



# Numerical solution of an optimal control of linear singular systems via Runge-Kutta methods based on various means

S. Sekar<sup>1,\*</sup>, R. Muthukrishnan<sup>2</sup> and S. Subbulakshmi<sup>3</sup>

<sup>1</sup>Department of Mathematics, Government Arts College (Autonomous), Cherry Road, Salem-636 007, Tamil Nadu, India.

<sup>2</sup>Department of Mathematics, Panimalar Institute of Technology, Nasarathpettai, Chennai – 600 123, Tamil Nadu, India.

<sup>3</sup>Department of Mathematics, Dharamapuram Gnanambigai Government Arts College for Women, Mayiladuthurai – 609 001, Tamil Nadu, India.

## ARTICLE INFO

### Article history:

Received: 18 September 2011;

Received in revised form:

22 October 2011;

Accepted: 13 November 2011;

### Keywords

Optimal control problem,  
Time-invariant systems,  
Singular systems,  
Runge-Kutta methods.

## ABSTRACT

In this article, the problem of optimal control of time-invariant linear singular systems with quadratic performance index has been studied using Runge-Kutta (RK) method based on various means. The obtained discrete solutions are compared with the exact solutions of the time-invariant optimal control of linear singular systems. It is observed that the result obtained using Runge-Kutta arithmetic mean (RKAM) and RKCeM (Runge-Kutta Centroidal Mean) are closer to the true solutions of the problem. Error graphs for the simulated results and exact solutions are presented in a graphical form to highlight the efficiency of this RKAM and RKCeM. This RKAM and RKCeM can be easily implemented in a digital computer and the solution can be obtained for any length of time for this type of optimal control of time-invariant linear singular systems and it is an added advantage of this method.

© 2011 Elixir All rights reserved.

## Introduction

The problem of optimal control of singular systems has invoked immense interest, especially among the researchers in the field of computational mathematics to study the existing problems in the field of control theory and to compute the value of the control vector numerically which controls the state vector. Chen and Hsiao [6], Chen and Shih [7], applied Walsh series to study the problem of optimal control of time-invariant and time-varying linear systems. It is to be noted that from the study of past literature that Cobb [5] and Pandolfi [11] seems to have been the first authors to consider the optimal regulator problem of continuous time singular systems. Both of them used state feed backs and their results were derived by the aid of Riccati-type matrix equations.

Walsh functions have been widely used to study the problem of optimal control of linear systems with quadratic performance index [8-10]. Palanisamy [10] has analyzed the optimal control of linear systems via STWS approach. Balachandran and Murugesan [2] have applied the STWS method to optimal control of linear singular systems.

Runge-Kutta methods have become very popular both as computational techniques as well as subject for research, which were discussed by Alexander and Coyle [1], Butcher [3,4], Shampine [13] and Yaakub and Evans [14-16]. Butcher [4] derived the best RK pair along with an error estimate and by all statistical measures it appeared as the RK-Butcher algorithms. This RK-Butcher algorithm is nominally considered sixth order since it requires six function evaluations (it looks like a sixth order method, but it is a fifth order method only), but in actual practice the “working order” is closer to five (fifth order) but still the accuracy of results (while solving the problems) exceeds all the other algorithms examined including RK-Fehlberg, RK-Merson, RK-Centroidal Mean (RKCeM) and RK-Arithmetic Mean.

Park et al [12] applied the RK-Butcher algorithms to compute the numerical solution of an optimal control of time-invariant linear singular systems. In this article, we consider the same time-invariant optimal control of linear singular systems with quadratic performance index (discussed by Park et al [12]) with more accuracy using RK methods based on various means. An elaborate, well composed comparison has been carried out with the aid of the obtained results and graphs.

## Extended Runge - Kutta Method Based on AM

The general p-stage RK method for solving  $\dot{x} = f(t, x)$  is defined by

$$x_{n+1} = x_n + h \sum_{i=1}^p b_i k_i$$

$$\text{where } k_i = f \left( t_n + c_i h, x_n + h \sum_{i=1}^p a_{ij} k_j \right)$$

$$c_i = \sum_{j=1}^p a_{ij}, \quad i = 1, 2, \dots, p,$$

where b and c are p-dimensional vectors and the matrix A = (a<sub>ij</sub>) is of order (p x p).

Hence the fourth order RK method for solving an IVP of the form

$$\dot{x} = f(t, x) \quad \text{with } x(0) = x_0$$

can be formulated as

$$x_{n+1} = x_n + \frac{h}{3} \sum_{i=1}^3 \left[ \frac{K_i + K_{i+1}}{2} \right]$$

i.e.,

$$x_{n+1} = x_n + \frac{h}{3} \left[ \frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2} \right]$$

$$x_{n+1} = x_n + \frac{h}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

In the initial iteration, we get

$$x(1) = x(0) + \Delta x$$

where

$$\Delta x = \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(t(0), x(0))$$

$$k_2 = f\left(t(0) + \frac{h}{2}, x(0) + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t(0) + \frac{h}{2}, x(0) + \frac{h}{2}k_2\right)$$

$$k_4 = f(t(0) + h, x(0) + hk_3)$$

**Extended Runge - Kutta method based on CEM**

In [14-16], Evans and Yaakub have developed a new RK method of order 4 based on Centroidal mean to solve first order equation and it is to be noted that the Centroidal Mean of two points  $x_1$  and  $x_2$  is defined as

$$\frac{2}{3} \left( \frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2} \right)$$

Consider the first order equation (2.1) of the form

$$y' = f(x, y)$$

with

$$y(x_0) = y_0.$$

Let  $h$  denote the interval between equidistant values of  $x$ . The fourth order RKAM formula (2.21) can be written as

$$y_{n+1} = y_n + \frac{h}{3} \left( \frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2} \right)$$

$$y_{n+1} = y_n + \frac{h}{3} \left( \sum_{i=1}^3 \frac{k_i + k_{i+1}}{2} \right)$$

and substituting the arithmetic mean (AM) of  $k_i, 1 \leq i \leq 6$  with their Centroidal Means we obtain a new formula, similar to the above equation, as

$$y_{n+1} = y_n + \frac{h}{3} \left[ \sum_{i=1}^3 \frac{2(k_i^2 + k_i k_{i+1} + k_{i+1}^2)}{3(k_i + k_{i+1})} \right]$$

to obtain the fourth order formula in the form,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + a_1 h, y_n + h a_1 k_1)$$

$$k_3 = f(x_n + (a_2 + a_3)h, y_n + h a_2 k_1 + h a_3 k_2)$$

$$k_4 = f(x_n + (a_4 + a_5 + a_6)h, y_n + h a_4 k_1 + h a_5 k_2 + h a_6 k_3)$$

$$y_{n+1} = y_n + \frac{h}{3} \left[ \frac{2(k_1^2 + k_1 k_2 + k_2^2)}{3(k_1 + k_2)} + \frac{2(k_2^2 + k_2 k_3 + k_3^2)}{3(k_2 + k_3)} + \frac{2(k_3^2 + k_3 k_4 + k_4^2)}{3(k_3 + k_4)} \right]$$

that is  $y_{n+1} = y_n + \frac{\text{UPPER}}{\text{LOWER}}$

where,  
UPPER =

$$\frac{2h}{9} \left[ (k_1^2 + k_1 k_2 + k_2^2)(k_2 + k_3)(k_3 + k_4) + (k_2^2 + k_2 k_3 + k_3^2)(k_1 + k_2)(k_3 + k_4) + (k_3^2 + k_3 k_4 + k_4^2)(k_1 + k_2)(k_2 + k_3) \right]$$

and,

$$\text{LOWER} = (k_1 + k_2)(k_2 + k_3)(k_3 + k_4),$$

while the Taylor series expansion of  $y(x_{n+1})$  may be given as,  
TAYLOR =

$$y_n + hf + \frac{h^2}{2} ff_y + \frac{h^3}{6} (ff_y^2 + f^2 f_{yy}) + \frac{1}{24} h^4 (f^3 f_{yyy} + ff_y^3 + 4f^2 f_y f_{yy}) + \dots$$

Hence  $\text{ERROR} = \text{TAYLOR} - \frac{\text{UPPER}}{\text{LOWER}}$

or,  $(\text{TAYLOR} \times \text{LOWER}) - \text{UPPER} = (\text{LOWER} \times \text{ERROR})$ .

**Extended Runge - Kutta method based on HAM**

In the development of methods for solving ordinary differential equations, it is not clear whether the arithmetic mean is always the best choice. Naturally RK formulae, based on arithmetic mean, are the most convenient and flexible to apply. But there is no guarantee that they would yield more accurate results for all type of problems. Hence, the use of harmonic means in the functional values instead of the usual arithmetic mean may result in better accuracy for a certain class of problems. It may be noted that the harmonic mean of two quantities  $x_1$  and  $x_2$  is given by

$$\frac{2x_1x_2}{x_1 + x_2}$$

In [14], it has been shown that the use of harmonic means in the functional values, instead of the usual arithmetic mean in the trapezoidal formula has also produced a formula with an accuracy of order -2.

i.e.,  $x_{n+1} = x_n + h \left( \frac{2f_n f_{n+1}}{f_n + f_{n+1}} \right)$  (1)

The local truncation error (LTE) for the eq. (1) is given by

$$\text{LTE} = \left( \frac{-\frac{2}{12} + \left(\frac{2}{4}\right)^2}{\&#x27E; } \right) h^3 + O(h^4)$$

It is possible to establish a 4-stage non-linear RK formula based on harmonic mean (RKHM) in the form

$$x_{n+1} = x_n + \frac{h}{3} \sum_{i=1}^3 \left( \frac{2k_i k_{i+1}}{k_i + k_{i+1}} \right)$$

i.e.,

$$x_{n+1} = x_n + \frac{h}{3} \left( \frac{2k_1 k_2}{k_1 + k_2} + \frac{2k_2 k_3}{k_2 + k_3} + \frac{2k_3 k_4}{k_3 + k_4} \right)$$

as a direct extension of eq. (20), where

$$k_1 = f(x_n)$$

$$k_2 = f(x_n + h a_1 k_1)$$

$$k_3 = f(x_n + h(a_2 k_1 + a_3 k_2))$$

$$k_4 = f(x_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3)).$$

**Optimal Control of Singular Systems**

The linear time-invariant singular system represented in the following form

$$K\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

with initial condition  $x(0) = x_0$

where  $K$  is an  $n \times n$  singular matrix,  $A$  and  $B$  are  $n \times n$  and  $n \times p$  constant matrices respectively.  $x(t)$  is an  $n$ -state vector and  $u(t)$  is the  $p$ -input control vector. This singular system has many aspects and applications.

Assuming that  $\det(sK - A) \neq 0$ ,  $B = \begin{bmatrix} 0 \\ I_p \end{bmatrix}$ ,  $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$  (3)

Where  $K_1 = \begin{bmatrix} I_{n-p} & 0 \end{bmatrix}$

Now the problem can be stated as follows: Given the initial state  $x(0^-) = x_0$  find a control vector  $u(t)$  that generates a state  $x(t)$  such that  $x(t_f) = x_f$ , where  $t_f$  is a prescribed time and  $x_f$  is a fixed vector, and minimizes the cost functional

$$J = \int_0^{t_f} L(x, u) dt \tag{4}$$

where  $L = \frac{1}{2}(x^T Qx + u^T Ru)$ ,  $Q$  and  $R$  denote given real symmetric constant matrices. In case the initial state  $x(0^-)$  is not known, the method developed by El-Tohami *et al.* [8] may be used to reconstruct the state. It has been proved by Lovass-Nagy *et al.* [9], which the problem of finding an optimal control reduces to the solution of a two-point boundary value problem.

Let  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,

$x_1$  is  $(n-p) \times 1$  and  $x_2$  is  $p \times 1$ ,  $K_2 = \begin{bmatrix} K_{21} & K_{22} \end{bmatrix}$ , where  $K_{21}$  is  $p \times (n-p)$  and  $K_{22}$  is  $p \times p$  and

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, A_1 = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}, A_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}$$

Where  $A_{11}, A_{12}, A_{21}, A_{22}$  are respectively  $(n-p) \times (n-p), (n-p) \times p, p \times (n-p), p \times p$ .

Further take  $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$  where  $Q_1$  and  $Q_2$  are  $(n-p) \times n$  and  $p \times n$  respectively. Then we have the following equations (Lovass-Nagy *et al.* [9])

$$\frac{dx_1}{dt} = A_{11}x_1 + A_{12}x_2 \tag{5}$$

$$K_{21} \frac{dx_1}{dt} + K_{22} \frac{dx_2}{dt} - A_{21}x_1 - A_{22}x_2 = u \tag{6}$$

$$\frac{dp_1}{dt} = -A_{11}^T p_1 + K_{21}^T R \frac{du}{dt} + A_{21}^T Ru - Q_1 x \tag{7}$$

$$A_{12}^T p_1 = K_{22}^T R \frac{du}{dt} + A_{22}^T Ru - Q_2 x \tag{8}$$

where  $p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^T$  is the co-state vector corresponding to equations (2). From the equations (5)-(8), optimal state and optimal control can be calculated.

The governing equations for determining  $u(t)$  and  $x(t)$  for the time-invariant and time-varying optimal control problem can be obtained using the set of equations (5) to (8), which have been mentioned earlier. It is to be noted that the above governing equations may not suit all types of time-invariant and time-varying optimal control problems. Hence, it is necessary to investigate further to derive the governing equations exclusively (a generalized form) for the time-invariant.

**Formulation of Optimal Control for Time-Invariant Linear Singular Systems**

Rearranging Equations (5)-(8), we have the following system.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{22}^T & 0 \\ 0 & 0 & -K_{21}^T & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 1 & 0 \\ Q_{12}^T & Q_{22} & -A_{22}R & A_{12}^T \\ -Q_{11} & -Q_{12} & A_{21}^T & -A_{11}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ p_1 \end{bmatrix} \tag{9}$$

which can be written in the form

$$K\dot{y}(t) = My(t) \tag{10}$$

Where  $K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{22}^T & 0 \\ 0 & 0 & -K_{21}^T & 1 \end{bmatrix}$ ,

$$M = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 1 & 0 \\ Q_{12}^T & Q_{22} & -A_{22}R & A_{12}^T \\ -Q_{11} & -Q_{12} & A_{21}^T & -A_{11}^T \end{bmatrix}$$

and  $y = \begin{bmatrix} x_1 & x_2 & u & p_1 \end{bmatrix}^T$ . Where the matrix  $K$  is singular so that it is called as "time-invariant singular systems" and it can not be written in the standard form.

**Example of the optimal control for time-invariant linear singular systems**

The linear singular system [9] is considered

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{11}$$

with initial condition  $x(0) = \begin{bmatrix} -1/\sqrt{2} \\ 1 \end{bmatrix}$

The performance index  $J = \frac{1}{2} \int_0^{t_f} (x^T x + u^2) dt$  (12)

Let  $z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$

Then we obtain  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$  (13)

is obtained, since  $\frac{1}{2}(x^T x + u^2) = \frac{1}{2}(z^T z + u^2)$  minimization of  $L(z, u) = \frac{1}{2}(z^T z + u^2)$  with respect to  $z$  and  $u$  will yield

the same result as minimization of  $L(x, u)$  with respect to  $x$  and  $u$ . The exact solution of the system (23) is

$$\left. \begin{aligned} z_1(t) &= -\frac{1}{\sqrt{2}} \exp(-\sqrt{2}t) \\ z_2(t) &= \exp(-\sqrt{2}t) \end{aligned} \right\} \quad (14)$$

and the optimal control is  $u(t) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}t)$  (15)

Using RK methods based on various means in the optimal control of singular system, the discrete time solutions and the exact solutions of  $z(t)$  are calculated and is presented in Table 1 and Table 2 along with the solution obtained by RK method based on various means. The corresponding optimal control  $u(t)$  is calculated by using RK method based on various means and the results are presented in Table 3.

Applying the formula of RKAM, RKCeM and RKHaM discussed in 2 - 4, the discrete solutions of (13) have been obtained, taking the step-size as  $h = 0.25$ , for different values of (13).

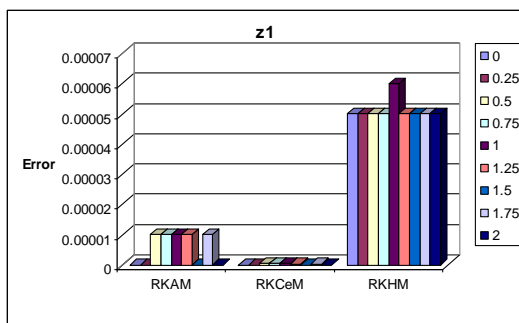


Figure 1 Error graph for the state  $z_1(t)$

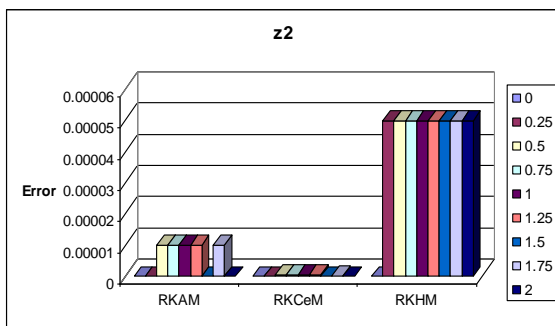


Figure 2 Error graph for the state  $z_2(t)$

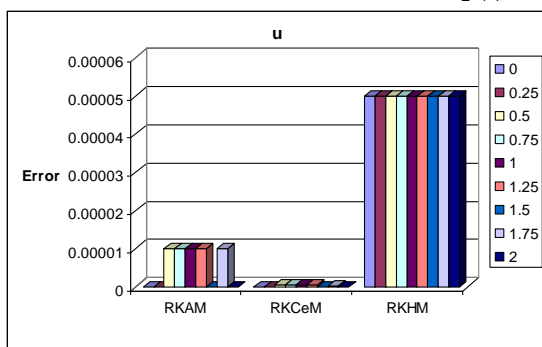


Figure 3 Error graph for the control input  $u(t)$

**Conclusions**

The obtained results of the time-invariant optimal control of linear singular systems with quadratic performance index show that the RKAM and RKCeM works well for finding the state

vector  $x(t)$  and the control input vector  $u(t)$ . From the tables 1 - 3, it can be observed that for most of the time intervals, the absolute error is less (almost no error) in RKAM and RKCeM when compared to the RKHM method, which yields a little error, along with the exact solutions of the problem.

From the results shown in the figures 1 - 3, it can be said that the error is very less in RKAM and RKCeM when compared to the RKHM method. Moreover, the RKAM and RKCeM method is highly stable because it is based on the Taylor series and hence one can get the results for any length of time.

**References**

- [1] Alexander R. K, and Coyle J. J, "Runge-Kutta methods for differential- algebraic systems", SIAM J. of Numer. Anal., Vol. 27(3), pages 736-752, 1990.
- [2] Balachandran K, and Murugesan K, "Optimal Control of Singular Systems via Single Term Walsh Series", Intern. J. Comp. Math., Vol.43, pages 153-159, 1990.
- [3] Butcher J. C, *The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods*. John Wiley & Sons, U.K. 1987.
- [4] Butcher J. C, *The Numerical Analysis of Ordinary Differential Equations*, John Wiley & Sons, U.K. 2003.
- [5] Cobb D, "Descriptor variable systems and optimal state regulation", IEEE Trans. Auto. Control, Vol. 28, pages 601-611, 1983.
- [6] Chen C. F, and Hsiao C. H, "Walsh series analysis in optimal control", Int. J. Control, Vol. 21, pages 881-897, 1975.
- [7] Chen, W. L, and Shih Y. P, "Analysis and optimum control of time varying linear systems via Walsh functions", Int. J. Control, Vol. 27, pages 917-932, 1978.
- [8] El-Tohami M, Lovass-Nagy V, and Mukundan R, "On the design of observers for generalized state-space systems using singular value decomposition", Int. J. Control, Vol. 38, pages 673-683, 1983.
- [9] Lovass-Nagy V, Schilling R, and Yan H. C, "A note on optimal control of generalized state-space (descriptor) systems", Intern. J. Control, Vol. 44, pages 613-624, 1986.
- [10] Palanisamy K. R, "Analysis and optimal control of linear systems via Single Term Walsh Series approach", Int. J. Systems Science, Vol. 12, pages 443-454, 1981.
- [11] Pandolfi L, "On the regulator problem for linear degenerate control systems", J. Optimi Theory and Appl., Vol. 33, pages 241-254, 1981.
- [12] Park J. Y, David J. Evans, Murugesan K, Sekar S, and Muruges V, "Optimal control of singular systems using the RK-Butcher algorithm", Intern. J. Comp. Math, Vol. 81, pages 239-249, 2004.
- [13] Shampine L. F, "Numerical Solution of Ordinary Differential Equations", Chapman & Hall, New York, U.S.A, 1994.
- [14] Yaakub, A. R. and David J. Evans, A new 4<sup>th</sup> order Runge-Kutta method based on the root-mean square formula, Rep. No. 862., Loughborough University of Technology, U.K , 1993.
- [15] Yaakub, A. R. and David J. Evans, "The Runge-Kutta Contraharmonic Mean method with extrapolation for the parallel solutions of O.D.E's", Intern. J. Comp. Math., Vol. 6, pages 111-128, 1995.
- [16] Yaakub A. R, and David. J.Evans, "A fourth orders Runge-Kutta RK (4,4) method with error control", Intern. J. Comp. Math, Vol. 71, pages 383-411, 1999.

**Table 1 Solution of the optimal control (time-invariant) system for  $z_1(t)$** 

Solution Number	Time t	$z_1(t)$			
		Exact Solution	RKAM Solution	RKCeM Solution	RKHM Solution
1	0.00	-0.70711	-0.70711	-0.70711	-0.70716
2	0.25	-0.49652	-0.49652	-0.49652	-0.49657
3	0.50	-0.34865	-0.34864	-0.34865	-0.34870
4	0.75	-0.24482	-0.24481	-0.24482	-0.24487
5	1.00	-0.17190	-0.17190	-0.17191	-0.17196
6	1.25	-0.12071	-0.12070	-0.12071	-0.12076
7	1.50	-0.08476	-0.08476	-0.08476	-0.08481
8	1.75	-0.05952	-0.05951	-0.05952	-0.05957
9	2.00	-0.04179	-0.04179	-0.04179	-0.04184

**Table 2 Solution of the optimal control (time-invariant) system for  $z_2(t)$** 

Solution Number	Time t	$z_2(t)$			
		Exact Solution	RKAM Solution	RKCeM Solution	RKHM Solution
1	0.00	1.00000	1.00000	1.00000	1.00000
2	0.25	0.70219	0.70218	0.70219	0.70224
3	0.50	0.49307	0.49305	0.49307	0.49312
4	0.75	0.34623	0.34621	0.34623	0.34628
5	1.00	0.24312	0.24310	0.24312	0.24317
6	1.25	0.17071	0.17070	0.17071	0.17076
7	1.50	0.11987	0.11986	0.11987	0.11992
8	1.75	0.08417	0.08417	0.08417	0.08422
9	2.00	0.05911	0.05910	0.05911	0.05916

**Table 3 Solution of the optimal control (time-invariant) system for  $u(t)$** 

Solution Number	Time t	$u(t)$			
		Exact Solution	RK Solution	STHW Solution	STHW Solution
1	0.00	0.70711	0.70711	0.70711	0.70716
2	0.25	0.49652	0.49652	0.49652	0.49657
3	0.50	0.34865	0.34864	0.34865	0.34870
4	0.75	0.24482	0.24481	0.24482	0.24487
5	1.00	0.17191	0.17190	0.17191	0.17196
6	1.25	0.12071	0.12070	0.12071	0.12076
7	1.50	0.08476	0.08476	0.08476	0.08481
8	1.75	0.05952	0.05951	0.05952	0.05957
9	2.00	0.04179	0.04179	0.04179	0.04184