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Solution of general lotka-volterra system by using differential transform method

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ABSTRACT
In this study the differential transform method is applied to solve the general lotka- volterra system of ordinary differential equations. Firstly, we stated the definition of the one dimensional transform method, and some related theorems. Then some illustrative examples are given, the numerical results of these examples compared with those obtained by the A domain decomposition method are found to be the same.

Introduction

The differential transform method (DTM) is one of the approximate methods which can be easily applied to many linear and nonlinear problems and is capable of reducing the size of computational work. The concept of the differential transform method has been introduced to solve linear and nonlinear initial value problem in electric circuit analysis [4,7,8].

Differential transform method is a semi-numerical analytic technique that formalizes the Taylor series in a totally different manner. With this method, the given differential equation and related initial conditions are transformed into a recurrence equation, that finally leads to a system of algebraic equations which can easily be solved. In this method no need for linearization or perturbations, large computational work and round- off errors are avoided In resent years many researchers apply the DTM for solving differential equations $[1,2]$.

This method constructs, for differential equations an analytical solution in the form of a polynomial. Not like the traditional high order Taylor series method that requires symbolic computations. Another important advantage is that this method reduces the size of computational work while the Taylor series method is computationally taking long time for large orders. This method is well addressed in $[3,5]$.

Differential transform:

The basic definitions and fundamental theorems of one dimensional differential transform method are defined and proved in $[7]$ and will be stated brief in this paper.

Differential transform of function $y(x)$ is defined as follows:

$$y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \qquad (1)$$

Where $y(x)$ the original is function and $y(k)$ is the transformed function, which is also colled the T- function.

The inverse differential transform of $y(k)$ is defined as.

$$y(x)=\sum_{k=0}^{\infty} y(k)\, x^k \qquad (2)$$

Combining eqs (1) and (2) we have

$$y(x)=\sum_{k=0}^{\infty}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}\frac{x^k}{k!} \qquad (3)$$

The fundamental theorems of the one dimensional differential transform are:

Theorems [7]:

(1) $If\ z(x)=u(x)\pm v(x)\ \ Then\ \ z(k)=u(k)\pm v(k)$

(2) $If\ z(x)=cu(x)\ \ Then\ \ z(k)=cu(k)$

(3) $If\ z(x)=\frac{du(x)}{dx}\ \ Then\ \ z(k)=(k+1)u(k+1)$

(4) $If\ z(x)=\frac{d^n u(x)}{dx^n}\ Then\ z(k)=\frac{(k+n)!}{k!}u(k+n)$

(5) $If\ \ z(x)=u(x)\,v(x)\ \ Then\ \ z(k)=\sum_{m=0}^{k} u(m)\,v(k-m)$

(6) $If\ z(x)=u_1(x)u_2(x).....u_n(x) Then$

$$z(k)=\sum_{k_{n-1}=0}^{k}\ \sum_{k_{n-2}=0}^{k_{n-1}} ......\ \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} u_1(k_1)u_2(k_2-k_1).....u_{n-1}(k_{n-1}-k_{n-2})u_n(k-k_{n-1})$$

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(7) If $z(x)=x^n$ Then $z(k)=\delta(n-k)=\begin{cases}1 & k=0\\ 0 & k\neq n\end{cases}$

Note that c is Constant and n is a nonnegative integer.

Volterra model:

In this section we study the lotka-Volterra model which is Composed of a pair of differential equations that describe predator- prey (herbivore-plant or parasitoid – host) dynamics in their simplest case (one predator population). It was developed in dependently by Alfred lotka and veto volterra in (1920).

Solution of prey and predator problem:

One of the simplest models of lotke-volterra is used to investigate predator and prey population dynamic. This model is a system of two equations one describes the prey's population and the other describes the predator's population

$$\begin{cases}\frac{dx_1(t)}{dt}=x_1(t)\left[a_1(t)-b_1(t)x_2(t)\right]\\ \frac{dx_2(t)}{dt}=x_2(t)\left[a_2(t)-b_2(t)x_1(t)\right]\end{cases} \tag{4}$$

With the initial Conditions $x_1(0)=\alpha_1, x_2(0)=\alpha_2$

Where $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are respectively the growth rate of the prey, the efficiency of the predator's ability to capture prey, the death rate of Predator and the growth rate of the predator $x_1(t)$ and $x_2(t)$ are respectively the population of rabbits and the foxes at time t , $x_1(t)$. $x_2(t)$ indicate the amount of the two organisms encounter each other.

By using the differential transform method in to equation (4) we have

$$\begin{cases}x_1(k+1)=\frac{1}{k+1}\left[\sum_{m=0}^{k}a_1(m)x_1(k-m)-\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}b_1(m)x_1(k_1-m)x_2(k-k_1)\right]\\ x_2(k+1)=\frac{1}{k+1}\left[\sum_{m=0}^{k}a_2(m)x_2(k-m)-\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}b_2(m)x_1(k_1-m)x_2(k-k_1)\right]\end{cases}$$

Replacing $(k+1)$ by k and the above system can be writhen in the following.

$$x_i(k)=\frac{1}{k}\left[\sum_{m=0}^{k-1}a_i(m)x_i(k-1-m)-\sum_{k_1=0}^{k-1}\sum_{m=0}^{k_1}b_i(m)x_1(k_1-m)x_2(k-1-k_1)\right] i=1,2$$

Substituting $x_i(k)$ into eg (2) we have

$$x_i(t)=\alpha_i+\sum_{k=0}^{\infty}\frac{1}{k}\left[\sum_{m=0}^{k-1}a_i(m)x_i(k-1-m)-\sum_{k=0_1}^{k-1}\sum_{m=0}^{k_1}b_i(m)x_1(k_1-m)x_2(k-1-k_i)\right]t^k$$

Example:

Consider the following system

$$\begin{cases}\frac{dx(t)}{dt}=x[-2x-3y]\\ \frac{dy(t)}{dt}=y[-x-2y]\end{cases} \tag{5}$$

With the initial conditions $x(0)=1$, $y(0)=-1$

Applying the differential transform method to equation (5) we get.

$$\begin{cases}x(k+1)=\frac{1}{k+1}\left[-2\sum_{m=0}^{k}x(m)x(k-m)-3\sum_{m=0}^{k}x(m)y(k-m)\right]\\ y(k+1)=\frac{1}{k+1}\left[-\sum_{m=0}^{k}x(m)y(k-m)-2\sum_{m=0}^{k}y(m)y(k-m)\right]\end{cases} \tag{6}$$

Substituting $k=0,1,2,........$, m into equation (6) series coefficients can be obtained as following.

$x(1)=1$, $x(2)=1$, $x(3)=1$

$y(1)=-1$, $y(2)=-1$, $y(3)=-1$

And so on in general $x(k)=1,\ y(k)=-1.$

Substituting $x(k)$ and $y(k)$ into eg(2) we have

$$x(t)=\sum_{k=0}^{\infty}x(t)\,t^k=\sum_{k=0}^{\infty}t^k=\frac{1}{1-t}$$

$$y(t)=\sum_{k=0}^{\infty}y(t)\,t^k=-\sum_{k=0}^{\infty}t^k=\frac{-1}{1-t}$$

Example:

Consider the following system.

$$\begin{cases}\frac{dx(t)}{dt}=x\left[\frac{1}{1-t}-xe^t-ye^t\right]\\ \frac{dy(t)}{dt}=y\left[\frac{1}{1-t}-xe^{-t}-ye^{-t}\right]\end{cases} \tag{7}$$

With the initial conditions $x(0)=1$, $y(0)=-1$

Applying the DTM to eg (7) yields

$$\begin{cases}x(k+1)=\frac{1}{k+1}\left[\sum_{m=0}^{k}A(m)x(k-m)-\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}B(m)x(k-m)x(k-k_1)-\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}B(m)x(k_1-m)y(k-k_1)\right]\\ y(k+1)=\frac{1}{k+1}\left[\sum_{m=0}^{k}A(m)y(k-m)-\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}C(m)x(k-m)y(k-k_1)-\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}C(m)y(k_1-m)y(k-k_1)\right]\end{cases}$$

(8)

Where $A(t), B(t)$ and $c(t)$ correspond to the differential transformation of $\frac{1}{1-t}$, e^t and e^{-t} respectively and this leads to

$$A(k)=1\ ,\ B(k)=\frac{1}{k!}\ ,\ C(k)=\frac{(-1)^k}{k!}$$

Substituting these values into eq (8) and $k=0,1,2,....$, m gives

$x(k)=1$, $y(k)=-1$

Substituting $x(k)$ and $y(k)$ into eq (2) we get

$x(t)=(1-t)^{-1}$, $y(t)=-(1-t)^{-1}$

General Lotka-Voiterra model:-

Consider the following system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1\left[r_1 - b_{11}x_1 - b_{12}x_2 - \ldots\ldots - b_{1n}x_n\right] \\ \dfrac{dx_2(t)}{dt} = x_2\left[r_2 - b_{21}x_1 - b_{22}x_2 - \ldots\ldots - b_{2n}x_n\right] \\ \vdots \\ \dfrac{dx_n(t)}{dt} = x_n\left[r_n - b_{n1}x_1 - b_{n2}x_2 - \ldots\ldots - b_{nn}x_n\right] \end{cases} \tag{9}$$

With the initial Condition $x_i(0) = \alpha_i$, $i = 1,2,\ldots\ldots,n$

Applying the DTM to eq (9) we get

$$\begin{cases} x_1(k+1) = \dfrac{1}{k+1}\left[\sum_{m=0}^{k} r_1(m)x_1(k-m) - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{11}(m)x_1(k_1-m)x_1(k-k_1) - \ldots\ldots \right. \\ \qquad\qquad \left. - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{1n}(m)x_1(k-m)x_n(k-m)\right] \\ x_2(k+1) = \dfrac{1}{k+1}\left[\sum_{m=0}^{k} r_2(m)x_2(k-m) - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{21}(m)x_2(k_1-m)x_1(k-k_1) - \ldots\ldots \right. \\ \qquad\qquad \left. - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{2n}(m)x_2(k-m)x_n(k-m)\right] \\ \vdots \\ x_n(k+1) = \dfrac{1}{k+1}\left[\sum_{m=0}^{k} r_n(m)x_n(k-m) - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{n1}(m)x_n(k_1-m)x_1(k-k_1) - \ldots\ldots \right. \\ \qquad\qquad \left. - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{nn}(m)x_n(k-m)x_n(k-m)\right] \end{cases}$$

The above system can be written is the following

$$x_i(k+1) = \frac{1}{k+1}\left[\sum_{m=0}^{k} r_i(m)x_i(k-m) - \sum_{j=1}^{n}\sum_{k_1=0}^{k}\sum_{m=0}^{k_1} b_{ij}(m)x_i(k_1-m)x_j(k-k_1)\right]$$

$$i = 1,2,\ldots\ldots,n$$

Substituting $x_i(k)$ into eg (2) we have.

$$x_i(t) = \alpha_i + \sum_{k=1}^{\infty}\frac{1}{k}\left[\sum_{m=0}^{k-1} r_i(m)x_i(k-1-m) - \sum_{j=1}^{n}\sum_{k_1=0}^{k-1}\sum_{m=0}^{k_1} b_{ij}(m)x_i(k_1-m)x_j(k-1-k_1)\right].t^k$$

Example:-

Consider the following system

$$\begin{cases} \dfrac{dx(t)}{dt} = x\left[x+y-z\right] \\ \dfrac{dy(t)}{dt} = y\left[x-y+z\right] \\ \dfrac{dz(t)}{dt} = z\left[-x+y+z\right] \end{cases} \tag{10}$$

With initial conditions

$$x(0) = 1\,,\ y(0) = 1\ ,\ z(0) = 1 \tag{11}$$

By using the $DT\ M$ into eq (10) we have

$$\begin{cases} x(k+1) = \dfrac{1}{k+1}\left[\sum_{m=0}^{k} x(m)x(k-m) + \sum_{m=0}^{k} x(m)y(k-m) - \sum_{m=0}^{k} x(m)z(k-m)\right] \\ y(k+1) = \dfrac{1}{k+1}\left[\sum_{m=0}^{k} x(m)y(k-m) - \sum_{m=0}^{m=0} y(m)y(k-m) + \sum_{m=0}^{k} y(m)z(k-m)\right] \\ z(k+1) = \dfrac{1}{k+1}\left[-\sum_{m=0}^{k} x(m)z(k-m) + \sum_{m=0}^{k} y(m)z(k-m) + \sum_{m=0}^{k} z(m)z(k-m)\right] \end{cases} \tag{12}$$

Substituting eg (11) into eg (12) and $k = 0,1,2\ldots\ldots m$ we have.

$$x(k) = y(k) = z(k) = 1$$

Substituting $x(k)$, $y(k)$ and $z\,(k)$ in to eg (2) gives.

$$x(t) = y(t) = z(t) = (1-t)^{-1}$$

Example:

Consider the following:

$$\begin{cases} \dfrac{dx(t)}{dt} = x\left[2t + y^2 - z\right] \\ \dfrac{dy(t)}{dt} = y\left[1 - \dfrac{1}{2}e^{-t}z\right] \\ \dfrac{dz(t)}{dt} = z\left[2 - e^{-\frac{1}{2}t}y\right] \end{cases} \tag{13}$$

With initial Conditions

$$x(0) = 1,\ y(0) = 1\ ,\ z(0) = 1 \tag{14}$$

Applying the differential transform method to eg (13) we have.

$$\begin{cases} x(k+1) = \dfrac{1}{k+1}\left[2x(k-1) + \sum_{k_1=0}^{k}\sum_{m=0}^{k_1} x(m)y(k_1-m)y(k-k_1) - \sum_{m=0}^{k} x(m)z(k-m)\right] \\ y(k+1) = \dfrac{1}{k+1}\left[\ y(k) - \dfrac{1}{2}\sum_{k_1=0}^{k}\sum_{m=0}^{k_1}\dfrac{(-1)^m}{m!}y(k_1-m)z(k-k_1)\ \right] \\ z(k+1) = \dfrac{1}{k+1}\left[2z(k) - \sum_{k_1=0}^{k}\sum_{m=0}^{k_1}\dfrac{(-1/2)^m}{m!}z(k_1-m)y(k-k_1)\right] \end{cases} \tag{15}$$

Substituting eg (14) in to eg (15) we have

$$x(1) = 0\ ,\ x(2) = 1\ ,\ x(3) = 0\ ,\ x(4) = \frac{1}{2!}\ ,\ x(5) = 0\ ,\ x(6) = \frac{1}{3!}$$

$$y(1) = \frac{1}{2}\ ,\ y(2) = \frac{1}{8}\ ,\ y(3) = \frac{1}{48}\ ,\ y(4) = \frac{1}{384}$$

$$z(1) = 1\ ,\ z(2) = \frac{1}{2!}\ ,\ z(3) = \frac{1}{3!}\ ,\ z(4) = \frac{1}{4!}$$

And so on In general we find

$$x(k) = \begin{bmatrix} \dfrac{1}{(k/2)!} & if\ k\ is\ even \\ 0 & if\ \ k\ \ is\ \ odd \end{bmatrix},$$

$$y(k) = \frac{\left(\frac{1}{2}\right)^k}{k!}\ ,\ z(k) = \frac{1}{k!}$$

Substituting $x(k)$, $y(k)$ and $z(k)$ into eg (2) yields

$$x(t)=\sum_{k=0}^{\infty}x(k)\,t^{k}=\sum_{k=0,2,4,...}^{\infty}\frac{t^{k}}{(k/2)!}=\sum_{k=0}^{\infty}\frac{t^{2k}}{k!}=e^{t^{2}}$$

$$y(t)=\sum_{k=0}^{\infty}y(k)\,t^{k}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}t\right)^{k}}{k!}=e^{\frac{1}{2}t}$$

$$z(t)=\sum_{k=0}^{\infty}z(k)\,t^{k}=\sum_{k=0}^{\infty}\frac{t^{k}}{k!}=e^{t}$$

Conclusion:-

One dimensional differential transform have been applied to non lienear system of ordinary differential equation .The results for all example can be obtained in Tayler's series form, All the calculations in the method are very easy. In summary, using one dimensional differential transformation to solve ODE, consists of three main steps. First, transformaction ODE in to algebra equation, second, solve the equations, finally inverting the solution of algebraic equations to obtain a closed form series solution.

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