# Three-dimensional dispersion analysis of a transversely isotropic solid cylinder of elliptic inner and outer cross-section immersed in a fluid 

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#### Abstract

ARTICLE INFO

\section*{Article history:}

Received: 2 August 2011; Received in revised form: 23 September 2011; Accepted: 30 September 2011;

\section*{Keywords}

Wave propagation in isotropic cylinders of arbitrary cross-sections; Free vibration of transversely isotropic cylinder; Elastic rods loaded with fluid; Wave propagation in a cylinders immersed in a fluid; Wave propagation in a transversely isotropic plate immersed in a fluid.


#### Abstract

The wave propagation in an infinite, homogeneous transversely isotropic solid cylinder of elliptic inner and outer cross-section immersed in a fluid is studied using the Fourier expansion collocation method, within the framework of the linearized, three-dimensional theory of elasticity. The equation of motion of solid and fluid are respectively formulated using the constitutive equations of a transversely isotropic cylinder and the constitutive equation of an inviscid fluid. Three displacement potentional functions are introduced to uncouple the equation of motion. The frequency equations of longitudinal and flexural (symmetric and antisymmetric) modes are analyzed numerically for an elliptic crosssectional transversely isotropic solid cylinder of elliptic inner and outer cross-section immersed in a fluid. To compare the model with the existing literature, the results of a fluidloaded transversely is otropic cylinder are obtained and they are compared with the results of Berlinear and Solecki (1996). It shows very good degree of agreement. The computed nondimensional wave numbers are presented in the form of dispersion curves for the material zinc.


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## Introduction

In many structural applications requiring high strength-toweight and stiffness-to-weight ratios, the isotropic cylinders are being replaced by the cylinders of composite materials. So the study of frequency equations of wave propagation in anisotropic cylinders must be developed. A through knowledge of various wave propagation characteristics, as a function of material and geometrical parameters is necessary for a wide range of applications, from geophysical prospecting in cased holes, nondestructive evaluation of oil and gas pipelines, to the insulated fiber optic cables for data transmission.

The most general form of harmonic waves in a hollow cylinder of circular cross section of infinite length has been analyzed by Gazis (1959). Mirsky (1964) analyzed the wave propagation in transversely isotropic circular cylinders of infinite length and presented the frequency equation in Part I and numerical results in Part II. A method, for solving wave propagation in arbitrary cross-sectionl cylinders and plates and to find out the phase velocities in different modes of vibrations namely longitudinal, torsional and flexural, by constructing frequency equations was devised by Nagaya (1982, 1983, 1984 and 1985) He formulated the Fourier expansion collocation method for this purpose. Following Nagaya, Paul and Venkatesan (1989) studied the wave propagation in an infinite piezoelectric solid cylinder of arbitrary cross section using Fourier expansion collocation method.

Guided waves in a transversely isotropic cylinder immersed in a fluid is analyzed by Ahmad (2001). Following Ahmad, Nagay (1995) have studied the longitudinal guided wave propagation in a transversely isotropic rod immersed in fluid, later, Nagy with Nayfeh (1996) discussed the viscosity-induced attenuation of longitudinal guided waves in fluid-loaded rods. Easwaran and Munjal (1995) reported a note on the effect of
wall compliance on lowest-order mode propagation in fluidfilled/submerged impedance tubes Sinha et. al. (1992) have discussed the axisymmetric wave propagation in circular cylindrical shell immersed in fluid, in two parts. In Part I, the theoretical analysis of the propagating modes is discussed and in Part II, the axisymmetric modes excluding torsional modes are obtained theoretically and experimentally and are compared. Berlinear and Solecki (1996) have studied the wave propagation in fluid loaded transversely isotropic cylinder. In that paper, Part I consists of the analytical formulation of the frequency equation of the coupled system consisting of the cylinder with inner and outer fluid and Part II gives the numerical results.
Venkatesan and Ponnusamy $(2002,2003)$ have obtained the frequency equation of the free vibration of a solid cylinder of arbitrary cross section immersed in a fluid using Fourier expansion collocation method. The frequency equations are obtained for longitudinal and flexural vibrations and are studied numerically for elliptical and cardioidal cross-sectionl cylinders.

In this paper, the wave propagation in a cylinder of elliptic inner and outer cross section immersed in an invicid fluid is analyzed. The frequency equations of longitudinal and flexural modes are analyzed numerically for cylinders with elliptic crosssection using Fourier expansion collocation method. The computed non-dimensional plotted in the form of dispersion curves.

## Formulation of the problem

We consider a homogeneous transversely isotropic infinite cylinder of both inner and outer elliptic cross-section immersed in inviscid fluid. The system is assumed to be linear so that the linearized three-dimensional stress equation of motion is used for both the cylinder and the fluid. The system displacements and stresses are defined by the cylindrical

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coordinates $r, \theta$ and $z$. In cylindrical coordinates, the threedimensional stress equations of motion and strain-displacement relations in the absence of body force for a linear elastic medium are

$$
\begin{gather*}
\sigma_{r r, r}+r^{-1} \sigma_{r \theta, \theta}+\sigma_{r z, z}+r^{-1}\left(\sigma_{r r}-\sigma_{\theta \theta}\right)=\rho u_{r, t t}  \tag{1a}\\
\sigma_{r \theta, r}+r^{-1} \sigma_{\theta \theta, \theta}+\sigma_{\theta z, z}+2 r^{-1} \sigma_{r \theta}=\rho u_{\theta, t t}  \tag{1b}\\
\quad \sigma_{r z, r}+r^{-1} \sigma_{\theta z, \theta}+\sigma_{z z, z}+r^{-1} \sigma_{r z}=\rho u_{z, t t} \tag{1c}
\end{gather*}
$$

where

$$
\begin{align*}
& \sigma_{r r}=c_{11} e_{r r}+c_{12} e_{\theta \theta}+c_{13} e_{z z}  \tag{2a}\\
& \sigma_{\theta \theta}=c_{12} e_{r r}+c_{11} e_{\theta \theta}+c_{13} e_{z z}  \tag{2b}\\
& \sigma_{z z}=c_{13} e_{r r}+c_{13} e_{\theta \theta}+c_{33} e_{z z}  \tag{2c}\\
& \sigma_{r \theta}=2 c_{66} e_{r \theta}, \sigma_{\theta z}=2 c_{44} e_{\theta z}, \sigma_{r z}=2 c_{44} e_{r z} \tag{2c}
\end{align*}
$$

where $\sigma_{r r}, \sigma_{\theta \theta,} \sigma_{z z}, \sigma_{r \theta}, \sigma_{\theta z}, \sigma_{r z}$ are the stress components, $e_{r r}, e_{\theta \theta}, e_{z z}, e_{r \theta}, e_{\theta z}, e_{r z}$ are the strain components, $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$ and $c_{66}=\left(c_{11}-c_{12}\right) / 2$ are the five independent elastic moduli, $\rho$ is the mass density of the material.
The strain $e_{i j}$ are related to the displacements are given by

$$
\begin{gather*}
e_{r r}=u_{r, r}, \quad e_{\theta \theta}=r^{-1}\left(u_{r}+u_{\theta, \theta}\right), e_{z z}=u_{z, z}  \tag{3a}\\
2 e_{z r}=\left(u_{r, z}+u_{z, r}\right), 2 e_{\theta z}=\left(u_{z, \theta}+r^{-1} u_{\theta, z}\right) \tag{3b}
\end{gather*}
$$

in which $u_{r}, u_{\theta}$ and $u_{z}$ are the displacement components along radial, circumferential and axial directions respectively. The comma in the subscripts denotes the partial differentiation with respect to the variables.

Substituting the Eqs. (3) and (2) in the Eq. (1), results in the following three-dimensional equation of motion are obtained as follows:

$$
\begin{align*}
& c_{11}\left(u_{r, r r}+r^{-1} u_{r, r}-r^{-2} u_{r}\right)-r^{-2}\left(c_{11}+c_{66}\right) u_{\theta, \theta}+r^{-2} c_{66} u_{r, \theta \theta}  \tag{4a}\\
& +c_{44} u_{r, z z}+\left(c_{44}+c_{13}\right) u_{z, r z}+r^{-1}\left(c_{66}+c_{12}\right) u_{\theta, r \theta}=\rho u_{r, t t} \\
& r^{-1}\left(c_{12}+c_{66}\right) u_{r, r \theta}+r^{-2}\left(c_{66}+c_{11}\right) u_{r, \theta}+c_{66}\left(u_{\theta, r r}+r^{-1} u_{\theta, r}-r^{-2} u_{\theta}\right) \\
& +r^{-2} c_{11} u_{\theta, \theta \theta}+c_{44} u_{\theta, z z}+r^{-1}\left(c_{44}+c_{13}\right) u_{z, \theta z}=\rho u_{\theta, t t}  \tag{4b}\\
& c_{44}\left(u_{z, r r}+r^{-1} u_{z, r}+r^{-2} u_{z, \theta \theta}\right)+r^{-1}\left(c_{44}+c_{13}\right)\left(u_{r, z}+u_{\theta, \theta z}\right)  \tag{4c}\\
& +\left(c_{44}+c_{13}\right) u_{r, r z}+c_{33} u_{z, z z}=\rho u_{z, t t}
\end{align*}
$$

## Solution to the equation of motion

The Eq. (4) is coupled partial differential equations of the three displacement components. This systemof equations can be uncoupled by eliminating two of the three displacement components through two of the three equations, but these results in partial differential equations of fourth order. To uncouple the Eq. (4), we follow Mirsky (1964) and assuming the solution of Eq. (4) as follows:
$u_{r}(r, \theta, z, t)=\sum_{n=0}^{\infty} \varepsilon_{n}\left[\left(\phi_{n, r}+r^{-1} \psi_{n, \theta}\right)+\left(\bar{\phi}_{n, r}+r^{-1} \bar{\psi}_{n, \theta}\right)\right] e^{i(k z+\omega t)}$
$u_{\theta}(r, \theta, z, t)=\sum_{n=0}^{\infty} \varepsilon_{n}\left[\left(r^{-1} \phi_{n, \theta}-\psi_{n, r}\right)+\left(r^{-1} \bar{\phi}_{n, \theta}-\bar{\psi}_{n, r}\right)\right] e^{i(k z+\omega t)}$
$u_{z}(r, \theta, z, t)=(i / a) \sum_{n=0}^{\infty} \varepsilon_{n}\left[W_{n}+\bar{W}_{n}\right] e^{i(k z+\omega t)}$
where $\varepsilon_{n}=\frac{1}{2}$ for $n=0, \varepsilon_{n}=1$ for $n \geq 1, i=\sqrt{-1}, k$ is the wavenumber, $\omega$ is the frequency, $\phi_{n}(r, \theta), W_{n}(r, \theta)$, $\psi_{n}(r, \theta), \bar{\phi}_{n}(r, \theta), \quad \bar{W}_{n}(r, \theta)$, and $\bar{\psi}_{n}(r, \theta)$ are the displacement potentials and $a$ is the geometrical parameter of the cylinder.
By introducing the dimensionless quantities such as $\varsigma=k a$,
$\Omega^{2}=\rho \omega^{2} a^{2} / c_{44}, \quad \bar{c}_{11}=c_{11} / c_{44}, \quad \bar{c}_{13}=c_{13} / c_{44}$,
$\bar{c}_{33}=c_{33} / c_{44}, T=t \sqrt{c_{44} / \rho / a} \quad$ and $\quad x=r / a$ and
substituting Eq. (5) in Eq. (4), we obtain
$\left(\bar{c}_{11} \nabla^{2}+\left(\Omega^{2}-\varsigma^{2}\right)\right) \phi_{n}-\left(1+\bar{c}_{13}\right) W_{n}=0$
$\varsigma\left(1+\bar{c}_{13}\right) \phi_{n}+\left(\nabla^{2}+\left(\Omega^{2}-\bar{c}_{33} \varsigma^{2}\right)\right) W_{n}=0$
and
$\left(\nabla^{2}+\left(\Omega^{2}-\varsigma^{2}\right) / \bar{c}_{66}\right) \psi_{n}=0$
where $\nabla^{2} \equiv \partial^{2} / \partial x^{2}+x^{-1} \partial / \partial x+x^{-2} \partial^{2} / \partial \theta^{2}$
Eliminating $W_{n}$ from the Eq.(6), we obtain
$\left(A \nabla^{4}+B \nabla^{2}+C\right) \phi_{n}=0$
where
$A=\bar{c}_{11}, B=-\left[\left(1+\bar{c}_{11}\right) \Omega^{2}+\varsigma^{2}\left(\bar{c}_{13}^{2}+2 \bar{c}_{13}-\bar{c}_{11} \bar{c}_{33}\right)\right]$,
$C=\left(\Omega^{2}-\varsigma^{2}\right)\left(\Omega^{2}-\bar{c}_{33} \varsigma^{2}\right)$
Solving the Eq.(8), the solution for the symmetric mode are obtained as

$$
\begin{aligned}
& \phi_{n}=\sum_{i=1}^{2}\left[A_{i n} J_{n}\left(\alpha_{i} a x\right)+B_{i n} Y_{n}\left(\alpha_{i} a x\right)\right] \cos n \theta \\
& W_{n}=\sum_{i=1}^{2} d_{i}\left[A_{i n} J_{n}\left(\alpha_{i} a x\right)+B_{i n} Y_{n}\left(\alpha_{i} a x\right)\right] \cos n \theta
\end{aligned}
$$

where $J_{n}$ and $Y_{n}$ are Bessel functions of the first and second kind of order n . The solution for the antisymmetric modes $\bar{\phi}_{n}$ and $\bar{W}_{n}$ are obtained by replacing $\cos n \theta$ by $\sin n \theta$ in Eq. (10).

Here $\left(\alpha_{i} a\right)^{2}>0, \quad(i=1,2)$ are the roots of the algebraic equation $A(\alpha a)^{4}-B(\alpha a)^{2}+C=0$.

The Bessel functions $J_{n}$ and $Y_{n}$ are used when the roots $\left(\alpha_{i} a\right)^{2},(i=1,2)$ are real or complex and the modified Bessel function $I_{n}$ and $K_{n}$ are used when the roots $\left(\alpha_{i} a\right)^{2},(i=1,2)$ are imaginary.

The constants $d_{i}$ defined in the Eq. (10b) can be calculated from the equation
$d_{i}=\left[\Omega^{2}-\varsigma^{2}-\left(\alpha_{i} a\right)^{2} \bar{c}_{11}\right] / \varsigma\left(1+\bar{c}_{13}\right), \quad i=1,2$
Solving the Eq. (7), the solution to the symmetric mode is obtained as
$\psi_{n}=\left[A_{3 n} J_{n}\left(\alpha_{3} a x\right)+B_{3 n} Y_{n}\left(\alpha_{3} a x\right)\right] \sin n \theta$
where $\left(\alpha_{3} a\right)^{2}=\left(\Omega^{2}-\varsigma^{2}\right) / \bar{c}_{66}$. If $\left(\alpha_{3} a\right)^{2}<0$, the Bessel functions $J_{n}$ and $Y_{n}$ are replaced by the modified Bessel function $I_{n}$ and $K_{n}$. The solution for the antisymmetric mode $\bar{\psi}_{n}$ is obtained from Eq. (13) by replacing $\sin n \theta$ by $\cos n \theta$.

## Equation of motion of the fluid

In cylindrical polar coordinates $r, \theta$ and $z$ the acoustic pressure and radial displacement equation of motion for an invicid fluid (Achenbach (1973)) are of the form
$p^{f}=-B^{f}\left(u_{r, r}^{f}+r^{-1}\left(u_{r}^{f}+u_{\theta, \theta}^{f}\right)+u_{z, z}^{f}\right)$
and
$c_{f}^{-2} u_{r, t}^{f}=\Delta, r$
respectively where $B^{f}$, is the adiabatic bulk modulus, $\rho^{f}$ is the density, $c^{f}=\sqrt{B^{f} / \rho^{f}}$ is the acoustic phase velocity in the fluid, and $\left(u_{r}^{f}, u_{\theta}^{f}, u_{z}^{f}\right)$ is the displacement vector.
$\Delta=\left(u_{r, r}^{f}+r^{-1}\left(u_{r}^{f}+u_{\theta, \theta}^{f}\right)+u_{z, z}^{f}\right)$
Substituting
$u_{r}^{f}=\phi_{, r}^{f}, \quad u_{\theta}^{f}=r^{-1} \phi_{, \theta}^{f}$ and $u_{z}^{f}=\phi_{, z}^{f}$
and seeking the solution of (15) in the form
$\phi^{f}(r, \theta, z, t)=\sum_{n=0}^{\infty} \varepsilon_{n}\left[\phi_{n}^{f}(r) \cos n \theta+\bar{\phi}_{n}^{f}(r) \sin n \theta\right] e^{i(k z+\omega t)}$,
the oscillating waves propagating in the inner fluid located in the annulus is given by

$$
\phi_{n}^{f}=A_{4 n} J_{n}\left(\delta_{1} a x\right)
$$

where $\left(\delta_{1} a\right)^{2}=\Omega^{2} / \bar{\rho}_{1}^{f} \bar{B}_{1}^{f}-\varsigma^{2}$, in which $\bar{\rho}_{1}^{f}=\rho_{1} / \rho^{f}$, $\bar{B}_{1}^{f}=B_{1}^{f} / \mu, \quad \rho_{1}^{f}$ is the density of inner fluid, $B_{1}^{f}$ acoustic bulk modulus of inner fluid. If $\left(\delta_{1} a\right)<0$, the Bessel function $J_{n}$ in (19) is to be replaced by modified Bessel function $I_{n}$. Similarly, for the outer fluid that represents the oscillatory waves propagating away is given as
$\phi_{n}^{f}=B_{4 n} H_{n}^{(2)}\left(\delta_{2} a x\right)$
Where $\quad\left(\delta_{2} a\right)^{2}=\Omega^{2} / \bar{\rho}_{2}^{f} \bar{B}_{2}^{f}-\varsigma^{2}, \quad$ in $\quad$ which $\bar{\rho}_{2}^{f}=\rho_{2} / \rho^{f}, \bar{B}_{2}^{f}=B_{2}^{f} / \mu, \quad \rho_{2}^{f}$ is the density of outer fluid, $B_{2}^{f}$ acoustic bulk modulus of outer fluid. $H_{n}^{(2)}$ is the Hankel function of the second kind. If $\left(\delta_{2} a\right)^{2}<0$, then the Hankel function of second kind is to be replaced by $K_{n}$, where
$K_{n}$ is the modified Bessel function of the second kind. By substituting Eq.(17) in (14) along with (19) and (20), the acoustic pressure for the inner fluid can be expressed as
$p_{1}^{f}=A_{4 n} \Omega^{2} \bar{\rho}_{1} J_{n}\left(\delta_{1} a x\right) \cos n \theta e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}$
and for the outer fluid is
$p_{2}^{f}=B_{4 n} \Omega^{2} \bar{\rho}_{2} H_{n}^{(2)}\left(\delta_{2} a x\right) \cos n \theta e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}$
In the case of antisymmetric mode, the solutions for the inner and outer fluid are obtained by replacing $\cos n \theta$ by $\sin n \theta$ in the Eqs. (21) and (22).

## Boundary conditions and frequency equations

The frequency equation of the system is developed by coupling the cylinder to the fluid through the boundary conditions at the cylinder inner and outer surfaces. This approach is used to deal with the solid-fluid interfaces. The boundary conditions for the inner and outer boundaries of an infinite cylinder are obtained as follows:
For the inner boundary, the boundary conditions are
$\left(\sigma_{p p}+p_{1}^{f}\right)_{i}=\left(\sigma_{p q}\right)_{i}=\left(\sigma_{z p}\right)_{i}=\left(u-(\underline{u})^{f}\right)_{i}=0$
and for the outer boundary, the boundary conditions are
$\left(\sigma_{p p}+p_{2}^{f}\right)_{i}=\left(\sigma_{p q}\right)_{i}=\left(\sigma_{z p}\right)_{i}=\left(u+\mathcal{V} \bar{D}^{f}\right)_{i}=0$
where $p$ is the coordinate normal to the boundary and $q$ is the coordinate in the tangential direction, $\sigma_{p p}$ is the normal stress, $\sigma_{\mathrm{pq}}$ and $\sigma_{\mathrm{zp}}$ are the shearing stresses and ()$_{i}$ is the value at the $i-t h$ segment of the boundary. The first and last conditions in Eqs. (23) and (24) are due to the continuity ${ }^{(16)}$ of the stresses and displacements of the cylinder and fluid on the curved surfaces. Since the boundary of the cross section is iffegular in shape, it is difficult to satisfy the boundary conditions along both inner and outer surfaces of the cylinder directly. Hence, to satisfy the boundary conditions, the Fourier expansion collocation method due to Nagaya (1982, 1983, 1984 and 1985) is applied. If $\gamma_{i}$ is the angle between normal to the segment and the reference axis is assumed to be constant, then the transformed expres sions for the stresses are
(19)

$$
\begin{align*}
\sigma_{x r}= & \left(c_{11} \cos ^{2}\left(\theta-\gamma_{i}\right)+c_{12} \sin ^{2}\left(\theta-\gamma_{i}\right)\right) u_{r, r}+r^{-1}\left(c_{11} \sin ^{2}\left(\theta-\gamma_{i}\right)+c_{12} \cos ^{2}\left(\theta-\gamma_{i}\right)\right)\left(u_{r}+u_{\theta, \theta}\right)  \tag{25a}\\
& +c_{66}\left(r^{-1}\left(u_{\theta}-u_{r, \theta}\right)-u_{\theta}, r\right) \sin 2\left(\theta-\gamma_{i}\right)+c_{13} u_{z, z} \\
\sigma_{x y}= & c_{66}\left(\left(u_{r, r}-r^{-1}\left(u_{\theta, \theta}+u_{r}\right)\right) \sin 2\left(\theta-\gamma_{i}\right)+\left(r^{-1}\left(u_{r, \theta}-u_{\theta}\right)+u_{\theta, r}\right) \cos 2\left(\theta-\gamma_{i}\right)\right)  \tag{25b}\\
\sigma_{z x} & =c_{44}\left(\left(u_{r, z}+u_{z, r}\right) \cos \left(\theta-\gamma_{i}\right)-\left(u_{\theta, z}+r^{-1} u_{z, \theta}\right) \sin \left(\theta-\gamma_{i}\right)\right) . \tag{25c}
\end{align*}
$$

Substituting equations (5), (10), (13), (21) and (22) in the boundary conditions (23) and (24), the boundary conditions for the inner surface are transformed as follows :
$\left[\left(S_{x x}^{1}\right)_{i}+\left(\bar{S}_{x x}^{1}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0$,
$\left[\left(S_{x y}^{1}\right)_{i}+\left(\bar{S}_{x y}^{1}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0$
$\left[\left(S_{z x}^{1}\right)_{i}+\left(\bar{S}_{z x}^{1}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0$,
$\left[\left(S_{r}^{1}\right)_{i}+\left(\bar{S}_{r}^{1}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0$
and for the outer surface

$$
\begin{align*}
& {\left[\left(S_{x x}\right)_{i}+\left(\bar{S}_{x x}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0,} \\
& {\left[\left(S_{x y}\right)_{i}+\left(\bar{S}_{x y}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0} \\
& {\left[\left(S_{z x}\right)_{i}+\left(\bar{S}_{z x}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0,} \\
& {\left[\left(S_{r}\right)_{i}+\left(\bar{S}_{r}\right)_{i}\right] e^{i\left(\varsigma \bar{z}+\Omega T_{a}\right)}=0} \tag{27}
\end{align*}
$$

where

$$
\begin{align*}
S_{x x}^{1}= & 0.5\left(e_{0}^{1} A_{10}+e_{0}^{2} B_{10}+e_{0}^{3} A_{20}+e_{0}^{4} B_{20}+e_{0}^{7} B_{30}\right) \\
& +\sum_{n=1}^{\infty}\left(e_{n}^{1} A_{1 n}+e_{n}^{2} B_{1 n}+e_{n}^{3} A_{2 n}+e_{n}^{4} B_{2 n}+e_{n}^{5} A_{3 n}+e_{n}^{6} B_{3 n}+e_{n}^{9} B_{4 n}\right)  \tag{28a}\\
S_{x y}^{1}= & 0.5\left(f_{0}^{1} A_{10}+f_{0}^{2} B_{10}+f_{0}^{3} A_{20}+f_{0}^{4} B_{20}\right) \\
& +\sum_{n=1}^{\infty}\left(f_{n}^{1} A_{1 n}+f_{n}^{2} B_{1 n}+f_{n}^{3} A_{2 n}+f_{n}^{4} B_{2 n}+f_{n}^{5} A_{3 n}+f_{n}^{6} B_{3 n}\right)  \tag{28b}\\
S_{x 2}^{1}= & 0.5\left(g_{0}^{1} A_{10}+g_{0}^{2} B_{10}+g_{0}^{3} A_{20}+g_{0}^{4} B_{20}\right) \\
& +\sum_{n=1}^{\infty}\left(g_{n}^{1} A_{1 n}+g_{n}^{2} B_{1 n}+g_{n}^{3} A_{2 n}+g_{n}^{4} B_{2 n}+g_{n}^{5} A_{3 n}+g_{n}^{6} B_{3 n}\right)  \tag{28c}\\
S_{r}^{1}= & 0.5\left(h_{0}^{1} A_{10}+h_{0}^{2} B_{10}+h_{0}^{3} A_{20}+h_{0}^{4} B_{20}+h_{0}^{7} A_{40}\right) \\
& +\sum_{n=1}^{\infty}\left(g_{n}^{1} A_{1 n}+g_{n}^{2} B_{1 n}+g_{n}^{3} A_{2 n}+g_{n}^{4} B_{2 n}+g_{n}^{5} A_{3 n}+g_{n}^{6} B_{3 n}+h_{n}^{7} A_{4 n}\right)  \tag{28d}\\
S_{x x}= & 0.5\left(e_{0}^{1} A_{10}+e_{0}^{2} B_{10}+e_{0}^{3} A_{20}+e_{0}^{4} B_{20}+e_{0}^{8} B_{30}\right) \\
& +\sum_{n=1}^{\infty}\left(e_{n}^{1} A_{1 n}+e_{n}^{2} B_{1 n}+e_{n}^{3} A_{2 n}+e_{n}^{4} B_{2 n}+e_{n}^{5} A_{3 n}+e_{n}^{6} B_{3 n}+e_{n}^{8} B_{4 n}\right)  \tag{29a}\\
S_{x y}= & 0.5\left(f_{0}^{1} A_{10}+f_{0}^{2} B_{10}+f_{0}^{3} A_{20}+f_{0}^{4} B_{20}\right) \\
& +\sum_{n=1}^{\infty}\left(f_{n}^{1} A_{1 n}+f_{n}^{2} B_{1 n}+f_{n}^{3} A_{2 n}+f_{n}^{4} B_{2 n}+f_{n}^{5} A_{3 n}+f_{n}^{6} B_{3 n}\right)  \tag{29b}\\
S_{x 2}= & 0.5\left(g_{0}^{1} A_{10}+g_{0}^{2} B_{10}+g_{0}^{3} A_{20}+g_{0}^{4} B_{20}\right) \\
& +\sum_{n=1}^{\infty}\left(g_{n}^{1} A_{1 n}+g_{n}^{2} B_{1 n}+g_{n}^{3} A_{2 n}+g_{n}^{4} B_{2 n}+g_{n}^{5} A_{3 n}+g_{n}^{6} B_{3 n}\right)  \tag{29c}\\
S_{r}^{1}= & 0.5\left(h_{0}^{1} A_{10}+h_{0}^{2} B_{10}+h_{0}^{3} A_{20}+h_{0}^{4} B_{20}+h_{0}^{8} A_{40}\right) \\
& +\sum_{n=1}^{\infty}\left(g_{n}^{1} A_{1 n}+g_{n}^{2} B_{1 n}+g_{n}^{3} A_{2 n}+g_{n}^{4} B_{2 n}+g_{n}^{5} A_{3 n}+g_{3}^{6} B_{3 n}+h_{n}^{8} A_{4 n}\right)
\end{align*}
$$

$$
\begin{equation*}
\bar{S}_{x x}^{1}=0.5\left(e_{(-50}^{-5} \bar{A}_{30}+e_{0}^{-6} \bar{B}_{30}\right) \tag{29d}
\end{equation*}
$$

$$
\begin{equation*}
+\sum_{n=1}^{\infty}\left(e_{n}^{-1} \bar{A}_{I_{n 1}}-e_{n}^{-2} \bar{B}_{1 n}+e_{n}^{-3} \bar{A}_{2 n}+e_{n}^{-4} \bar{B}_{2 n}+e_{n}^{-5} \bar{A}_{3 n}+e_{n}^{-6} \bar{B}_{3 n}+e_{n}^{-7} \bar{A}_{s_{n}}\right) \tag{30a}
\end{equation*}
$$

$$
\bar{S}_{x y}^{1}=0.5\left(\bar{f}_{0}^{5} \bar{A}_{30}+\bar{f}_{0}^{6} \bar{B}_{30}\right)
$$

$$
\begin{equation*}
+\sum_{n=1}^{\infty}\left(\bar{f}_{n}^{1}-\bar{A}_{1 n}+\bar{f}_{n}^{2} \bar{B}_{1 n}+\bar{f}_{n}^{3} \bar{A}_{2 n}+\bar{f}_{n}^{4} \bar{B}_{2 n}+\bar{f}_{n}^{5} \bar{A}_{3 n}+\bar{f}_{n}^{6} \bar{B}_{3 n}\right) \tag{30b}
\end{equation*}
$$

$$
\begin{align*}
\bar{S}_{x 2}^{1}= & 0.5\left(\bar{g}_{0}^{5} \bar{A}_{30}+\bar{g}_{0}^{6} \bar{B}_{30}\right) \\
& +\sum_{n=1}^{\infty}\left(\bar{g}_{n}^{1} \bar{A}_{1 n}+\bar{g}_{n}^{2} \bar{B}_{1 n}+\bar{g}_{n}^{3} \bar{A}_{2 n}+\bar{g}_{n}^{4} \bar{B}_{2 n}+\bar{g}_{n}^{-5} \bar{A}_{3 n}+\bar{g}_{n}^{6} \bar{B}_{3 n}\right)  \tag{30c}\\
\bar{S}_{r}^{1}= & 0.5\left(\bar{h}_{0}^{5} \bar{A}_{30}+\bar{h}_{0}^{6} \bar{B}_{30}\right) \\
+ & \sum_{n=1}^{\infty}\left(\bar{h}_{n}^{1} \bar{A}_{1 n}+\bar{h}_{n}^{2} \bar{B}_{1 n}+\bar{h}_{n}^{3} \bar{A}_{2 n}+\bar{h}_{n}^{4} \bar{B}_{2 n}+\bar{h}_{n}^{5} \bar{A}_{3 n}+\bar{h}_{n}^{6} \bar{B}_{3 n}+\bar{h}_{n}^{7} \bar{A}_{5 n}\right) \tag{30d}
\end{align*}
$$

$\bar{S}_{x x}=0.5\left(\bar{e}_{0}^{-5} \bar{A}_{30}+e_{0}^{-6} \bar{B}_{30}\right)$

$$
\begin{equation*}
+\sum_{n=1}^{\infty}\left(e_{n}^{-1} \bar{e}_{1 n}+e_{n}^{-2} B_{1 n}+e_{n}^{-3} \bar{A}_{2 n}+e_{n}^{-4} \bar{B}_{2 n}+e_{n} A_{3 n}+e_{n}^{-6} \bar{B}_{3 n}+e_{n}^{-8} \bar{B}_{5 n}\right) \tag{31a}
\end{equation*}
$$

$$
\bar{S}_{x y}=0.5\left(\bar{f}_{0}^{5} \bar{A}_{30}+\bar{f}_{0}^{6} \bar{B}_{30}\right)
$$

$$
\begin{equation*}
+\sum_{n=1}^{\infty}\left(\bar{f}_{n}^{1} \bar{A}_{1 n}+\bar{f}_{n}^{2} \bar{B}_{1 n}+\bar{f}_{n}^{3} \bar{A}_{2 n}+\bar{f}_{n}^{4} \bar{B}_{2 n}+\bar{f}_{n}^{5} \bar{A}_{3 n}+\bar{f}_{n}^{6} \bar{B}_{3 n}\right) \tag{31b}
\end{equation*}
$$

$$
\bar{S}_{x z}=0.5\left(\bar{g}_{0}^{5} \bar{A}_{30}+\bar{g}_{0}^{6} \bar{B}_{30}\right)
$$

$$
\begin{equation*}
+\sum_{n=1}^{\infty}\left(\bar{g}_{n}^{1} \bar{A}_{1 n}+\bar{g}_{n}^{2} \bar{B}_{1 n}+\bar{g}_{n}^{3} \bar{A}_{2 n}+\bar{g}_{n}^{4} \bar{B}_{2 n}+\bar{g}_{n}^{5} A_{3 n}+\bar{g}_{n}^{6} B_{3 n}\right) \tag{31c}
\end{equation*}
$$

$$
\bar{S}_{r}=0.5\left(\bar{h}_{0}^{5} \bar{A}_{30}+\bar{h}_{0}^{6} \bar{B}_{30}\right)
$$

$$
\begin{equation*}
+\sum_{n=1}^{\infty}\left(\bar{h}_{n}^{1} \bar{A}_{1 n}+\bar{h}_{n}^{2} \bar{B}_{1 n}+\bar{h}_{n}^{3} \bar{A}_{2 n}+\bar{h}_{n}^{4} \bar{B}_{2 n}+\bar{h}_{n}^{5} \bar{A}_{3 n}+\bar{h}_{n}^{6} \bar{B}_{3 n}+\bar{h}_{n}^{8} \bar{B}_{5 n}\right) \tag{31d}
\end{equation*}
$$

The equations for $e_{n}^{1} \sim \bar{h}_{n}^{8}$ are given in Appendix A. The boundary conditions along both the inner and outer arbitrary surface cannot be satisfied directly. Therefore, performing the Fourier series expansion to (23) and (24) along the boundary, the boundary conditions are expanded in the form of double Fourier series. In the symmetric mode, the necessary boundary conditions for the inner surface are obtained as
$\sum_{m=0}^{\infty} \varepsilon_{m}\left[E_{m 0}^{1} A_{10}+\mathbb{E}_{m 0}^{2} B_{10}+\mathbb{E}_{m 0}^{3} A_{20}+\mathcal{E}_{m 0}^{4} B_{20}+\mathbb{E}_{m 0}^{7} A_{50}\right.$
$\left.+\sum_{n=1}^{\infty}\left(\mathbb{E}_{m m}^{1} A_{1 n}+\mathbb{E}_{m m}^{2} B_{1 n}+\mathbb{E}_{m m}^{3} A_{2 n}+\mathbb{E}_{m n}^{4} B_{2 n}+\mathbb{E}_{m m}^{5} A_{3 n}+\mathbb{E}_{m m}^{6} B_{3 n}+\mathbb{E}_{m m}^{7} A_{5 n}\right)\right] \cos m \theta=0$
$\sum_{m=1}^{\infty}\left[\mathcal{F}_{m 0}^{1} A_{10}+\mu_{m 0}^{2} B_{10}+\mu_{m 0}^{3} A_{20}+\mu_{m 0}^{4} B_{20}\right.$
$\left.+\sum_{n=1}^{\infty}\left(\mathcal{F}_{m n}^{1} A_{1 n}+\mathcal{F}_{m n}^{2} B_{l n}+\mathcal{F}_{m n}^{3} A_{2 n}+\mathcal{F}_{m m}^{4} B_{2 n}+\mu_{n m}^{5} A_{3 n}+\mathcal{F}_{m m}^{6} B_{3 n}\right)\right] \sin m \theta=0$

$\left.+\sum_{n=1}^{\infty}\left(\bigotimes_{m m}^{1} A_{1 n}+\mathbb{G}_{m n}^{2} B_{1 n}+\mathfrak{G}_{m n}^{3} A_{2 n}+\mathbb{\bigotimes}_{m n}^{4} B_{2 n}+\bigotimes_{m n}^{5} A_{3 n}+\bigotimes_{m n}^{6} B_{3 n}\right)\right] \cos m \theta=0$
$\sum_{m=0}^{\infty} \varepsilon_{m}\left[\boldsymbol{H}_{m 0}^{1} A_{10}+\boldsymbol{H}_{m 0}^{2} B_{10}+\boldsymbol{H}_{m 0}^{3} A_{20}+\boldsymbol{H}_{m 0}^{4} B_{20}+\boldsymbol{H}_{m 0}^{7} A_{50}\right.$
$+\sum_{n=1}^{\infty}\left(\boldsymbol{H}_{m n}^{1} A_{1 n}+\boldsymbol{H}_{m m}^{2} B_{1 n}+\boldsymbol{H}_{m m}^{3} A_{2 n}+\boldsymbol{\mu}_{m m}^{4} B_{2 n}+\boldsymbol{\mu}_{m m}^{5} A_{3 n}+\boldsymbol{H}_{m n}^{6} B_{3 n}+\boldsymbol{H}_{m m}^{7} A_{5 n}\right) \cos m \theta=0$ and for the outer surface
$\sum_{m=0}^{\infty} \varepsilon_{m}\left[E_{m 0}^{1} A_{10}+E_{m 0}^{2} B_{10}+E_{m 0}^{3} A_{20}+E_{m 0}^{4} B_{20}+E_{m 0}^{8} B_{50}\right.$ $\left.+\sum_{n=1}^{\infty}\left(E_{m n}^{1} A_{l n}+E_{m n}^{2} B_{1 n}+E_{m n}^{3} A_{2 n}+E_{m n}^{4} B_{2 n}+E_{m n}^{5} A_{3 n}+E_{m n}^{6} B_{3 n}+E_{m n}^{8} B_{5 n}\right)\right] \cos m \theta=0$
$\sum_{m=1}^{\infty}\left[F_{m 0}^{1} A_{10}+F_{m 0}^{2} B_{10}+F_{m 0}^{3} A_{20}+F_{m 0}^{4} B_{20}+\right.$
$\left.+\sum_{n=1}^{\infty}\left(F_{m n}^{1} A_{1 n}+F_{m n}^{2} B_{1 n}+F_{n n}^{3} A_{2 n}+F_{m n}^{4} B_{2 n}+F_{m n}^{5} A_{3 n}+F_{m n}^{6} B_{3 n}\right)\right] \sin m \theta=0$
$\sum_{m=0}^{\infty} \varepsilon_{m}\left[G_{m 0}^{1} A_{10}+G_{m 0}^{2} B_{10}+G_{m 0}^{3} A_{20}+G_{m 0}^{4} B_{20}\right.$
$\left.+\sum_{n=1}^{\infty}\left(G_{m n}^{1} A_{1 n}+G_{m n}^{2} B_{1 n}+G_{m n}^{3} A_{2 n}+G_{m n}^{4} B_{2 n}+G_{n n}^{5} A_{3 n}+G_{n n}^{6} B_{3 n}\right)\right] \cos m \theta=0$
$\sum_{m=0}^{\infty} \varepsilon_{m}\left[H_{m 0}^{1} A_{10}+H_{m 0}^{2} B_{10}+H_{m 0}^{3} A_{20}+H_{m 0}^{4} B_{20}+H_{m 0}^{8} B_{50}\right.$
$\left.+\sum_{n=1}^{\infty}\left(H_{m n}^{1} A_{1 n}+H_{m n}^{2} B_{1 n}+H_{m n}^{3} A_{2 n}+H_{m n}^{4} B_{2 n}+H_{m n}^{5} A_{3 n}+H_{m n}^{6} B_{3 n}+H_{m n}^{8} B_{5 n}\right)\right] \cos m \theta=0$
Similarly, for the anti symmetric mode, the boundary conditions for the inner surface are
$\sum_{m=1}^{\infty}\left[\bar{E}_{m 0}^{-5} \bar{A}_{30}+\bar{E}_{m 0}^{-6} \bar{B}_{30}\right.$

$\sum_{m=0}^{\infty}\left[F_{m 0}^{-5} \bar{A}_{30}+\bar{F}_{m 0}^{-6} \bar{B}_{30}\right.$
$+\sum_{n=1}^{\infty}\left(\overline{\mathcal{H}}_{n m}^{1} \bar{A}_{1 n}+\overline{\mathcal{H}}_{m n}^{2} \bar{B}_{1 n}+\bar{\mu}_{n n}^{3} \bar{A}_{2 n}+\overline{\mathcal{F}}_{m B}^{4} \bar{B}_{2 n}+\bar{\mu}_{n m}^{5} \bar{A}_{3 n}+\overline{\mathcal{H}}_{n m}^{6} \bar{B}_{3 n}\right) \cos m \theta=0$
$\sum_{m=1}^{\infty}\left[\bar{G}_{m 0}^{-5} \bar{A}_{30}+\bar{G}_{m 0}^{-6} \bar{B}_{30}\right.$
$+\sum_{n=1}^{\infty}\left(\overline{\boldsymbol{\oiint}}_{m n} \bar{A}_{1 n}+\bar{\oiint}_{m n}^{2} \bar{B}_{1 n}+\bar{\oiint}_{m n}^{3} \bar{A}_{2 n}+\bar{\oiint}_{m n}^{4} \bar{B}_{2 n}+\bar{\oiint}_{m n}^{5} \bar{A}_{3 n}+\bar{\oiint}_{m n}^{6} \bar{B}_{3 n}\right) \sin m \theta=0$
$\sum_{m=1}^{\infty}\left[\overline{\boldsymbol{H}}_{n 0}^{5} \bar{A}_{30}+\bar{H}_{m 0}^{6} \bar{B}_{30}\right.$
$+\sum_{n=1}^{\infty}\left(\overline{\boldsymbol{H}}_{m m}^{1} \boldsymbol{A}_{1 n}+\overline{\boldsymbol{H}}_{m m}^{2} \bar{B}_{1 n}+\overline{\boldsymbol{H}}_{m m}^{3} \bar{A}_{2 n}+\overline{\boldsymbol{H}}_{n m}^{4} \bar{B}_{2 n}+\overline{\boldsymbol{H}}_{n m}^{5} \bar{A}_{3 n}+\overline{\boldsymbol{H}}_{n m}^{6} \bar{B}_{3 n}+\overline{\boldsymbol{H}}_{m m}^{\prime} \bar{A}_{5 n}\right) \sin m \theta=0$
and for the outer surface are
$\sum_{m=1}^{\infty}\left[\bar{E}_{m 0}^{5} \bar{A}_{30}+\bar{E}_{m 0}^{6} \bar{B}_{30}\right.$
$\left.+\sum_{n=1}^{\infty}\left(\bar{E}_{m m}^{1} \bar{A}_{1 n}+\bar{E}_{m m}^{2} \bar{B}_{1 n}+\bar{E}_{m m}^{3} \bar{A}_{2 n}+\bar{E}_{m m}^{4} \bar{B}_{2 n}+\bar{E}_{m n}^{5} \bar{A}_{3 n}+\bar{E}_{m n}^{6} \bar{B}_{3 n}+\bar{E}_{m n}^{8} \bar{B}_{5 n}\right)\right) \sin m \theta=0$
$\sum_{m=0}^{\infty}\left[\bar{F}_{m 0}^{5} \bar{A}_{30}+\bar{F}_{m 0}^{6} \bar{B}_{30}\right.$
$\left.+\sum_{n=1}^{\infty}\left(\bar{F}_{n n}^{1} \bar{A}_{1 n}+\bar{F}_{m n}^{2} \bar{B}_{1 n}+\bar{F}_{m n}^{3} \bar{A}_{2 n}+\bar{F}_{m n}^{4} \bar{B}_{2 n}+\bar{F}_{m n}^{5} \bar{A}_{3 n}+\bar{F}_{m n}^{6} \bar{B}_{3 n}\right)\right] \cos m \theta=0$
$\sum_{m=1}^{\infty}\left[\bar{G}_{m 0}^{5} \bar{A}_{30}+\bar{G}_{m 0}^{6} \bar{B}_{30}\right.$
$\left.+\sum_{n=1}^{\infty}\left(\bar{G}_{m n}^{1} \bar{A}_{1 n}+\bar{G}_{m n}^{2} \bar{B}_{1 n}+\bar{G}_{n n}^{3} \bar{A}_{2 n}+\bar{G}_{m n}^{4} \bar{B}_{2 n}+\bar{G}_{m n}^{5} \bar{A}_{3 n}+\bar{G}_{m n}^{6} \bar{B}_{3 n}\right)\right] \sin m \theta=0$
$\sum_{m=1}^{\infty}\left[\bar{H}_{m 0}^{5} \bar{A}_{30}+\bar{H}_{m 0}^{6} \bar{B}_{30}\right.$
$\left.+\sum_{n=1}^{\infty}\left(\bar{H}_{m n}^{1}-\bar{A}_{1 n}+\bar{H}_{m B}^{2} \bar{B}_{1 n}+\bar{H}_{m n}^{3} \bar{A}_{2 n}+\bar{H}_{m n}^{4} \bar{B}_{2 n}+\bar{H}_{m n}^{5} \bar{A}_{3 n}+\bar{H}_{m n}^{6} \bar{B}_{3 n}+\bar{H}_{m n}^{8} \bar{B}_{3 n}\right)\right] \sin m \theta=0$

The frequency equations are obtained from the inner and outer boundary conditions of the equations (32) and (33), for the symmetric mode, and for the antisymmetric mode, the frequency equations are obtained from the equations (34) and (35) by truncating the series to $\mathrm{N}+1$ terms, and equating the determinant of the coefficients of the amplitudes $A_{i n}, B_{i n}, \bar{A}_{i n}$ and $\bar{B}_{i n}$ $(i=1,2,3,4)$ to zero. Thus the frequency equation for the symmetric mode is obtained as


Similarly, the frequency equation for antisymmetric mode of vibration is given by

where
$\boldsymbol{E}_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} e_{n}^{j}\left(\boldsymbol{R}_{i}, \theta\right) \cos m \theta d \theta$,
$\mathbf{F}_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} f_{n}^{j}\left(\mathbf{K}_{i}, \theta\right) \sin m \theta d \theta$
$\boldsymbol{G}_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} g_{n}^{j}\left(\boldsymbol{R}_{i}, \theta\right) \cos m \theta d \theta$,
$\boldsymbol{H}_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} h_{n}^{j}\left(\boldsymbol{K}_{i}, \theta\right) \cos m \theta d \theta$
$E_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} e_{n}^{j}\left(R_{i}, \theta\right) \cos m \theta d \theta$,
$F_{n n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{l}} f_{n}^{j}\left(R_{i}, \theta\right) \sin m \theta d \theta$
$G_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} g_{n}^{j}\left(R_{i}, \theta\right) \cos m \theta d \theta$,
$H_{m n}^{j}=\left(2 \varepsilon_{n} / \pi\right) \sum_{i=1}^{I} \int_{\theta_{i-1}}^{\theta_{i}} h_{n}^{j}\left(R_{i}, \theta\right) \cos m \theta d \theta$
where $j=1,2,3,4,5,6,7$ and $8, \varepsilon_{m}=1 / 2$ for $m=0$ and $\varepsilon_{m}=1$ for $m \geq 0, I$ is the number of segments, $\boldsymbol{R}_{i}^{\mathbf{l}}$ is the coordinate $r$ at the inner boundary, and $R_{i}$ is the coordinate $r$ at the outer boundary. The equations for $\overline{\boldsymbol{A}}_{m n}^{j} \sim \overline{\boldsymbol{M}}_{m n}^{j}$ can be obtained by replacing $\cos n \theta$ by $\sin n \theta$ and $\sin n \theta$ by $\cos n \theta$ in Eqs.(38) and (39).

## Particular case

For an isotropic materials $c_{11}=c_{33}=\lambda+2 \mu, c_{44}=c_{13}=\lambda$, $c_{12}=c_{13}$ and $c_{66}=\left(c_{11}-c_{12}\right) / 2$, where $\lambda$ and $\mu$ are Lames, constants. Using the values in various relevant relations and equations, the preceding analysis will be reduced to free vibrations of an isotropic hollow cylinder of both inner and outer arbitrary cross section immersed in fluid.

## Numerical results and discussion

In order to illustrate the nature and general behavior of the solution, some numerical examples are considered in this section. The resulting frequency equations of the symmetric and antisymmetric cases of the cylinder of general cross section immersed in a fluid is given in (36) and (37) are transcendental in nature with respect to the dimensionless frequency $\Omega$ and dimensionless wavenumber $\varsigma$. The analysis is carried out for elliptic, cardioid cross sections by fixing the dimensionless frequencies $\Omega$ and the dimensionless complex wavenumbers $\varsigma$ are obtained. The computation of cylindrical Bessel functions of complex arguments are performed using the method provided by Zhang and Jin (1996). The computation of Fourier coefficients given in (38) and (39) are carried out using the five point Gaussian quadrature. To obtain the roots of the frequency equation, the secant method applicable for the complex roots (Antia (2002) ) is employed. The material chosen for the numerical calculation is Zinc, its properties are as follows: for the solid the elastic constants are $c_{11}=1.628 \times 10^{11} \mathrm{Nm}^{-2}, c_{12}=0.362 \times 10^{11} \mathrm{Nm}^{-2}$,
$c_{13}=0.508 \times 10^{11} \mathrm{Nm}^{-2}, \quad c_{33}=0.627 \times 10^{11} \mathrm{Nm}^{-2}$, $c_{44}=0.385 \times 10^{11} \mathrm{Nm}^{-2}$ and density $\rho=7.14 \times 10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$ and for the fluid the density $\rho^{f}=1000 \mathrm{~kg} \mathrm{~m}^{-3}$ and phase velocity $c=1500 \mathrm{~ms}^{-1}$. The same fluid is assumed to be inside and outside the cylinder for the numerical calculations.

In the present problem, three kinds of basic independent modes of wave propagation have been considered, namely, the longitudinal and two flexural (symmetric and antisymmetric) modes for geometries having more than one symmetry. For geometries having only one symmetry, two modes of wave propagations are studied since the two flexural modes are coupled in this case.

## Eliptic cross-section with elliptic cavity

The geometrical relations of an elliptic cross-sectional cylinder given by Nagaya (1983) and are used for numerical calculation and are given below:
$R_{l} / b_{1}=\left(a_{2} / b_{1}\right) /\left[\cos ^{2} \theta+\left(a_{2} / b_{2}\right)^{2} \sin ^{2} \theta\right]^{1 / 2}$
$\gamma_{l}=\pi / 2-\tan ^{-1}\left[\left(b_{2} / a_{2}\right)^{2} / \tan \theta_{l}^{*}\right]$, for $\theta_{l}^{*}<\pi / 2$
$\gamma_{l}=\pi / 2, \theta_{l}^{*}=\pi / 2$
$\gamma_{l}=\pi / 2+\tan ^{-1}\left[\left(b_{2} / a_{2}\right)^{2} /\left|\tan \theta_{l}^{*}\right|\right]$, for $\theta_{l}^{*}>\pi / 2$
for the outer surface and
$R_{l} / b_{1}=\left(a_{1} / b_{1}\right) /\left[\cos ^{2} \theta+\left(a_{1} / b_{1}\right)^{2} \sin ^{2} \theta\right]^{1 / 2}$
$\gamma_{l}=\pi / 2-\tan ^{-1}\left[\left(b_{1} / a_{1}\right)^{2} / \tan \theta_{l}^{*}\right]$, for $\theta_{l}^{*}<\pi / 2$
$\gamma_{l}=\pi / 2, \theta_{l}^{*}=\pi / 2$
$\gamma_{l}=\pi / 2+\tan ^{-1}\left[\left(b_{1} / a_{1}\right)^{2} /\left|\tan \theta_{l}^{*}\right|\right]$, for $\theta_{l}^{*}>\pi / 2$
for the inner surface, where $a_{1}$ and $a_{2}$ are the length of inner and outer semi major axis, and $b_{1}$ and $b_{2}$ are the length of semi minor axis of an elliptic cross section. Also $\theta_{l}^{*}=\left(\theta_{l}+\theta_{l-1}\right) / 2$ and $R_{l}$ is the coordinate $r$ at the boundary $\Gamma_{l}, \gamma_{l}$ is the angle between the reference axis and the normal to the segment.

The Eq. given in Eq. ( ) are used directly for the frequency analysis, and three kinds of basic independent modes of wave propagation are studied. In case of the longitudinal mode of elliptical cross section, the cross section vibrates along the axis of the cylinder, so that the vibration and displacements in the cross section is symmetrical about both major and minor axes. Hence, the frequency equation is obtained by choosing both terms of $n$ and $m$ as $0,2,4,6, \ldots$ in (36) for the numerical calculations. In this method, the boundary in the range $\theta=0$ and $\theta=\pi$ is divided into 20 segments, such that distance between any two segments is negligible and integration is performed for each segment numerically by using the five point Gaussian quadrature. The nondimensional complex wavenumbers are computed for $0<\Omega \leq 1.0$, for different aspect ratios by fixing the frequency $\Omega$, using the secant method applicable for the complex roots.

In the case of flexural mode of elliptical cross section, the vibration and displacements are antisymmetrical about the major axis and symmetrical about the minor axis. Hence, the frequency equations may be obtained from (37) by choosing $n, m=1,3,5, \ldots$. Two kinds of flexural (symmetric and antisymmetric) modes are considered in the case of elliptic hollow cylinder immersed in a fluid.

## Dispersion curves

The results of longitudinal and flexural (symmetric and antisymmetric) modes are plotted in figures. The notations Lm, Fs and Fa represents the longitudinal mode, flexural (symmetric and antisymmetric) modes respectively. Similarly Re and Im represent the real and imaginary modes in the order. The $1,2,3$ and 4 refers to the first, second, third and forth modes respectively. Fig.1, shows, the dispersion curve drawn between non-dimensional $\delta / \pi$ versus $\Omega / \pi$ for a longitudinal modes of
transversely isotropic fluid-loaded cylinder. From the Fig.1, it is observed that, the behaviour of present method coincides with the graph obtained by Berlinear and Solecki (1996). Similar comparison is made also with fluid-loaded transversely isotropic and isotropic circular cylinders. It is observed that both cases the dispersion behaviors are same.


I
Fig. 1 Comparision of author's method with M. J. Berliner and R. Solecki (1996). Dispersion curves for dimensionless versus dimensionless
A graph is drawn between non-dimensional frequency $\Omega$ versus dimensionless wave number $|\varsigma|$ for an elliptic cylinder for the longitudinal and flexural (symmetric and antisymmetric) modes of vibrations for the aspect ratios $a_{1} / b_{1}=a_{2} / b_{2}=1.5$ with thickness $b_{1} / b_{2}=0.5$ is shown in Fig.2. From the Fig.2, it is observed that as the non-dimensional frequency $\Omega$ increases, the dimensionless wave number $|\varsigma|$ are also increases.


Fig. 2 Non-dimensional frequency $\Omega$ versus dimensionless wavenumber $|\varsigma|$ for $a_{1} / b_{1}=a_{2} / b_{2}=1.5$ with thickness

$$
b_{1} / b_{2}=0.5
$$

The energy displacement is more in the first two modes of vibrations Lm1, Lm2 as compared to the other modes of vibrations Lm3 and Lm4. This is the proper physical behavior of a cylinder immersed in fluid.

A comparison made between the real and imaginary modes of vibration for longitudinal modes of elliptic cross-sectional cylinders for the aspect ratios $a_{1} / b_{1}=a_{2} / b_{2}=1.5$ with thickness $b_{1} / b_{2}=0.5$ is shown in Fig.3.


Fig. 3 Non-dimensional frequency $\Omega$ versus dimensionless real $(\varsigma)$ and imaginary $(\varsigma)$ of wavenumber for

$$
a_{1} / b_{1}=a_{2} / b_{2}=1.5 \text { with thickness } b_{1} / b_{2}=0.5
$$

From the Fig.3, it is observed that as the frequency increases, the wave number increases for the real part, while it decreases for the imaginary part. The Fig. 4, shows the nondimensional frequency $\Omega$ versus dimensionless wave number $|\varsigma|$ for an elliptic cylinder for the flexural (symmetric and antisymmetric) modes of vibrations for the aspect ratios $a_{1} / b_{1}=a_{2} / b_{2}=1.5$ with thickness $b_{1} / b_{2}=0.5$. From the Fig. 4, it is observed that, the first modes of flexural (symmetric and antisymmetric) modes, the energy transfer from solid into fluid is very high in comparing with the second and third modes of vibrations. This shows that at the point of contact between solid and the fluid, the displacement of particles is more than the second and third mode of vibrations.


Fig. 4 Non-dimensional frequency $\Omega$ versus dimensionless wavenumber $|\varsigma|$ for $a_{1} / b_{1}=a_{2} / b_{2}=1.5$ with thickness

## $b_{1} / b_{2}=0.5$ for flexural (symmetric and antisymmetric)

 modes of an elliptic cross-section.A dispersion curve is drawn between the non-dimensional frequencies $\Omega$ versus dimensionless wave number $|\varsigma|$ for the longitudinal modes of elliptic cross-sectional cylinder for different thickness $b_{1} / b_{2}=0.3,0.5,0.7,0.9,1.5$ and 2.0 is shown in Fig.5. It is observed that as the thickness of the cylinder is increased, with increasing the dimensionless frequency, the non-dimensional wave number is decreased. This is the proper physical behavior of a cylinder with increasing thickness.


Fig. 5 Non -dimensional frequency $\Omega$ versus dimensionless wavenumber $|\varsigma|$ for the longitudinal mode of vibration for different thickness of elliptic cylinder for the aspect ratio

$$
a_{1} / b_{1}=a_{2} / b_{2}=1.5
$$

## Conclusions

In this paper, the wave propagation in a cylinder of both inner and outer elliptic cross section immersed in fluid is analyzed by satisfying the boundary condition on the irregular boundary using the Fourier expansion collocation method and the frequency equation for the longitudinal and flexural vibrations are obtained. Numerically the frequency equations are analyzed for elliptic cross sectional cylinder immersed in fluid. The computed non-dimensional wave numbers are plotted for
the material Zinc. The results of circular section are compared with exact results and they show very good agreement. The method proposed in this paper can be used to analyze the vibration of a cylinder of any cross section with appropriate geometric relation.

## References

Achenbach, J. D., 1973. Wave Propagation in Elastic Solids, Elsevier Publishing Company Inc., New York.
Ahmad, F., 2001. Giuded waves in a transversely isotropic cylinder immersed in a Fluid, J. Acoust. Soc. Am. Vol. 109(3), 886-890.
Antia H.M., 2002. Numerical Methods for Scientists and Engineers, Hindustan Book Agency, New Delhi.
Berliner, J. and Solecki, R., 1996. Wave Propagation in a fluidloaded, transversely isotropic cylinders. Part I. Analytical formulation; Part II Numerical results, J. Acoust. Soc. Am. Vol. 99, 1841-1853.
Easwaran, V. and Munjal, M.L.,1995. A note on the effect of wall compliance on lowest-order mode propagation in fluidfilled/submerged impedance tubes, J. Acoust. Soc. Am. Vol. 97(6), 3494-3501.
Gazis, D.C., 1959. Three-dimensional investigation of the propagation of waves in hollow circular cylinders, I. Analytical Formulation; II Numerical Results, J. Acoust. Soc. Am. Vol.31, 568-578.
Mirsky, I., 1964. Wave propagation in transversely isotropic circular cylinders, J. Acoust. Soc.Am. Vol. 36, 41-51.
Nagaya, K., 1982. Stress wave propagation in a bar of arbitrary cross section, J. Acoust. Soc. Am. Vol. 49,157-164, 1982.
Nagaya, K., 1983.Direct method on determination of eigenfrequencies of arbitrary shaped plates, ASME .J. Vib. Stress Rel. Des. Vol. 105, 132-136.
Nagaya, K., 1984. Wave propagation in an infinite long bar of arbitrary cross section and with a circular cylindrical cavity, J. Acoust. Soc. Am. Vol. 75(3), 834-841.
Nagaya, K., 1985. Wave propagation in a rod with an arbitrary shaped outer boundary and a cylindrical cavity of arbitrary shape, J. Acoust. Soc. Am. Vol.77(5), 1824-1833.
Nagy, B., 1995. Longitudinal guided wave propagation in a transversely isotropic rod Immersed in a fluid, J. Acoust. Soc. Am. Vol. 98(1), 454-457.
Nagy, B. and Nayfeh, H., 1996. Viscosity - induced attenuation of longitudinal guided Waves in a fluid loaded rods, J. Acoust. Soc. Am. Vol. 100(3), 1501-1508.
Paul, H. S. and Venkates an, M. 1989. M., Wave propagation in a piezoelectric ceramic cylinder of arbitrary cross section with a circular cylindrical cavity, J. Acoust. Soc. Am. Vol.85(1), 163170.

Sinha, K. Plona, J. Kostek, S. and Chang, S., 1992. Axisymmetric wave propagation in a fluid-loaded cylindrical shells. I: Theory; II Theory versus experiment, J. Acoust. Soc. Am. Vol. 92,1132-1155.
Venkatesan, M. and Ponnusamy, P., 2002. Wave propagation in a solid cylinder of arbitrary cross-section immersed in a fluid, J. Acoust. Soc. Am. Vol. 112, 936-942.
Venkatesan, M. and Ponnusamy, P., 2003. Wave propagation in a solid cylinder of polygonal cross section immersed in a fluid, Indian J. pure Appl. Math., Vol. 34(9),1381-1391.

Zhang S. and Jin J., 1996. Computation of Special Functions, A Wiley-International Publication, New York.

## Appendix $A$

$$
\begin{equation*}
f_{n}^{8}=0 \tag{A11}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}^{i}=\bar{c}_{H}\left(\xi+d_{i}\right)\left\{\cos \left(\overline{n-1 \theta}+\gamma_{i}\right) J_{n}\left(\alpha_{i}, \alpha x\right)-\left(\alpha_{i}\right)\right)_{n+1}\left(\alpha_{i}, \alpha\right) \cos \left(\theta-\gamma_{i}\right) \cos n \theta_{j}, i=1,2 \tag{A12}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}^{3}=\left\{\left\{n \cos \left(\overline{n-1} \theta+\gamma_{i}\right) J_{n}\left(\alpha_{3} a x\right)-\left(\alpha_{3} a\right) J_{n+1}\left(\alpha_{3} a x\right) \sin \left(\theta-\gamma_{i}\right) \sin n \theta\right\}\right. \tag{A13}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}^{i}=\bar{c}_{4}\left(\zeta+d_{i}\right)\left\{n \cos \left(\overline{n-1 \theta}\left(\gamma_{i}\right) J_{n}\left(\alpha_{i}, \alpha x\right)-\left(\alpha_{i}\right)\right) J_{n+1}\left(\alpha_{i}, \alpha x\right) \cos \left(\theta-\gamma_{i}\right) \cos n \theta\right), i=5,6 \tag{A14}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}^{7}=\left\{\left\{n \cos \left(\overline{n-1} \theta+\gamma_{i}\right) J_{n}\left(\alpha_{2} a x\right)-\left(\alpha_{r} a\right) J_{n+1}\left(\alpha_{7} a x\right) \sin \left(\theta-\gamma_{i}\right) \sin n \theta\right\}\right. \tag{A15}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}^{4}=0 \tag{A16}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}^{8}=0 \tag{A17}
\end{equation*}
$$

The quantities $\bar{e}_{n}^{i}: \bar{g}_{n}^{i}$ are obtained by replacing $\cos n \theta$ by $\sin n \theta$ and $\sin n \theta$ by $\cos n \theta$ in the relevant equations.

$$
\begin{align*}
& e_{n}^{i}=2 \bar{C} \sigma\left[\left\{n(n-1) J_{n}\left(\alpha_{i} a x\right)+\left(\alpha_{i} \alpha x\right) J_{n+1}\left(\alpha_{i} \alpha x\right)\right\} \cos 2\left(\theta-\gamma_{i}\right)\right. \\
& -\left\{\left(\alpha_{i} a\right)^{2}\left[\bar{c}_{11} \cos ^{2}\left(\theta-\gamma_{i}\right)+\bar{c}_{12} \sin ^{2}\left(\theta-\gamma_{i}\right)\right]+\bar{c}_{1 S} d_{i} d_{i} J_{n}\left(\alpha_{i} a x\right)\right] \cos n \theta  \tag{A1}\\
& +2 n \overline{c o s}\left\{(n-1) J_{n}\left(\alpha_{i} a x\right)-\left(\alpha_{i} a\right) J_{n+1}\left(\alpha_{i} a x\right)\right) \sin 2\left(\theta-\gamma_{i}\right) \sin n \theta, \quad i=1,2 \\
& e_{n}^{3}=2 n \overline{n_{6}}\left(\{n-1) J_{n}\left(\alpha_{3} a x\right)-\left(\alpha_{3} a\right) J_{n+1}\left(\alpha_{3} a x\right)\right) \cos 2\left(\theta-\gamma_{i}\right) \cos n \theta \\
& +\overline{C_{\sigma}}\left[2\left(\alpha_{3} a\right) J_{n+1}\left(\alpha_{3} a x\right)-\left\{\left(\alpha_{3} a\right)^{2}-2 n(n-1) J_{n}\left(\alpha_{3} a x\right)\right\}\right] \sin n \theta \sin 2\left(\theta-\gamma_{i}\right)  \tag{A2}\\
& e_{n}^{i}=\overline{2} \overline{\cos }\left\{\left\{n(n-1) Y_{n}\left(\alpha_{i} \alpha x\right)+\left(\alpha_{i} a x\right) Y_{n+1}\left(\alpha_{i} a x\right)\right\} \cos 2\left(\theta-\gamma_{i}\right)\right. \\
& \left.-\left\{\left(\alpha_{i}\right)\right)^{2}\left[\bar{c}_{11} \cos ^{2}\left(\theta-\gamma_{i}\right)+\bar{c}_{12} \sin ^{2}\left(\theta-\gamma_{i}\right)\right]+\bar{c}_{1 S} d_{i} d_{i} Y_{n}\left(\alpha_{i} a x\right)\right] \cos n \theta  \tag{A3}\\
& +2 n \overline{C_{66}}\left((n-1) Y_{n}\left(\alpha_{i} \alpha x\right)-\left(\alpha_{i}\right) Y_{n+1}\left(\alpha_{i} \alpha x\right)\right) \sin 2\left(\theta-\gamma_{i}\right) \sin n \theta, \quad i=5,6 \\
& e_{n}^{\gamma}=2 n n_{c o s}\left((n-1) Y_{n}\left(\alpha_{7} \alpha x\right)-\left(\alpha_{7} a\right) Y_{n+1}\left(\alpha_{7} \alpha x\right)\right\} \cos 2\left(\theta-\gamma_{i}\right) \cos n \theta \\
& +\bar{c}_{\sigma 6}\left[2\left(\alpha_{1} a\right) Y_{n+1}\left(\alpha_{7}, a x\right)-\left\{\left(\alpha_{r} a\right)^{2}-2 n(n-1) Y_{n}\left(\alpha_{\tau} \alpha x\right)\right\}\right] \sin n \theta \sin 2\left(\theta-\gamma_{i}\right)  \tag{A4}\\
& e_{n}^{4}=\Omega^{2} \bar{\rho}_{1} J_{n}\left(\delta_{1} a x\right) \cos n \theta  \tag{A5}\\
& e_{n}^{8}=\Omega^{2} \bar{\rho}_{2} H_{n}^{(2)}\left(\delta_{2} a x\right) \cos n \theta  \tag{A6}\\
& f_{n}^{i}=\left[2\left(\alpha_{i}\right) J_{n+1}\left(\alpha_{i} \alpha x\right)-\left\{\left(\left(\alpha_{i} a\right)^{2}-2 n(n-1)\right) J_{n}\left(\alpha_{i} \alpha x\right)\right\}\right] \sin 2\left(\theta-\gamma_{i}\right) \cos n \theta  \tag{A7}\\
& +2 n\left\{(n-1) J_{n}\left(\alpha_{i} a x\right)+\left(\alpha_{i}\right) J_{n+1}\left(\alpha_{i} a x\right)\right\} \cos \left(\theta-\gamma_{i}\right) \sin n \theta, i=1,2 \\
& f_{n}^{3}=2 n\left\{(n-1) J_{n}\left(\alpha_{3} a x\right)-\left(\alpha_{3} a\right) J_{n+1}\left(\alpha_{3} a x\right)\right) \sin 2\left(\theta-\gamma_{i}\right) \cos n \theta \\
& +\left[\left\{\left(\alpha_{3} a\right)^{2}-2 n(n-1)\right\} J_{n}\left(\alpha_{3} a x\right)-2\left(\alpha_{3} a\right) J_{n+1}\left(\alpha_{3} a x\right)\right] \cos 2\left(\theta-\gamma_{i}\right) \sin n \theta  \tag{A8}\\
& f_{n}^{i}=\left[2\left(\alpha_{i}\right) Y_{n+1}\left(\alpha_{i} a x\right)-\left\{\left(\left(\alpha_{i} a\right)^{2}-2 n(n-1)\right) Y_{n}\left(\alpha_{i} \alpha x\right)\right\}\right] \sin 2\left(\theta-\gamma_{i}\right) \cos n \theta  \tag{A9}\\
& +2 n\left\{(n-1) Y_{n}\left(\alpha_{i} a x\right)+\left(\alpha_{i} a\right) Y_{n+1}\left(\alpha_{i} a x\right)\right\} \cos \left(\theta-\gamma_{i}\right) \sin n \theta, i=5,6 \\
& \left.f_{n}^{7}=2 n(n-1) Y_{n}\left(\alpha_{1} a x\right)-\left(\alpha_{7} a\right) Y_{n+1}\left(\alpha_{7} \alpha x\right)\right) \sin 2\left(\theta-\gamma_{i}\right) \cos n \theta \\
& +\left[\left\{\left(\alpha_{i} a\right)^{2}-2 n(n-1)\right\} Y_{n}\left(\alpha_{7}, \alpha x\right)-2\left(\alpha_{r},\right) Y_{n+1}\left(\alpha_{i}, \alpha x\right)\right] \cos 2\left(\theta-\gamma_{i}\right) \sin n \theta \tag{A10}
\end{align*}
$$

