Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 38 (2011) 4228-4231

ISSN: 2229-712X

Numerical strategy of the harmonic oscillators using single-term HAAR wavelet series

S. Sekar[1], A. Manonmani[2] and E. Paramanathan[3]

[1]Department of Mathematics, Government Arts College (Autonomous), Cherry Road, Salem-636 007, TamilNadu, India.
[2]Department of Mathematics, LRG Government Arts College for Women, Palladam Road, Tirupur-638 604, TamilNadu, India.
[3]Research and Development Centre, Bharathiar University, Coimbatore – 641 046, TamilNadu, India.

ARTICLE INFO

Article history:
Received: 4 July 2011;
Received in revised form:
23 August 2011;
Accepted: 28 August 2011;

Keywords
Harmonic oscillators,
ODE45 in Matlab,
Runge-Kutta method,
Single-term Walsh series,
Single-term Haar wavelet series.

ABSTRACT

In this article an interesting and famous realistic problem harmonic oscillators is discussed using the single-term Haar wavelet series (STHW) method. The results (approximate solutions) obtained very accurate using classical Runge-Kutta (RK) method, single-term Walsh Series (STWS) and STHW methods are compared with the ODE45 in Matlab. It is found that the solution obtained using STHW is closer to the ODE45 in Matlab. The high accuracy and the wide applicability of STHW approach will be demonstrated with numerical example. Solution graphs for discrete exact solutions are presented in a graphical form to show the efficiency of the STHW. The results obtained show that STHW is more useful for solving harmonic oscillators and the solution can be obtained for any length of time.

© 2011 Elixir All rights reserved.

Introduction

Most of the realistic singular non Most of the realistic singular non-linear systems do not admit any analytical solution and hence a numerical procedure has to be used. In the last few years substantial progress has been made in finding the numerical solution of special classes of nonlinear singular systems of differential equations.

A general numerical procedure for their solution has not previously existed. Hence it is important to understand the structure of such systems and develop efficient methods for solving them.

The conventional methods such as Euler, Runge-Kutta and Adams-Moulton are restricted to very small step size in order that the solution is stable.

Runge-Kutta methods have become very popular, both as computational techniques as well as subject for research, which were discussed by Butcher [2-4] and Shampine [22]. This method was derived by Runge about the year 1894 and extended by Kutta a few years later.

They developed algorithms to solve differential equations efficiently and yet are the equivalent of approximating the exact solutions by matching 'n' terms of the Taylor series expansion. The beauty of the RK pair is that it requires no extra function evaluations, which is the most time consuming aspect of all ODE solvers. This breakthrough initiated a search for RK algorithms of higher and higher order and better error estimates.

Nandhakumar et al. [5-6] introduce Haar Wavelet Series to numerical investigation of an industrial robot arm control problem.

Sekar et al. [7-21] introduced the STHW to study in the various real world applications and calculations. In this paper, we consider the same STHW method for harmonic oscillators but presenting a different approach by the STHW with more accuracy.

Tele:
E-mail addresses: sekar_nitt@rediffmail.com

© 2011 Elixir All rights reserved

STHW

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1 in some intervals and zeros elsewhere Sekar et al. [7-21]. In general,

$$h_n(t) = h_1\left(2^j t - k\right), \text{ where } n = 2^j + k,$$

$$j \geq 0,\ 0 \leq k < 2^j,\ n,\ j,\ k \in Z$$

$$h_1(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \end{cases}$$

Namely, each Haar wavelet contains one and just one square wave, and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems.

Any function *y(t)*, which is square integrable in the interval [0,1]. Can be expanded in a Haar series with an infinite number of terms

$$y(t) = \sum_{i=0}^{\infty} c_i h_i(t), i = 2^j + k, \qquad (1)$$

where $i = 2^j + k,\ j \geq 0, 0 \leq k < 2^j, n, j, t \in [0,1]$

where the Haar coefficients

$$c_i = 2^j \int_0^1 y(t) h_i(t) dt$$

are determined such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt, \text{ where } m = 2^j, j \in \{0\} \cup N$$

Usually, the series expansion Eq. (1) contains an infinite number of terms for a smooth *y(t)*. If *y(t)* is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (1) will be terminated after m terms, that is

$$y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = c_{(m)}^T h_{(m)}(t), t \in [0,1]$$

$$c_{(m)}(t) = [c_0 c_1 \ldots c_{m-1}]^T, \qquad (2)$$

$$h_{(m)}(t) = [h_0(t) h_1(t) \ldots h_{m-1}(t)]^T,$$

where "T" indicates transposition, the subscript m in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix *P[3]*.

$$\int h_{(m)}(\tau) d\tau \approx P_{(m\times m)} h_{(m)}(t), t \in [0,1]$$

where the m-square matrix P is called the operational matrix of integration and single-term $P_{(1\times 1)} = \frac{1}{2}$. Let us define [7-21]

$$h_{(m)}(t) h_{(m)}^T(t) \approx M_{(m\times m)}(t), \qquad (3)$$

and $M_{(1\times 1)}(t) = h_0(t)$. Eq.(3) satisfies

$$M_{(m\times m)}(t) c_{(m)} = C_{(m\times m)} h_{(m)}(t),$$

where $c_{(m)}$ is defined in Eq.(2)

and $C_{(1\times 1)} = c_0$.

Harmonic Oscillators

Unforced harmonic oscillators can be modeled by the following second order homogeneous differential equation Blanchard et al.[1],

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad (4)$$

where $m, k > 0$ and $b \geq 0$. If b = 0, then the system is undamped. However, if $b > 0$, then different types of behavior are possible. For the above harmonic oscillatory equation, the characteristic equation is,

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

The three different possibilities for the roots of the characteristic equation are,

- If $b^2 - 4mk < 0$, then we have complex roots and the harmonic oscillator is said to be under damped. In this case, we expect the system to oscillate about its equilibrium position.
- If $b^2 - 4mk = 0$, then we have repeated roots and the oscillator is critically damped.
- IF $b^2 - 4mk > 0$, then the roots are real and distinct, and the oscillator is said to be over-damped, and the system will move to its equilibrium position without any oscillations.

We now consider the second order homogenous differential equation given as an initial value problem, with m = 1, k = 1 and b = 0.01t,

$$\frac{d^2y}{dt^2} + 0.01t\frac{dy}{dt} + y = 0, \; y(0) = -1, \dot{y}(0) = 2 \qquad (5)$$

We note that for Equation (5), at time t = 0 the system is undamped, and the critical damping value is given by t^* = 200. The interval of time $(0,200)$ and $(200,\infty)$ corresponds to the system being under damped and over damped respectively.

To solve Equation (5) with numerical integrators, we can rewrite this second order homogeneous differential equation as a system of first order differential equations, by using the substitution $dy/dt = v$, and a vector $Y(t) = [y(t), v(t)]$,

$$\dot{Y}(t) = A(t) Y(t) \qquad (6)$$

where the matrix $A(t)$ and the initial condition $Y(0)$ are given by,

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -0.01t \end{bmatrix}, Y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Numerical Strategies

Firstly, we solved the matrix differential equation with the classical methods like the Runge-Kutta method (RK) and STWS. Then, we solved the same differential equation with STHW method. All 3 integrators are highly stable, so when step size h is share equally, their global error rations should change roughly by a factor of 16. This can be observed from the numerical results, for the integration period from $t_0 = 0$ to $t_f = 0.1$. Also, it turns out that if we plot such results on a log-log scale of global error against step size h, then solutions from the STHW method have a better accuracy in comparison with RK or STWS methods. This can be observed from Figure 1.

Next, we turn our attention to the numerical solutions from integrating Equation (6) along a certain interval with fixed step size h = 1/20. We integrated the differential equation firstly over the integration period t = [0, 40], with the 3 integrators RK, STWS, and the STHW, all methods are stable. Then we also integrate over the same period with the highly accurate ODE45 in Matlab, with a specified tolerance of 10^{-14}. Comparisons can be then be done between numerical solutions from the 3 integrators under consideration, and that of Matlab's ODE45 integrator. Note that the interval t = [0.40] is below the critical damping value of t^* = 200 in Equations (6). This implies that the dynamical system is under damped, and we should expect oscillations to occur. Similarly, we can integrate over the interval t = [220,230], which is now above t^* = 200. So we would not expect any oscillations in the numerical solutions since the system is over damped in this case.

Figure 2 and Figure 3 contain the results from the experiments mentioned above. As seen from the top 3 graphs in Figure 2, solutions $y(t)$ are oscillating against time t throughout the integration period in the under damped system, while the results from the over damped system in the top 3 graphs of Figure 3 move toward the equilibrium at $y(t)$ = 0, with no oscillations against t in the integration interval of t = [220,230].

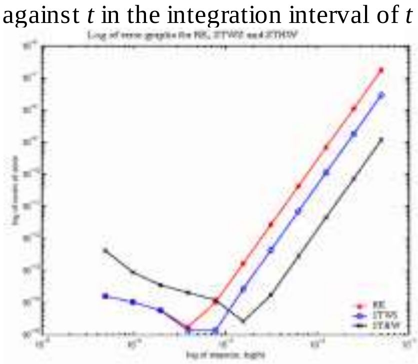

Figure 1 Log-log plot of global error against step size for results from RK, STWS and STHW. They solved the initial value problem in Equation (6), from $t_0 = 0$, $t_f = 0.1$

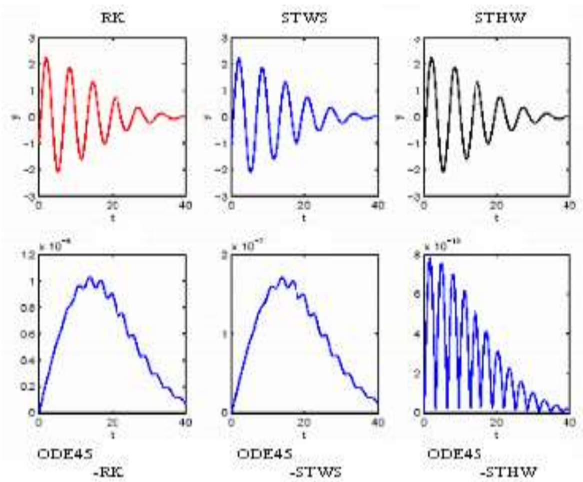

Figure 2 Solving the under damped system in Equation (6) for the interval $t \in [0,40]$, for which $t < t^* = 200$.

The top 3 graphs are plots of numerical solutions *y(t)* against time *t*, with *h* = 1/20. The bottom 3 graphs are the differences between numerical solutions from each of the 3 integrators and the highly accurate ODE45.

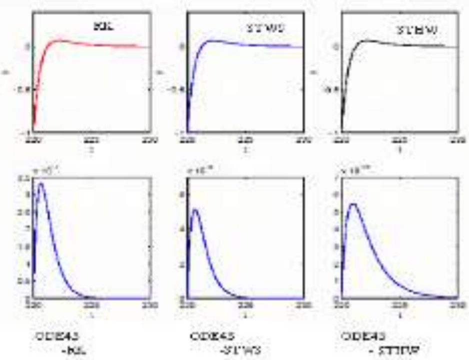

Figure 3 Solving the over damped system in Equation (6) for the interval $t \in [220,230]$, for which $t < t^* = 200$.

The top 3 graphs are plots of numerical solutions *y(t)* against time *t*, with *h* = 1/20. The bottom 3 graphs are the differences between numerical solutions from each of the 3 integrators and the highly accurate ODE45.

Conclusions

The accuracy achieved from the STHW method is higher than that of the RK and the STWS methods. This can be observed if we compare results from the 3 integrators and that ODE45. The differences in this comparison are also plotted in the bottom 3 graphs of Figure 2 and Figure 3. For both the under damped and over damped systems, the difference between ODE45 and STHW are several degrees smaller in magnitude than the differences between smaller in magnitude than the differences between numerical solutions from RK and STWS methods against ODE45's. With a relatively low computational cost, and a relatively good accuracy for fixed step size *h*, these brief experiments suggest the suitability of using the STHW to integrate systems of first order differential equations describing the dynamics of an unforced harmonic oscillator. Hence the STHW method is more suitable for studying the harmonic oscillators.

References

[1] Blanchard, Olivier and Charles Kahn, "*The Solution of Linear Difference Models under Rational Expectations*," Econometrica 45, July 1985, 1305-1311.

[2] J. C. Butcher, "On Runge-Kutta processes of higher order", *Journal of the Australian Mathematical Society*, Vol. 4, 1964, pp. 179.

[3] J. C. Butcher, "*The Numerical Analysis of Ordinary Differential Equations*", 1987, John Wiley & Sons, U.K.

[4] J. C. Butcher, "*The Numerical Methods for Ordinary Differential Equations*", 2003, John Wiley & Sons, U.K.

[5] S. Nandhakumar, V. Selladurai, and S. Sekar, "Numerical investigation of an Industrial robot arm control problem using Haar Wavelet Series", *American Journal of Engineering and Applied Sciences*, Vol. 2, No. 4, 2009, pp. 584-589.

[6] S. Nandhakumar, V. Selladurai, V. Muthukumaran and S. Sekar, "Haar wavelet approach to time varying model robot arm control problem", *International Journal of Academic Research*, Vol. 2, Nos. 4, 2010, pp. 66-74.

[7] S. Sekar and A. Manonmani, "A study on time-varying singular nonlinear systems via single-term Haar wavelet series", *International Review of Pure and Applied Mathematics*, Vol. 5, No. 2, 2009, pp. 435-441.

[8] S. Sekar, G. Balaji, R. Kumar, A. Manonmani and S. Nandhakumar, "A study on second order mechanical vibratory systems using single-term Haar wavelet series", *International Journal of Logic Based Intelligent Systems*, Vol. 3, No. 2, 2009, pp. 197-202.

[9] S. Sekar and G. Balaji, "Analysis of the differential equations on the sphere using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, Vol. 4, Nos. 3-4, 2010, pp. 387-393.

[10] S. Sekar and G. Balaji, "A study on RLC smoothing circuit using single-term Haar wavelet series", *International Journal of Physical Sciences*, Vol. 22, No. 2, 2010, pp. 381-386.

[11] S. Sekar and M. Duraisamy, "A study on cnn based hole-filler template design using single-term Haar wavelet series", *International Journal of Logic Based Intelligent Systems*, Vol. 4, Nos. 1, 2010, pp. 17-26.

[12] S. Sekar and M. Duraisamy, "Simulation of cellular neural networks using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, Vol. 4, Nos. 3-4, 2010, pp. 361-369.

[13] S. Sekar and M. Duraisamy, "Time-multiplexing scheme for cellular neural networks based image processing via single-term Haar wavelet series", *International Journal of computer Science and Information Technology*, Vol. 2, No. 1, 2011, pp. 25-36.

[14] S. Sekar and K. Jaganathan, "Analysis of the singular and stiff delay systems using single-term Haar wavelet series", *International Review of Pure and Applied Mathematics*, Vol. 6, No. 1, 2010, pp. 135-142.

[15] S. Sekar and K. Jaganathan, "Numerical investigation of time-varying transistor circuit using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, Vol. 4, Nos. 1-2, 2010, pp. 99-104.

[16] S. Sekar and K. Jaganathan, "Single-term Haar wavelet series approach to stiff systems of higher dimension", *International Journal of Physical Sciences*, Vol. 22, No. 3M, 2010, pp. 719-724.

[17] S. Sekar and R. Kumar, "Numerical investigation of nonlinear volterra-hammerstein integral equations via single-term Haar wavelet series", *International Journal of Current Research*, Vol. 3, Issue 2, 2011, pp. 099-103.

[18] S. Sekar and E. Paramanathan, "A study on periodic and oscillatory problems using single-term Haar wavelet series",

International Journal of Current Research, Vol. 2, Issue 1, 2011, pp. 097-105.
[19] S. Sekar and K. Prabakaran, "Numerical solution of nonlinear problems in the calculus of variations using single-term Haar wavelet series", *International Journal Mathematics and Computing Applications*, Vol. 2, Nos. 1, 2010, pp. 25-33.
[20] S. Sekar and M. Vijayarakavan, "Analysis of the non-linear singular systems from fluid dynamics using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, Vol. 4, Nos. 1-2, 2010, pp. 15-29.
[21] S. Sekar and M. Vijayarakavan, "Simulation of fed-batch fermentation process using single-term Haar wavelet series", *Elixir Discrete Mathematics online Journal*, Vol. 35, 2011, pp. 2969-2973.
[22] L. F. Shampine, "Some practical Runge-Kutta formulas", *Mathematics of Computations,* Vol. 46, 1985, pp. 135-150.