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Bifurcation analysis in a Lotka-Volterra model with delay

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ABSTRACT

In this paper, a Lotka-Volterra model with time delay is considered. The stability of the equilibrium of the model is investigated and the existence of Hopf bifurcation is proved. Numerical simulations are performed to justify the theoretical results. Finally, main conclusions are included.

Introduction

The Lotka-Volterra model is very important in population modelling and has been studied by many authors (see, [2,4,7-9]). As is known to us, the past history has an important effect on the stability of system. It is generally recognized that some kinds of time delays are inevitable in population interactions and tend to be destablizing in the sense that longer delays may destroy the stability of positive equilibrium[1].

Considering that generally consumption of prey by the predator throughout its past history governs the present birth rate of the predator, we think that time delay due to the gestation is a common example. In this paper, we consider the following Lotka-Volterra model with time delays.

$$\begin{cases} \dot{N}(t) = N(t)[R-\alpha N(t-\tau)-M(t-\tau)], \\ \dot{M}(t) = M(t)[-\gamma+N(t-\tau)], \end{cases} \quad (1)$$

where $N(t)$ and $M(t)$ denote the density(per suquare unit of the habitat) of prey and predator population, respectively; R,α,γ are positive constants.

It is well known that delay may have very complicated impact on the dynamics of a system.

To obtain a deep and clear understanding of dynamics of Lotka-Volterra model with delay, in this paper, we will study the Hopf bifurcation of model (1). Choosing the delay τ as the bifurcation parameter, we shall investigated the effect of the delay τ on the dynamics of system (1).

The remainder of the paper is organized as follows. In Section 2, we discuss the stability of the equilibrium and the existence of Hopf bifurcations occurring at the equilibrium.

In Section 3, numerical simulations are carried out to validate our main results. Some main conclusions are drawn in Section 4.

Stability of the equilibrium and local Hopf bifurcations

It is easy to see that if $R>\alpha\gamma$, then system (1) has a unique positive equilibrium $E_0(N^*,M^*)=(\gamma,R-\alpha\gamma)$.

Let $x(t)=N(t)-N^*, y(t)=M(t)-M^*$, Then the linearization of Eq. (1) at $E_0(N^*,M^*)=(\gamma,R-\alpha\gamma)$ takes the form

$$\begin{cases} \dot{x}(t) = -\alpha N^* x(t-\tau) - N^* y(t-\tau), \\ \dot{y}(t) = M^* y(t-\tau) \end{cases} \quad (2)$$

which the characteristic equation gives

$$\lambda^2+\alpha N^*\lambda e^{-\lambda\tau}+M^*N^*e^{-2\lambda\tau}=0, \quad (3)$$

i.e.,

$$\lambda^2 e^{\lambda\tau}+\alpha N^*\lambda+M^*N^*e^{-\lambda\tau}=0. \quad (4)$$

The stability of the positive equilibrium of system (1) depends on the locations of the roots of the characteristic equation (4), when all roots of Eq.(4) locate in the left half of complex plane, the trivial solution is stable, otherwise, it is instable.

In the following, we will investigate the distribution of roots of (4).

For $\tau=0$, (4) becomes

$$\lambda^2+\alpha N^*\lambda+M^*N^*=0. \quad (5)$$

It is easy to see that all roots of (5) have a negative real part.

For $\omega>0$, $i\omega$ is a root of (4), then

$$-\omega^2 e^{i\omega\tau}+\alpha N^*\omega i+M^*N^*e^{-i\omega\tau}=0.$$

Then

$-\omega^2(\cos\omega\tau+i\sin\omega\tau)+\alpha N^*\omega i+M^*N^*(\cos\omega\tau-i\sin\omega\tau)=0.$ Separating the real and imaginary parts, we get

$$\begin{cases} (M^*N^*-\omega^2)\cos\omega\tau=0, \\ (M^*N^*+\omega^2)\sin\omega\tau=\alpha N^*\omega. \end{cases} \quad (6)$$

Case 1. If $\cos\omega\tau=0$, then

$$\omega\tau=j\pi+\frac{\pi}{2}, j=0,\pm1,\pm2,\mathrm{L} \quad (7)$$

Hence $\sin\omega\tau=1$. From the second equation of (6), we have

$$\omega^2-\alpha N^*\omega+M^*N^*=0,$$

Therefore

$$\omega_{\pm}=\frac{\alpha N^*\pm\sqrt{\alpha N^*-4M^*N^*}}{2}. \quad (8)$$

It follows from (7) that

$$\tau_j^{\pm}=\frac{1}{\omega_{\pm}}\left[j\pi+\frac{\pi}{2}\right], j=0,1,2,\mathrm{L} \cdot \quad (9)$$

Case 2. If $M^*N^*-\omega^2=0$, then $\omega=\sqrt{M^*N^*}$.

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From the second equation of (6), we have

$$\tau_k = \frac{1}{\omega}\left[\arcsin\frac{\alpha\omega}{2M^*} + 2k\pi\right], k = 0,1,2,\ldots \qquad (10)$$

It is easy to see that when $\tau = \tau_j^{\pm}$ (or τ_k), Eq.(4) has a pair of purely imaginary roots $\pm i\omega_{\pm}$ (or ω). Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq.(4) such that $\alpha(\tau_j^{\pm}) = 0, \omega(\tau_j^{\pm}) = \omega_{\pm}$ (or $\alpha(\tau_k) = 0,$ $\omega(\tau_k) = \omega$). Due to functional differential equation theory, for every $\tau_j^{\pm}, j = 0,1,2,\ldots$ ($\tau_k, k = 0,1,2,\ldots$), there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_j^{\pm}| < \varepsilon$ ($|\tau - \tau_k| < \varepsilon$). Substituting $\lambda(\tau)$ into the left hand side of (4) and taking derivative with respect to τ, we have

$$\left[(2\lambda + \tau)e^{\lambda\tau} - M^*N^*\tau e^{-\lambda\tau} + \alpha N^*\right]\frac{d\lambda}{d\tau} = -\lambda e^{\lambda\tau} + M^*N^*e^{-\lambda\tau}\lambda,$$

which leads to

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2\lambda e^{\lambda\tau} + \alpha N^*}{\lambda(M^*N^*e^{-\lambda\tau} - e^{\lambda\tau})} - \frac{\tau}{\lambda},$$

which, together with (6), leads to

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^{\pm}} = \mathrm{Re}\left\{\frac{2\lambda e^{\lambda\tau} + \alpha N^*}{\lambda(M^*N^*e^{-\lambda\tau} - e^{\lambda\tau})}\right\}_{\tau=\tau_j^{\pm}}$$

$$= \frac{\alpha N^* - 2\omega_{\pm}\sin\omega_{\pm}\tau_j^{\pm} + i2\omega_{\pm}\cos\omega_{\pm}\tau_j^{\pm}}{\omega_{\pm}(M^*N^*+1)\sin\omega_{\pm}\tau_j^{\pm} + i\omega_{\pm}(M^*N^*-1)\cos\omega_{\pm}\tau_j^{\pm}}$$

$$= \frac{\omega_{\pm}\sin\omega_{\pm}\tau}{\Delta_1}\left[\alpha N^*(M^*N^*+1) - 2\omega_{\pm}\sin\omega_{\pm}\tau\right],$$

where

$$\Delta_1 = [\omega_{\pm}(M^*N^*+1)\sin\omega_{\pm}\tau_j^{\pm}]^2 + [\omega_{\pm}(M^*N^*-1)\cos\omega_{\pm}\tau_j^{\pm}]^2.$$

Notice that $\sin\omega_{\pm}\tau_j^{\pm} = 1$, then we get

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^{\pm}} = \frac{\omega_{\pm}}{\Delta_1}\left[\alpha N^*(M^*N^*+1) - 2\omega_{\pm}\right].$$

Thus, if $\Delta_1 \neq 0$, we obtain

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j^{+}}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^{+}}\right\} > 0$$

Similarly,

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_k} = \mathrm{Re}\left\{\frac{2\lambda e^{\lambda\tau} + \alpha N^*}{\lambda(M^*N^*e^{-\lambda\tau} - e^{\lambda\tau})}\right\}_{\tau=\tau_k}$$

$$= \frac{\alpha N^* - 2\omega\sin\omega\tau_k + i2\omega\cos\omega\tau_k}{\omega(M^*N^*+1)\sin\omega\tau_k + i\omega(M^*N^*-1)\cos\omega\tau_k}$$

$$= \frac{\omega\sin\omega\tau_k}{\Delta_2}\left[\alpha N^*(M^*N^*+1) - 2\omega\sin\omega\tau_k\right]$$

where

$$\Delta_2 = [\omega(M^*N^*+1)\sin\omega\tau_k]^2 + [\omega(M^*N^*-1)\cos\omega\tau_k]^2.$$

Notice that $\sin\omega\tau_k = \frac{\alpha\omega}{2M^*}$, then we get

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^{\pm}} = \frac{\alpha\omega^2}{2\Delta_2 M}\left[\alpha N^*(M^*N^*+1) - \frac{\alpha\omega^2}{M^*}\right].$$

Thus, if $\Delta_2 \neq 0$, we obtain

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_k}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_k}\right\} > 0.$$

According to above analysis and the results of Ruan and Wei[6],Yang[5] and Hale[3], we have the following result.

Theorem 2.1. Let $\tau_j^{\pm}(j = 0,1,2,\ldots)$ and $\tau_k(k = 0,1,2,\ldots)$ be defined by (11) and ((12), respectively, and $\tau = \min\{\tau_0^{\pm}, \tau_0\}$. If $(\alpha N^*)^2 > 4M^*N^*$ and $\alpha M^*(N^*)^2 > \sqrt{(\alpha N^*)^2 - 4M^*N^*}$, hold, then the equilibrium $E_0(N^*, M^*)$ of system (1) is asymptotically stable for $\tau \in [0, \tau_0)$. System (1) undergoes a Hopf bifurcation at the equilibrium $E_0(N^*, M^*)$ when $\tau = \tau_j^{\pm}(j = 0,1,2,\ldots)$ and $\tau = \tau_k(k = 0,1,2,\ldots)$.

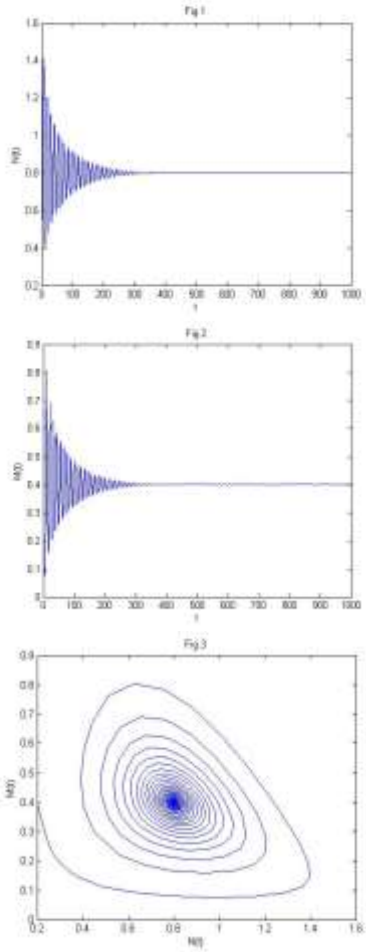
Numerical Examples

In this section, we will prove some numerical results of system
(1) to illustrate our results obtained in Section 2. We consider the system

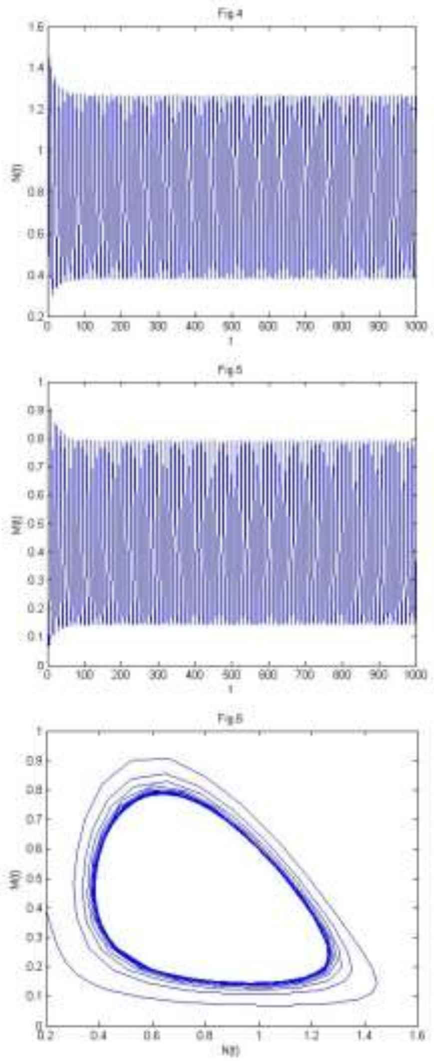
$$\begin{cases} \dot{N}(t) = N(t)[0.8 - 0.5N(t-\tau) - M(t-\tau)], \\ \dot{M}(t) = M(t)[-0.8 + N(t-\tau)], \end{cases} \qquad (11)$$

which has a positive equilibrium $E_0(0.8, 0.4)$ and satisfies the conditions indicated in Theorem 2.1. When $\tau = 0$, the positive equilibrium

$E_0(0.8, 0.4)$ is asymptotically stable. The positive equilibrium $E_0(0.8, 0.4)$ is stable when $\tau < \tau_0 \approx 0.625$ which is illustrated by the computer simulations (see Figs.1-3). When τ passes through the critical value τ_0, the positive equilibrium $E_0(0.8, 0.4)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the positive equilibrium $E_0(0.8, 0.4)$ (see Figs.4-6).



Figs.1-3 When $\tau = 0.6 < \tau_0 \approx 0.625$. The positive equilibrium $E_0(0.8, 0.4)$ of system (12) is asymptotically stable. The initial value is (0.2,0.4)



Figs.4-6 When $\tau = 0.7 > \tau_0 \approx 0.625$. Hopf bifurcation of system (12) occurs from the positive equilibrium $E_0(0.8, 0.4)$. The initial value is (0.2, 0.4)

Conclusions

In this paper, the local stability of the positive equilibrium $E_0(N^*, M^*)$ and local Hopf bifurcation in a Lotka-Volterra model with time delay are investigated. We found that if the conditions $(\alpha N^*)^2 > 4M^*N^*$ and $\alpha M^*(N^*)^2 > \sqrt{(\alpha N^*)^2 - 4M^*N^*}$ hold, the positive equilibrium $E_0(N^*, M^*)$ of system (2) is asymptotically stable for all $\tau \in [0, \tau_0)$. As the delay τ increases, the positive equilibrium $E_0(N^*, M^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(N^*, M^*)$, i.e., a family of periodic orbits bifurcate from the positive equilibrium $E_0(N^*, M^*)$. Some numerical simulations are performed to verify our theoretical results found.

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References

[1] Y.K. Li, M. Hu, "Stability and Hopf bifurcation analysis in a stage-structure predator-prey system with two time delays", International Journal of Computational and Mathematical Sciences, 5(3), 2011.

[2] W. Han, M.X. Liu, "Stability and bifurcation analysis for a discrete-time model of Lokta-Volterra type with delay," Applied Mathematics and Computation, 217(12), 5449-5457, 2011.

[3] J. Hale, "Theory of Functional Differential Equation", Springer-Verlag, 1977.

[4] Y. Muroya, "Persistence and global stability in discrete models of Lokta-Volterra type," Journal of Mathematical Analysis and Applications, 330(1), 24-33, 2007.

[5] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics. Academic Press, INC, 1993.

[6] S. G. Ruan and J. J. Wei, "On the zero of some transcendential functions with applications to stability of delay differential equations with two delays," Dynamics of Continuous, Discrete and Impulsive Systems, Series A, 10, 863-874, 2003.

[7] G.I. Bischi, F. Tramontana, "Three-dimensional discrete Lotka-Volterra models with an application to industrial clusters," Communications in Nonlinear Science and Numerical Simulation, 15(10), 3000-3014, 2010.

[8] X.H. Tang, D.M. Cao, X.F. Zou, "Global attractivity of positive periodic solution to periodic Lotka–Volterra competition systems with pure delay," Journal of Differential Equations, 228(2) 580-610, 2006.

[9] J.R. Yan, A. Zhao, J.J. Nieto, "Existence and global attractivity of positive periodic solution of periodic single-species impulsive Lotka-Volterra systems ," Mathematical and Computer Modelling, 40(5-6), 509-518, 2004.