G$ is not edge - graceful. Hence, $G$ is not global-edge graceful.
Case 2: $\mathrm{q} \equiv \frac{p}{2}(\bmod p)$
Then, $q=s p+\frac{p}{2}$ for some $s$.
So, $q(q+1)=\left(s p+\frac{p}{2}\right)\left(s p+\frac{p}{2}+1\right)$
$=\frac{p(2 s+1)}{2}\left(s p+\frac{p}{2}+1\right)$.
Here, $s p+\frac{p}{2}+1$ is even and so $q(q+1) \equiv 0(\bmod p)$. As in case $1, q(q+1) \neq \frac{p(p+1)}{2}(\bmod p)$. So by Lo's condition, $G$ is not edge - graceful. Hence, $G$ is not global edge - graceful.

We now examine the global edge - gracefulness of a $(p, q)$ graph where $p$ is of the form $4 t, t \geq 1$. We split this into 4 cases namely $p \equiv 4(\bmod 16), p \equiv 12(\bmod 16), p \equiv 0(\bmod 16)$ and $p \equiv$ $8(\bmod 16)$. Fortunately, $p \equiv 0$ or $8(\bmod 16)$ can be considered as $p \equiv 0(\bmod 8)$ and hence the number of cases boils down to 3 .

Further we observe that in all the above cases, the global edge - gracefulness depends on the following condition.

$$
q(q+1) \equiv q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)
$$

On these facts, we have some observations.

## Observation

If $p \equiv 4(\bmod 16)$ then $\mathrm{q} \equiv \frac{p}{4}(\bmod p)$ if and only if

$$
q(q+1) \equiv q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)
$$

Proof
Assume $\quad \mathrm{q} \equiv \frac{p}{4} \quad(\bmod \quad p)$. Then we have, $q+1 \equiv \frac{p}{4}+1(\bmod p)$.

Now, $q(q+1)-\frac{p}{2}=\frac{p}{4}\left(\frac{p}{4}+1\right)-\frac{p}{2}=\frac{p^{2}}{16}-\frac{p}{4}=\frac{p(p-4)}{16} \equiv 0(\bmod p)$, as $p \equiv 4(\bmod 16)$.
Thus, $q(q+1)=\frac{p}{2}(\bmod p) \equiv \frac{p(p+1)}{2}(\bmod p)$.
Again, $q_{1}=\frac{p(p-1)}{2}-q \equiv \frac{p}{2}-\frac{p}{4}(\bmod p) \equiv \frac{p}{4}(\bmod p)$
As proved above, we can see that $q_{1}\left(q_{1}+1\right) \equiv \frac{p}{2}(\bmod p)$.
Thus we have, $q(q+1) \equiv q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)$.
Conversely, Assume that $q(q+1) \equiv q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)$

$$
\begin{aligned}
& q(q+1) \equiv \frac{p}{2}(\bmod p) \Rightarrow q(q+1)=l p+\frac{p}{2} \\
\text { or } q= & \frac{-1 \pm \sqrt{1+4\left(l p+\frac{p}{2}\right)}}{2}
\end{aligned}
$$

Since q is an integer, choose 1 such that $1+4\left(l p+\frac{p}{2}\right)=s^{2}$.
As $\mathrm{q}>0, \mathrm{q}=\frac{s-1}{2}$.
Similarly, $q_{1}\left(q_{1}+1\right) \equiv \frac{p}{2}(\bmod p)$ implies $q_{1}=\frac{s-1}{2}$. So, $q$ $=q_{1}$.
Again, $q_{1}=\frac{p(p-1)}{2}-q \Rightarrow 2 q=\frac{p(p-1)}{2} \equiv \frac{p}{2}(\bmod p)$.
Therefore, $\mathrm{q} \equiv \frac{p}{4}(\bmod p)$.

## Observation

If $p \equiv 12(\bmod 16)$ then $q \equiv \frac{3 p}{4}(\bmod p)$ if and only if

$$
q(q+1) \equiv q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)
$$

## Proof

Assume $\mathrm{q} \equiv \frac{3 p}{4} \quad(\bmod p) . \quad$ Then we have, $q+1 \equiv \frac{3 p}{4}+1(\bmod p)$.
Now,
$q(q+1)-\frac{p}{2}=\frac{3 p}{4}\left(\frac{3 p}{4}+1\right)-\frac{p}{2}=\frac{p(9 p+4)}{16} \equiv 0(\bmod p)$.
$p \equiv 12(\bmod 16) \Rightarrow 9 p+4 \equiv 0(\bmod 16)$.
So, $q(q+1)=\frac{p}{2}(\bmod p) \equiv \frac{p(p+1)}{2}(\bmod p)$.
Again, $q_{1}=\frac{p(p-1)}{2}-q \equiv \frac{p(p-1)}{2}-\frac{3 p}{4}(\bmod p)$

$$
\equiv \frac{2 p^{2}-5 p}{4}(\bmod p) \equiv \frac{p(2 p-5)}{4}(\bmod p)
$$

Now,
$q_{1}-\frac{3 p}{4}=\frac{p(2 p-5)}{4}-\frac{3 p}{4}=\frac{p(2 p-8)}{4} \equiv 0(\bmod p)$ as $p$
$\equiv 12(\bmod 16)$.
Therefore, $\mathrm{q}_{1}=\frac{3 p}{4}(\bmod p)$.

By the arguments discussed for q , one can prove that $q_{1}\left(q_{1}+1\right) \equiv \frac{p}{2}(\bmod p)$.
By following the proof of observation 3.9, one can prove the converse by similar lines.

## Theorem

If a $(p, q)$ graph $G$ is global edge - graceful where $p \equiv$ $4(\bmod 16)$ then $q \equiv \frac{p}{4}(\bmod p)$.

## Proof

Suppose G is global edge - graceful. Then by Lo's condition we have, $q(q+1) \equiv q_{1}\left(q_{1}+1\right) \equiv \frac{p(p+1)}{2}(\bmod p)$. Now by observation $3.9, q \equiv \frac{p}{4}(\bmod p)$.

## Theorem

If a $(p, q)$ graph $G$ is global edge - graceful where $p \equiv$ $12(\bmod 16)$ then $q \equiv \frac{3 p}{4}(\bmod p)$.

## Proof

Follows from Lo's condition and observation 3.10.

## Theorem

If $p \equiv 4(\bmod 16)$ then the global edge - graceful graphs are precisely $\left(p, \frac{p}{4}+l p\right)$ graphs where $l \leq l \leq \frac{p-4}{2}$.

## Proof

Proof follows from Theorem 3.11 and by following the arguments of Theorem 3.4.

## Theorem

If $p \equiv 12(\bmod 16)$ then the global edge - graceful are precisely $\left(p, \frac{3 p}{4}+l p\right)$ graphs where $l \leq l \leq \frac{p-6}{2}$.

## Proof

Proof follows from Theorem 3.12 and by following the arguments of Theorem 3.4.

## Theorem

Any $(p, q)$ graph $G$ with $p \equiv 0(\bmod 8)$ is not global edge graceful

## Proof

Case 1: $q \equiv \frac{p}{2}(\bmod p)$
Then, $q(q+1) \equiv \frac{p}{2}\left(\frac{p}{2}+1\right)(\bmod p) \equiv \frac{p}{2}(\bmod p)$.
But, $q_{1}=\frac{P(p-1)}{2}-q \equiv \frac{p(p-1)}{2}-\frac{p}{2}(\bmod p)$

$$
\equiv \frac{p(p-2)}{2}(\bmod p) \equiv 0(\bmod p)
$$

$q_{1}\left(q_{1}+1\right) \equiv 0(\bmod p) \neq \frac{p(p+1)}{2}(\bmod p)$.
Therefore, $G^{c}$ is not edge - graceful. Thus, $G$ is not global edge - graceful.
Case 2: $q \equiv \frac{p}{2}-1(\bmod p)$
Then, $q(q+1) \equiv\left(\frac{p}{2}-1\right)\left(\frac{p}{2}\right)(\bmod p) \equiv\left(\frac{p}{2}\right)(\bmod p)($ Since $p \equiv$ $0(\bmod 8))$.

But, $q_{1}=\frac{p(p-1)}{2}-q=\frac{p(p-1)}{2}-\frac{p}{2}+1=\frac{p(p-2)}{2}+1 \equiv 1(\bmod p)$.
Now, $q_{1}\left(q_{1}+1\right) \equiv 2(\bmod p) \not \equiv \frac{p(p+1)}{2}(\bmod p)$.
Therefore, $G^{c}$ is not edge - graceful. Thus, $G$ is not global edge - graceful.
Case 3: Neither $q \equiv \frac{p}{2}(\bmod p){ }^{\text {nor }} q \equiv \frac{p}{2}-1(\bmod p)$.
In this case, one can prove that $q(q+1) \not \equiv \frac{p(p+1)}{2}(\bmod p)$ which shows that $G$ is not edge - graceful. Thus, $G$ is not global edge - graceful.

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