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# Some results on the supersolvability of finite groups

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**ABSTRACT**

A subgroup $H$ is said to be weakly $c^*$-normal in a group $G$ if there exists a subnormal subgroup $K \le G$ such that $HK = G$ and $H \cap K$ is $s$-quasinormally embedded in $G$. We give some conditions for supersolvability of finite groups $G$.



**Introduction**

In this paper the group is always finite. For some notions and notations, please refer to Huppert [9] and Robinson [17].

Ore [16] gave the conception of quasinormality, which is a generalization of normality. A subgroup $H$ is said to be quasinormal in $G$, if for every subgroup $K$ of $G$, then $HK = KH$. A subgroup $H$ of a group $G$ is said to be $s$-quasinormal ($\pi$-quasinormal) in $G$ if $H$ permutes with every Sylow subgroup of $G$. This concept was introduced by Kegel in [12], and is extensively studied by Deskins [7]. Ballester-Bolinches and Pedraza-Aguilera [6] introduced the conception of $s$- quasinormally embedded in $G$, if for each prime divisor p of $H$, a Sylow $p$-subgroup of $H$ is also a Sylow $p$-subgroup of some $s$-quasinormal subgroup of $G$. Wei and Wang [21] introduced the notion of $c^*$-normality, a subgroup $H$ of $G$ is said to be $c^*$-normal in $G$ if there exists a subgroup $K \le G$ such that $G = HK$ and $H \cap K$ is $s$-quasinormally embedded in $G$.

A class of finite group $\mathrm{F}$ is said to be a formation if every image of an $\mathrm{F}$-group is an $\mathrm{F}$-group and if $G/N_1 \cap N_2$ belongs to $\mathrm{F}$ whenever $G/N_1$ and $G/N_2$ belong to $\mathrm{F}$. Finite supersoluble groups are all formations. A formation $\mathrm{F}$ is said to be saturated if a finite group $G \in \mathrm{F}$ whenever $G/\Phi(G) \in \mathrm{F}$ (see [17, p277]). The class of supersoluble group is a saturated formation (see [17, 9.4.5]). Let $\mathrm{A}$ denote the class of supersoluble groups.

In this paper, we give some results by the conception of weakly $c^*$-normality.

**Some definitions and preliminary results**

**Definition 2.1** ([14]) A subgroup $H$ is said to be weakly $c^*$-normal in $G$ if there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \le H_{sG}$, where $H_{sG}$ is $s$-quasinormally embedded subgroup of $G$ contained in $H$.

**Remark 2.1** Weakly c-normality and $s$-quasinormally embedded implies weakly $c^*$-normality.

**Example 2.1** Every Sylow subgroup of any simple non-abelian group is $s$- quasinormally embedded but not weakly $c$-normal.

**Example 2.2** Let $G = S_4$, the symmetric group of degree 4. Let $a = (34)$, then $G = \langle a \rangle A_4$. Denote $\langle a \rangle$ by $P_0$. If $P_1$ be a Sylow 2-subgroup of $A_4$, $P_1$ is weakly $c$-normal but not $s$-quasinormally embedded in $G$. If $P_1$ is a Sylow 2-subgroup of some $s$-quasinormal subgroup $K$ of $G$, then $P_1Q$ is a subgroup of $G$, where $Q$ be a Sylow 3-subgroup of $G$. Since $P_0 < G$, $P_0P_1Q = QP_0P = P = QP$, where $P = P_0P_1$ is a Sylow 2-subgroup. Namely, $Q$ is s-quasinormal in $G$. By [12, Hilfssatz 7], $Q$ is normal in $G$ and so $G$ is 2-nilpotent, a contradiction.

**Lemma 2.1** ([6, Lemma 1]) Suppose that $\mathrm{A}$ is $s$-quasinormally embedded in a group $G$, and that $H \le G$ and $K < G$.

(1)If $U \le H$, then $U$ is $s$-quasinormally embedded in $H$

(2)If $UK$ is $s$-quasinormally embedded in $G$, then $UK/K$ is $s$-quasiormally embedded in $G/K$.

(3)If $K \le H$ and $H/K$ is $s$-quasinormally embedded in $G/K$, then $H$ is $s$-quasinormally embedded in $G$.

**Lemma 2.2** ([14, Lemma 2.2]) Let $G$ be a group. Then the following statements hold.

(1)Let $H$ is weakly $c^*$-normal in $G$ and $H \le M \le K$. Then $H$ is weakly $c^*$-normal in $M$.

(2)Let $N < G$ and $N \le H$. Then $H$ is weakly $c^*$-normal in

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$G$ if and only if $H/N$ is weakly $c^*$-normal in $G/N$.

(3)Let $\pi$ be a set of primes. $H$ is a $\pi$-subgroup of $G$ and $N$ a normal $\pi'$-subgroup of $G$, if $H$ is weakly $c^*$-normal in $G$, then $HN/N$ is weakly $c^*$-normal in $G/N$.

(4)Let $L \le G$ and $H \le \Phi(L)$.If $H$ is weakly $c^*$-normal in $G$, then $H$ is $s$-quasinormally embedded in $G$.

(5)Let $H$ is $c^*$-normal in G. Then $H$ is weakly $c^*$-normal in $G$.

**Lemma 2.3** Let $M$ be a maximal subgroup of $G$ and $P$ a normal Sylow $p$-subgroup of $G$ such that $G = PM$, where $p$ is a prime. Then $P \cap M$ is a normal subgroup of $G$.

**Lemma 2.4** ([21, Lemma 2.5]) Let $G$ be a group, $K$ an $s$-quasinormal subgroup of $G$, $P$ a Sylow $p$-subgroup of $K$, where $p$ is a prime divisor of $|G|$. If either $P \le O_p(G)$ or $K_G = 1$, then $P$ is $s$-quasinormal in $G$.

**Lemma 2.5** ([15, Lemma 2.2] Let $G$ be a group and $P$ a s-quasinormal $p$-subgroup of $G$, where p is a prime. Then $O^p(G) \le N_G(P)$.

**Lemma 2.6** ([21, Lemma 2.8]) Let $G$ be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1)If $N$ is normal in $G$ of order p, then $N$ is in $Z(G)$.

(2)If $G$ has cyclic Sylow $p$-subgroups, then $G$ is $p$-nilpotent.

(3)If $M \le G$ and $|G:M| = p$, then $M < G$.

**Lemma 2.7** ([8, Satz 1]) Suppose that $G$ is a group which is not supersolvable but whose proper subgroups are all supersolvable. Then

(1) $G$ has a normal Sylow $p$-subgroup $P$ for some prime $p$.

(2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(3)If $p \ne 2$, then $\exp(P) = p$.

(4)If $P$ is non-abelian and $p = 2$, then $\exp(P) = 4$.

(5)If $P$ is abelian, then $\exp(P) = p$.

**Lemma 2.8** Let G be a group and $p$ a prime number.

(1)If $P$ is a minimal normal p-subgroup of $G$, and $x \in P$ is weakly $c^*$-normal in $G$, then $P = \langle x \rangle$.

(2)Let $P$ be a normal p-subgroup of $G$ and x be an element of $P - \Phi(P)$. If $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $x$ is weakly $c^*$-normal in $G$, then $P = \langle x \rangle$.

Proof. (1) Since $\langle x \rangle$ is weakly $c^*$-normal in $G$, there exists a subnormal subgroup $K$ such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K$ is $s$-quasinormally embedded in $G$ and $G = PK$. Let $P_1 = P \cap K$. Since $P$ is a minimal normal subgroup of $G$, then $P_1$ is either trivial or $P$. If $P_1 = 1$, then $P = P \cap G = \langle x \rangle (P \cap K) = \langle x \rangle$. Otherwise $P_1 = P$ and hence $P \le K$, $\langle x \rangle = \langle x \rangle \cap P = \langle x \rangle \cap k \le \langle x \rangle_{sG}$. So $\langle x \rangle$ is a normal subgroup of $G$ and hence $P = \langle x \rangle$.

(2) Since $\langle x \rangle$ is weakly $c^*$-normal in $G$, then there exists a subnormal subgroup $K$ such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K \le \langle x \rangle_{sG}$, where $\langle x \rangle_{sG}$ is an $s$-quasinormally embedded subgroup of $G$. Let $P_1 = P \cap K$, Hence $P_1 < G$ and $P_1\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P_1\Phi(P)/\Phi(P)$ is either trivial or $P/\Phi(P)$. If the former, then $P_1 \le \Phi(P)$, and $P = P \cap G = \langle x \rangle (P \cap K) = \langle x \rangle \Phi(P) = \langle x \rangle$.

Otherwise $P_1 = P$ and $P \le K$, $G = PK = K$. Hence $\langle x \rangle = \langle x \rangle \cap P = \langle x \rangle \cap K \le \langle x \rangle_{sG}$, So $\langle x \rangle$ is a normal subgroup of $G$ and hence $P = \langle x \rangle$.

The Lemma is proved. □

**Lemma 2.9** Let $H$ be a subgroup of $G$. Then $H$ is weakly $c^*$-normal in $G$ if and only if there exists a subgroup $K$ such that $G = HK$ and $H \cap K = H_{sG}$.

**Proof.** $\Leftarrow$ It is clear.

$\Rightarrow$By Definition 2.1, there exists a subnormal subgroup L of $G$ such that $G = HL$ and $H \cap L \le H_{sG}$. If $H \cap L < H_{sG}$, note that $K = LH_{sG}$,then $HK = HLH_{sG} = HL = G$ and hence $H \cap K = H \cap LH_{sG} = H_{sG}(H \cap L) < H_{sG}$. □

**Main results and their applications**

**Theorem 3.1** Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H$ is supersolvable. If all maximal subgroups of any Sylow subgroups of $H$ are weakly $c^*$-normal in $G$, then $G$ is supersolvable.

Proof. Suppose that the theorem is false, we chose a minimal order group $G$ as a counterexample. We will prove the theorem by the following steps:

*Step 1*. Every proper subgroup of $G$ containing $H$ is supersolvable and $G$ is solvable.

Let $N$ is a proper subgroup of $G$ containing $H$. Then $N/H$ is supersolvable since $G/H$ is supersolvable. By hypothesis, all maximal subgroups of any Sylow subgroups of $H$ are weakly normal in $G$, then all maximal subgroups of any Sylow subgroups of $H$ are weakly $c^*$- normal in $M$ by Lemma 2.2(1). So $N$, $H$ satisfies the hypotheses of the theorem, the minimal choice of $G$ implies that N is supersolvable. Since every maximal subgroup of $G$ is supersolvable, then by [8, Hilfssatz C] $G$ is solvable.

The following. Let $L$ be a minimal normal subgroup of $G$ contained in $H$. Then, by Step 1 and [11, Lemma 8. 6, p102] L is an elementary abellian p-group for some prime divisor $p$ of $|G|$.

*Step 2*. $G/L$ is supersolvable and L is unique.

Since $(G/L)/(H/L)\cong G/H$ is supersolvable, and, by hypotheses, all maximal subgroups of any Sylow subgroups of $H$ are weakly $c^*$-normal in $G$, then all maximal subgroups of any Sylow subgroups of $H/L$ are weakly $c^*$-normal in $G/L$ by Lemma 2.2(2). Then $G/L$, $H/L$ satisfies the hypotheses of the theorem. So the minimal choice of $G$ implies that G=L is supersolvable. Since the class of supersolvable groups is a saturated formation by [10, Satz 10], then L is a unique minimal normal subgroup of $G$.

*Step 3*. $\Phi(G)=1$.

If $\Phi(G)\neq 1$, then there exists a Sylow $p$-subgroup $P$ of $\Phi(G)$, where $p$ is a prime divisor of $|\Phi(G)|$, $P$ char $\Phi(G)$ and so $P$ is normal in $G$. By Step 2, $G/P$ is supersolvable, then $G/\Phi(G)$ is supersolvable and so is $G$, a contradiction. Thus $\Phi(G)=1$.

*Step 4*. $H=G$.

If $H<G$, then $H$ is supersolvable by Step 1. Let $P_2$ be a Sylow $p$-subgroup of $H$, where $p$ is the largest prime divisor of $|H|$. Then by [9, VI-9.1], $P_2<H$. Let $P_1$ be a maximal subgroup of $P_2$. Then, by hypothesis, $P_1$ is weakly $c^*$-normal in $G$ and so there exists a subnormal subgroup $T$ of $G$ such that $G=P_1T$ and $P_1\cap T$ is $s$-quasinormally embedded in $G$. Let $S$ be a Sylow $p$-subgroup of $P_1\cap T$. Then $S$ is also a Sylow $p$-subgroup of some $s$-quasinormal subgroup $K$ of $G$. Let $Q$ be a Sylow $q$-subgroup of $G$, where $q\neq p$ is a prime divisor of $|G|$. Then $Q$ is also a Sylow $q$-subgroup of $T$.

If $S_G=1$, then by [13, Lemma 2.1(3)], then $S$ is $\pi$-quasinormal in $G$. and so $SQ=QS\le T$. $QS$ is supersolvable by Step 1 and $S<QS$. Thus we have for every $q\neq p$, the Sylow $q$-subgroups commutes with $S$, then $S$ is $s$-quasinormal in $G$. Then by Lemma 2.5, we have $O^p(G)\le N_G(P)$ and so $S$ is normal in $G$.

If $S_G\neq 1$, then $S_G<S$, then $S_G$ is $\pi$-quasinormal. And so $QS_G=S_GQ\le T$. Obviously, $T<G$, then by Step 1, $QS_G$ is supersoluble, and $S_G<QS_G$. Then by Lemma 2.5, we have $O^p(G)\le N_G(S_G)$ and so $S_G$ is normal in $G$.

Thus $S=L=S_G$ is a minimal normal subgroup of $G$ by Step 2. By Step 3, $F(H)$ is the direct product of minimal normal subgrroup of $G$ contained in $H$ by [15, Lemma 2.6], then $F(H)=S=L=C_G(L)$ since $L$ is the minimally unique normal subgroup of $G$ and $G$ is solvable. By Lemma 2.7, $\exp(P)=p$ or 4 (if $p=2$), then $L$ is a cyclic subgroup of order $p$ or 4. So $G$ is supersolvable by [19, 2.16]. So we assume that $QS<H$. If $H>T$, then $G=P_1T=P_1H=H$ is supersolvable, and so $H\le T$. By Step 1, $T$ is a supersolvable group. Let $R$ be a Hall $p'$-subgroup of $T$ which is also a Hall $p'$-subgroup of $G$. Then $|T:R|=p$, so $R<T$. So we have $|S|=p$ or $S=1$. In the two cases, $G$ is supersolvable, a contradiction.

*Step 5*. The final conclusion.

By Step 4, we have $F(H)=F(G)$. By Step 1 and [15, Lemma 2.6], $F(H)$ is the direct products of minimal normal subgroups of $G$. Then $F(H)=F(H)=L=C_H(L)$ as $L$ is abelian and $L$ is the unique minimal normal subgroup of $G$. Then we have $F(H)=\langle x_1,x_2,\ldots,x_n\rangle$, where $\langle x_i\rangle$ char $L$ and $x_i\in L\setminus\{1\}$, and so $F(H)=\langle x_i\rangle=L$ by Lemma 2.8. By Step 2, $G/L$ is supersolvable and by Step 1 and Lemma 2.7, $\exp(L)=p$ or 4 (when $p=2$), $\langle x_i\rangle$ is a cyclic subgroup of order p or 4 (when $p=2$), and so by [19, 2.16], $G$ is supersolvable. The final contradiction.

This completes the proof.

**Remark 3.1** The condition of the theorem $G/H$ is supersoluble can't be replaced by "$G/H$ is soluble". Let $G=C_2\times A_4$, where $C_2$ is a cyclic group of order 2 and $A_4$ is the alternating group of degree 4. Obviously $G/C_2\cong A_4$ is soluble. Since every maximal subgroup of $A_4$ is weakly $c^*$-normal in $G$ the hypotheses of the theorem is satisfied, but $G$ is not supersoluble.

Corollary 3.1 ([2, Theorem 3.1])Let $\mathcal{L}$ be a complete set of Sylow subgroups of a group $G$. If the maximal subgroups of $G_p$ are $\mathcal{L}$-permuatble subgroups of $G$, for all $G_p\in\mathcal{L}$, then $G$ is supersoluble.

**Corollary 3.2** ([3, Theorem 4.1 ]) If $G/H$ is supersoluble and all maximal subgroups of any Sylow subgroup of $H$ are $\pi$-quasinormal in $G$, then $G$ is supersoluble.

**Corollary 3.3** ([21, Theorem 4.1]) Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H$ is supersolvable. If all maximal subgroup of any Sylow subgroup of $H$ is $c^*$-normal in $G$, then $G$ is supersolvable.

**Theorem 3.2** Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H$ is supersolvable. If all cyclic subgroups of $H$ with order p or 4 (if $p=2$) are weakly $c^*$-normal in $G$, then $G$ is supersolvable.

Proof. Suppose that the theorem is false, we chose a minimal order group $G$ as a counterexample. We will prove the theorem by the following steps:

*Step 1*. Every proper subgroup of $G$ containing $H$ is supersolvable and $G$ is solvable.

Let $M$ be a proper subgroup of $G$ containing $H$. $M/H$ is supersolvable Since $G/H$ is supersolvable. By hypothesis, all cyclic subgroups of $H$ of order p or 4 (if $p=2$) are weakly $c^*$-normal in $G$, then by Lemma 2.2(1), all cyclic subgroups of $H$ of order p or 4 (if $p=2$) are weakly $c^*$-normal in M. Thus (M, $H$) satisfies the hypotheses of the theorem.

The minimal choice of $G$ implies that M is supersolvable. Then $G$ is not supersolvable but all proper subgroups are supersolvable, so by Lemma 2.7, we have $G$ has a unique normal Sylow $p$-subgroup $P$ of $G$ for some prime divisor $p$ of $|G|$. And also $G$ is solvable by [8, Hilfssatz C ].

The following. Let $L$ be a minimal normal subgroup of $G$ contained in $H$. Then, by Step 1 and [11, Lemma 8.6, p102] L is an elementary abellian $p$-group for some prime divisor $p$ of $|G|$.

*Step 2*. $G/L$ is supersolvable and $L \not\leq \Phi(G)$.

By hypothesis, all cyclic subgroups of $H$ of order p or 4 (if $p=2$) are weakly $c^*$-normal in $G$, then by Lemma 2.2(2), all cyclic subgroups of $H/L$ of order $p$ or 4 (if $p=2$) are weakly $c^*$-normal in $G/L$. And since $(G/L)/(H/L) \cong G/H$ is supersolvable, then $G/L$, $H/L$ satisfies the hypotheses of the theorem. Thus the minimality of $G$ implies that $G/L$ is supersolvable. Since the class of supersolvable groups is a saturated formation by [10, Satz 10], then L is unique. If $L \leq \Phi(G)$, then $G/\Phi(G)$ is supersolvable. Thus $G$ is supersolvable, a contradiction.

*Step 3*. $H=G$ and so $F(H)=F(G)$.

If $H<G$, then $H$ is supersolvable by Step 1. Let $Q$ be a Sylow q-subgroup of $H$, where $q$ is the largest prime divisor of $|H|$. Then by [9, VI-9.1], $Q$ char $H$ and $H<G$, so $Q<G$. If $q \neq p$, then $G/1 \cong G/P \times G/Q$ is supersolvable, and so $G$ is supersolvable, a contradiction. So $p=q$ and $P=Q$ or $P=L$ by Step 1. If the latter, then by Lemma 2.8, we have $L$ is cyclic of order $p$ or 4 (if $p=2$). Thus $G$ is supersolvable since $G/L$ is supersolvable by [19, 2.16]. Then we assume that $P=Q$. By hypothesis, all cyclic subgroups $C$ of $P$ of order $p$ or (if $p=2$) is weakly $c^*$-normal in $G$. Then there exists a subnormal T such that $G=CT$ and $C \cap T \leq C_{sG}$. We will deal with this from the following cases:

Case 1: $C_{sG}=1$.

Obviously $T \geq H$, for otherwise, $T \ngeq H$, then if $T \cap H=1$, $C \leq H$ and so $G=CT=HT$, $C \cap T \leq H \cap T=1$, and $C=H$ is a normal cyclic subgroup of $G$.

Thus $G$ is supersolvable,a contradiction. If $T<H$,then $G=CT=CH$. Obviously $C \leq H$, then $G=CH=H$, a contradiction.

Then we have $H \leq T$. Then by Step 1, $T$ is supersolvable. But $G=CT$ is supersolvable by [4], a contradiction.

Case 2: $C \cap T=C_{sG}=C$, where $P$ is abelian or $p$ is an odd prime.

In this case, $C \leq T$, then $G=CT=T$ is supersolvable by Step 1, a contradiction.

Case 3: $1<C \cap T=C_{sG} \leq C$, where $P$ is non-abelian and $p=2$.

By hypothesis and Lemma 2.7(4) $|C|=4$, then $|C_{sG}|=2$. Since $C$ is $s$-quasinormally embedded in $G$, then there exists a s-quasinormal subgroup K of $G$ such that C is a Sylow 2-subgroup of $K$. Since for any Sylow $q$-subgroup $Q$ of $G$, where $q \neq 2$, such that $KQ=QK$ is a subgroup of $G$. If $KQ \geq H$, then by Step 1, $KQ$ is supersolvable, then $K$ is normal in $KQ$ and $C$ is a Sylow 2-subgroup of $KQ$, and so $C_{sG}<KQ$. We have $C_{sG}Q=QC_{sG}$ and so $C_{sG}$ is $s$-quasinormal in G. By Lemma 2.5, $O^p(G) \leq N_G(C_{sG})$. Thus $C_{sG}<G$ so $L=C_{sG}$ and $LQ=Q \times L$. Then $Q \leq C_{sG}(L) \leq L$, by [8, Hilfssatz C], $G$ is solvable, a contradiction.

*Step 4*. $H \cap \Phi(G)=1$.

If not, then there exists a prime divisor $r$ of $|\Phi(G)|$, and Let $R$ be a Sylow $r$-subgroup of $\Phi(G)$, then $R$ is normal in $G$. By Step 2, $G/R$ is supersolvable and so $G/\Phi(G)$ is supersolvable. Thus $G$ is supersolvable, a contradiction.

*Step 5*. Conclusion.

By Step 4 and [15, Lemma 2.6], then $F(H)$ is the direct products of minimal normal subgroups of $G$ containing in $H$. By Step 2, $F(H)=L$. By Step 1 and Lemma 2.7, $\exp(L)=p$ or 4. By hypothesis, cyclic of $L$ of order p or 4 (if $p=2$) are weakly $c^*$-normal in $G$. Thus by Lemma 2.8, $L=\langle x \rangle$, for some $x \in L \setminus \{1\}$, is a cyclic normal minimal subgroup of $G$. Then by [19, 2.16], $G$ is supersolvable since $G/L$ is supersolvable by Step 2.

The final contradiction completes the proof. □

**Corollary 3.4** ( [1, Theorem B])Let $G$ be a finite group. If there exists a normal subgroup $H$ such that G=H is supersoluble, $(|H|,2)=1$, and every minimal subgroup $G/H$ of $H$ is pronormal in $G$, then $G$ is supersoluble.

Corollary 3.5 ([18, Theorem 3.1]) Let $H$ be a normal $p$-subgroup of $G$ such that $G/H$ is supersolvable. Suppose that every cyclic subgroup of $H$ of order $p$ or 4 (if $p=2$) is $\pi$-quasinormal in $G$, then $G$ is supersolvable.

As a generalization of Theorem 1.1 and 1.2, we get:

**Theorem 3.3** Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H$ is supersolvable. then $G$ is supersolvable if one of the followings contains

(1)all maximal subgroups of any Sylow subgroups of $H$ are weakly $c^*$-normal in $G$;

(2)all cyclic subgroups of $H$ with order p or 4 (if p = 2) are weakly $c^*$-normal in $G$.

**Theorem 3.4** Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{A}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H\in\mathfrak{F}$. If every maximal subgroup of all Sylow subgroups of $H$ is weakly $c^*$-normal in $G$, then $G\in\mathfrak{F}$.

Proof. Suppose that the theorem is false, we chose a minimal order group $G$ as a counterexample. Then $G$ is not supersoluble but all proper subgroups are supersoluble. Thus by Lemma 2.7, $G$ has a normal Sylow p-subgroup for some prime p dividing $|G|$, and $G/P\in\mathfrak{F}$ by induction. By hypotheses, all maximal subgroups of $P$ are weakly $c^*$-normal in $G$, then by Theorem 1.1, $G\in\mathfrak{F}$, a contradiction.

This completes the proof.

**Corollary 3.6** [20] Let F be a saturated formation containing $\mathfrak{A}$ and let $H$ be a normal subgroup of a group $G$ such that $G/H\in\mathfrak{F}$. Suppose that every member of $\mathfrak{M}(H)$ is $c^*$-normal in $G$. Then $G$ is in $\mathfrak{F}$.

**Theorem 3.5** Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{A}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H\in\mathfrak{F}$. If every subgroup of $H$ of order $p$ or 4 (when $p=2$) is weakly $c^*$-normal in $G$, then $G\in\mathfrak{F}$.

Proof. Suppose that the theorem is false, we chose a minimal order group $G$ as a counterexample. Then $G$ is not supersoluble but all proper subgroups are supersoluble.

Thus by Lemma 2.7, G has a normal Sylow $p$-subgroup for some prime $p$ dividing $|G|$, and $G/P\in\mathfrak{F}$ by induction. By hypotheses, all subgroups of order $p$ or 4 (when $p=2$) of $P$ are weakly $c^*$-normal in $G$, then by Theorem 1.2, $G\in\mathfrak{F}$, a contradiction.

This completes the proof.

**Remark 3.2** The condition of Theorem2 3.4 and 3.5 " $\mathfrak{A}$ " can't be replaced by " $\mathfrak{N}$ ", the class of nilpotent groups. For example $G=S_3$, the symmetric group of degree 3. Let $P$ be the Sylow 3-subgroup. Obviously, $G$ ; $P$ satisfies the hypotheses, but $G$ is not in $\mathfrak{F}$.

**Corollary 3.7** ([5, Theorem 5])Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{A}$. Let $G$ be a group with abelian Sylow 2-subgroups. If $H$ is a normal subgroup of $G$ such that $G/H\in\mathfrak{F}$ and every minimal subgroup of $H$ is permutable in $G$, then $G\in\mathfrak{F}$.

**Corollary 3.8** ([5, Theorem 2]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{A}$, the class of all supersoluble groups. Assume that $G$ is a group with a normal subgroup $H$ such that $G/H\in\mathfrak{F}$. If every generator of $\Psi(H)$ is permutable in $G$, then $G\in\mathfrak{F}$.

By Theorems 3.4 and 3.5, we have

**Theorem 3.6** Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{A}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H\in\mathfrak{F}$,. then $G\in\mathfrak{F}$ if one of the followings contains

(1)all maximal subgroups of any Sylow subgroups of $H$ are weakly $c^*$-normal in $G$;

(2)all cyclic subgroups of $H$ with order p or 4 (if $p=2$) are weakly $c^*$-normal in $G$.

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