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The new integral transform "Tarig Transform "properties and applications to differential equations

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ABSTRACT

A new integral transform similar to Laplace and Sumudu transforms is introduced, to explain the use of Tarig transform in differential equations, an example of first and second order linear differential equations are presented. In this work we show that the applicability of this interesting new transform its very efficiency to solving differential equations with constant coefficients.

Introduction

There are numerous integral transforms to solve differential equations and integral equations. Of these, the Laplace transformations is the most widely used.

In view of many interesting properties which make visualization easier, we introduced a new integral transform, named as Tarig transform and applied it to the solution of differential equations subsequently , we derived Tarig transform of different functions and derivatives use in engineering problems.

In this paper we discussed the basic theory of Tarig transform with supporting examples and presented a table. Typically, Fourrier, Laplace, Sumudu and ELzaki transforms are the convenient mathematical tools for solving differential equations, also Tarig transform and some of its fundamental Properties are used to solve differential equations

Anew transform called Tarig transform defined for function of exponential order. We consider functions in the set A defined by:

$$A=\left\{ f(t): \exists M, k_1, k_2 > 0, \left|f(t)\right| < Me^{\frac{|t|}{k_j}}, \text{ if } t \in (-1)^j X[0,\infty) \right\} \qquad (1)$$

For a given function in the set A, the constant M must be finite number, and k_1 , k_2 may be finite or infinite

Tarig transform defined by the integral equations,

$$T\left[f(t)\right] = F(u) = \frac{1}{u}\int_0^\infty f(t) e^{\frac{-t}{u^2}} dt \ , \quad t \geq 0 \ , u \neq 0 \qquad (2)$$

The variable u in this transform is used to factor of the variable t in the argument of the function f . this transform has deeper Connection with the Laplace transform.

We also present many different of properties of this new transform and Sumudu transform.

The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

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Theorem 1:

If $f(t)$ is sectionally continuous is every finite interval $0 \leq t \leq k$ and of exponential order β for $t > k$, then its Tarig transform $F(u)$ exists for all $u > \beta$.

Proof:

We have for any positive number k ,

$$\int_0^\infty f(ut)\, e^{\frac{-t}{u}} dt = \int_0^k e^{\frac{-t}{u}} f(ut) dt + \int_k^\infty e^{\frac{-t}{u}} f(ut) dt$$

Since $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq k$,the first integral on the right side exists. Also the second integral on the right exists, since $f(t)$ is of exponential order β for $t > k$,to see this we have only to observe that in such case .

$$\left|\int_k^\infty e^{\frac{-t}{u}} f(ut)\, dt\right| \leq \int_k^\infty \left|e^{\frac{-t}{u}} f(ut)\right| dt \leq \int_0^\infty e^{\frac{-t}{u}} \left|f(tu)\right| dt \leq \int_0^\infty e^{\frac{-t}{u}} M e^{\frac{\beta t}{u}} dt = \frac{Mu}{1-\beta}, \quad \beta \neq 1$$

Theorem 2:

If $T\left[f(t)\right] = F(u)$, then $T\left[f(at)\right] = \frac{1}{a} F(au)$

Proof:

We have $T\left[f(at)\right] = \int_0^\infty e^{\frac{-t}{u}} f(aut)\, dt$. Let $x = at,$ then we get:

$$T\left[f(at)\right] = \frac{1}{a}\int_0^\infty e^{\frac{-x}{au}} f(ux) dx = \frac{1}{a} F(au)$$

Theorem 3:

If a,b are any constants and $f(t)$ and $g(t)$ are any functions, then:

$$T\left[a f(t)+b g(t)\right]=aT\left(f(t)\right)+bT\left(g(t)\right)$$

Proof:

$$T\left[f(t)\right]=\int_0^\infty f(ut)e^{\frac{-t}{u}}dt.$$ And

$$T\left[g(t)\right]=\int_0^\infty g(ut)e^{\frac{-t}{u}}dt.$$

Then:

$$T\left[a f(t)+b g(t)\right]=\int_0^\infty e^{\frac{-t}{u}}\left[a f(ut)+b g(ut)\right]dt$$

$$=a\int_0^\infty f(ut)e^{-\frac{t}{u}}dt+b\int_0^\infty g(ut)e^{\frac{-t}{u}}dt=aT\left[f(t)\right]+bT\left[g(t)\right]$$

Theorem 4:

If $T\left[f(t)\right]=F(u)$ then:

$$(i)\ T\left[f'(t)\right]=\frac{F(u)}{u^2}-\frac{1}{u}f(0)\quad (ii)\ T\left[f''(t)\right]=\frac{F(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

$$(iii)\ T\left[f^{(n)}(t)\right]=\frac{F(u)}{u^{2n}}-\sum_{i=1}^{n}u^{2(i-n)-1}f^{(i-1)}(0)$$

Proof:

$(i)\ T\left[f'(t)\right]=\frac{1}{u}\int_0^\infty f'(t)e^{\frac{-t}{u^2}}dt$. Integrating by parts to find that: $T\left[f'(t)\right]=\frac{1}{u}\left\{-f(0)+\frac{1}{u}T\left[f(t)\right]\right\}$

And $T\left[f'(t)\right]=\frac{F(u)}{u^2}-\frac{1}{u}f(0)$

$(ii)\ By(i)\ T\left[G'(t)\right]=\frac{T\left[G(t)\right]}{u^2}-\frac{1}{u}G(0)$.Let $G(t)=f'(t)$. then:

$$T\left[f''(t)\right]=\frac{T\left(f'(t)\right)}{u^2}-\frac{1}{u}f'(0)=\frac{1}{u^2}\left[\frac{F(u)}{u^2}-\frac{1}{u}f(0)\right]-\frac{1}{u}f'(0)\quad and$$

$$T\left[f''(t)\right]=\frac{F(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

The generalization to nth order derivatives in (iii) can be proved by using mathematical induction.

Theorem 5:

If $T\left[f(t)\right]=G(u)$ and $L\left[f(t)\right]=F(s)$ then: $G(u)=\frac{F\left(\frac{1}{u^2}\right)}{u}$ where $F(s)$ is Laplace transform of $f(t)$.

Proof:

$T\left[f(t)\right]=\int_0^\infty f(ut)e^{\frac{-t}{u}}dt=G(u)$ Let $w=ut$, then we have

$$G(u)=\int_0^\infty f(w)e^{\frac{-w}{u^2}}\frac{dw}{u}=\frac{F\left(\frac{1}{u^2}\right)}{u}$$

Application of Tarig Transform of Ordinary Differential Equations

As stated in the introduction of this paper, Tarig transform can be used as an effective tool. For analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data. The following examples illustrate the use of Tarig transform in solving certain initial value problems described by ordinary differential equations.

Example (1)

Consider the first – order ordinary differential equation.

$$\frac{dx}{dt}+px=f(t),t>0 \qquad (3)$$

With the initial condition,

$$x(0)=a \qquad (4)$$

Application of Tarig transform of this equation we get:

$\frac{\bar{x}(u)}{u^2}-\frac{1}{u}x(0)+p\bar{x}(u)=F(u)$ Or

$$\bar{x}(u)=\frac{u^2F(u)}{1+u^2p}+\frac{au}{1+u^2p}$$

Where $\bar{x}(u)$ is Tarig transform of $x(t)$

The inverse Tarig transform leads to the solution.

Example (2):

The second order linear ordinary differential equation has the general form,

$$y''+2py'+qy=f(x),x>0 \qquad (5)$$

The initial conditions are

$$y(0)=a\ ,\ y'(0)=b \qquad (6)$$

Where p,q,a and b are constants.

Application of Tarig transform to this general initial value problem gives.

$$\frac{\bar{y}(u)}{u^4}-\frac{1}{u^3}y(0)-\frac{1}{u}y'(0)+2p\left[\frac{\bar{y}(u)}{u^2}-\frac{1}{u}y(0)\right]+q\bar{y}(u)=F(u)$$

The use of (6) leads to the solution for $\bar{y}$ as:

$$\bar{y}(u)=\frac{u^4F(u)+au+bu^3+2pau^3}{1+2pu^2+qu^4}$$

The inverse transform gives the solution.

Example (3):

Consider the first order differential equation,

$$y'+y=0\ ,\ y(0)=1$$

Application of Tarig transform to this equation gives:

$$\frac{F(u)}{u^2}-\frac{1}{u}y(0)+F(u)=0\Rightarrow F(u)-u+u^2F(u)=0\Rightarrow F(u)=\frac{u}{1+u^2}$$

Then: $y(x)=e^{-x}$

Where $F(u)$ is the Tarig transform of $y(x)$.

Example (4):

Solve the differential equation,

$y'+2y=x, y(0)=1$

Tarig transform of this equation is,

$$\frac{F(u)}{u^2}-\frac{1}{u}y(0)+2F(u)=u^3\Rightarrow\ F(u)\left[1+2u^2\right]=u^5+u$$

$$F(u)=\frac{u^5}{1+2u^2}+\frac{u}{1+2u^2}=\frac{1}{2}u^3-\frac{1}{4}u+\frac{\frac{3}{4}u}{1+2u^2}$$

The inverse transform of this equation gives the solution in the form:

$$y(x)=\frac{1}{2}x-\frac{1}{4}+\frac{3}{4}e^{-2x}$$

Example (5):

Let us consider the second – order differential equation. $y''+y=0 \quad , \quad y(0)=y'(0)=1$

Application of Tarig transform to this equation gives.

$$\frac{F(u)}{u^4}-\frac{1}{u^3}y(0)-\frac{1}{u}y'(0)+F(u)=0 \Rightarrow F(u)-u-u^3+u^4F(u)=0$$

And $F(u)=\frac{u^3}{1+u^4}+\frac{u}{1+u^4}$

The inverse Tarig transform of this equation simply obtained as $y(x)=\sin x+\cos x$

Example (6):

Solve the boundary value problem:

$$y''+9y=\cos t, \quad if \quad y(0)=1 \ , \ y\left(\frac{\pi}{2}\right)=-1.$$

Since $y'(0)$ is not known, let $y'(0)=c$, and take Tarig transform of this equation and using the conditions, we have:

$$\frac{F(u)}{u^4}-\frac{1}{u^3}-\frac{c}{u}+9F(u)=\frac{u}{1+4u^4}$$

$$F(u)=\frac{u^5}{\left(1+9u^4\right)\left(1+4u^4\right)}+\frac{u}{1+9u^4}+\frac{cu^3}{1+9u^4}=\frac{\frac{-1}{5}u}{1+9u^4}+\frac{\frac{1}{5}u}{1+4u^4}+\frac{u}{1+9u^4}+\frac{cu^3}{1+9u^4}$$

And invert to find the solution

$$y(t)=\frac{4}{5}\cos 3t+\frac{1}{5}\cos 2t+\frac{c}{3}\sin 3t, \ But \quad y\left(\frac{\pi}{2}\right)=-1 \ \Rightarrow c=\frac{12}{5}$$

Then:

$$y(t)=\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t+\frac{1}{5}\cos 2t$$

Example (7):

Consider the following differential equation,

$$y''-3y'+2y=4e^{3t} \quad , \quad y(0)=-3 \quad , \quad y'(0)=5$$

Taking the Tarig transform of both sides of the differential equation.

$$\frac{F(u)}{u^4}-\frac{1}{u^3}y(0)-\frac{1}{u}y'(0)-3\left[\frac{F(u)}{u^2}-\frac{1}{u}y(0)\right]+2F(u)=\frac{4u}{1-3u^2}$$

$$F(u)\left[2u^4-3u^2+1\right]=\frac{4u^5}{1-3u^2}+14u^3-3u$$

And

$$F(u)=\frac{4u}{1-2u^2}+\frac{2u}{1-3u^2}-\frac{9u}{1-u^2}$$

Thus:

$$y(t)=4e^{2t}+2e^{3t}-9e^{t}$$

Example (8):

Find the solution of the following initial value problem. $y''+4y=9t \quad , \quad y(0)=0 \quad , \quad y'(0)=7$

Applying Tarig transform of this problem and using the given conditions we get,

$$\frac{F(u)}{u^4}-\frac{7}{u}+4F(u)=9u^3 \Rightarrow F(u)-7u^3+4u^4F(u)=9u^7$$

$$F(u)=\frac{9u^7}{1+4u^4}+\frac{7u^3}{1+4u^4}=\frac{9}{4}u^3-\frac{\frac{9}{4}u^3}{1+4u^4}+\frac{7u^3}{1+4u^4}$$

Inverting to find the solution in the form:

$$y(t)=\frac{9}{4}t+\frac{19}{8}\sin 2t$$

Conclusion

The definition and application of the new transform "Tarig transform" to the solution of ordinary differential equations has been demonstrated.

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Appendix Tarig Transform of Some Functions

S.N0.	$f(t)$	$F(u)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$
10	$H(t-a)$	$ue^{\frac{-a}{u^2}}$
11	$\delta(t-a)$	$\frac{1}{u}e^{\frac{-a}{u^2}}$
12	te^{at}	$\frac{u^3}{(1-au^2)^2}$
13	$e^{at}\sin bt$	$\frac{bu^3}{(1-au^2)^2+b^2u^4}$
14	$\int_0^t f(\omega)d\omega$	$u^2F(u)$
15	$(f*g)(t)$	$u\,M(u)N(u)$
16	$e^{at}\cos bt$	$\frac{u(1-au^2)}{(1-au^2)^2+b^2u^4}$
17	$e^{at}\cosh bt$	$\frac{u(1-au^2)}{(1-au^2)-b^2u^4}$
18	$e^{at}\sin bt$	$\frac{bu^3}{(1-au^2)^2-b^2u^4}$
19	$t\sin at$	$\frac{2au^5}{(1+a^2u^4)^2}$
20	$t\cos at$	$\frac{u^3(1-a^2u^4)}{(1+a^2u^4)^2}$
21	$t\sinh at$	$\frac{2au^5}{(1-a^2u^4)^2}$
22	$t\cosh at$	$\frac{u^3(1+a^2u^4)}{(1-a^2u^4)^2}$