Solution of linear and nonlinear partial differential equation by using projected differential transform method

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ARTICLE INFO	ABSTRACT
	In this paper we introduced the modified version of the differential transform method, which is called the projected differential transform method (PDTM) .This method can be easily applied to the initial value problems with less computational work. It can also be applied to solve linear and nonlinear partial differential equations. Firstly, we stated the definition of the projected transform method, and some related theorems. Then some illustrative examples are given, the numerical results of these examples compared with those obtained by the Differential transform method are found to be the same.

Introduction

The projected differential transform method can be easily applied to many linear and nonlinear problems and is capable of reducing the size of computational work .some problems in science and engineering fields can be described by the initial value problems .A variety of numerical and analytical methods have been developed to obtain accurate approximate and analytic solutions for the initial value problems in the literature. [3,5,8,9].

Projected differential transform method is a semi-numerical analytic technique that formalizes the Taylor series in a totally different manner. With this method, the given differential equation and related initial conditions are transformed into a recurrence equation, that finally transforms partial differential equations to algebraic equations which can easily be solved. In this method no need for linearization or perturbations, much computational work and round- off errors are avoided. In resent years many researchers apply the PDTM for solving partial differential equations [1,2].

This method constructs, for partial differential equations an analytical solution in the form of a polynomial. Not like the traditional high order Taylor series method that requires symbolic computations. Another important advantage is that this method reduces the size of computational work while the Taylor series method is computationally taking long time for higher orders. This method is well addressed in [4,6].

Projected differential transform:

The basic definitions and fundamental theorems of projected differential transform method are defined in [3] and will be stated briefly in this paper.

Projected differential transform of function $y\big(\phi(x_1, x_2, ..., x_n), t\big)$ is defined as follows:

$$y(\phi, k) = \frac{1}{k!}\left[\frac{d^k y(\phi, t)}{dt^k}\right]_{t=0} \quad (1)$$

Where $y\big(\phi(x_1, x_2, ..., x_n), t\big)$ the original is is function and $y(\phi, k)$ is the transformed function.

The inverse differential transform of $y(\phi, k)$ is defined as.

$$y(\phi, t) = \sum_{k=0}^{\infty} y(\phi, k)\, t^k \quad (2)$$

Combining eqs (1) and (2) we

$$y(\phi, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k y(\phi, t)}{dt^k}\right]_{t=0} t^k \quad (3)$$

The fundamental theorems of the projected differential transform are:

Theorems:

(1) If $z(\phi, t) = u(\phi, t) \pm v(\phi, t)$

Then $z(\phi, k) = u(\phi, k) \pm v(\phi, k)$

(2) If $z(\phi, t) = c\, u(\phi, t)$

Then $z(\phi, k) = cu(\phi, k)$

(3) If $z(\phi, t) = \dfrac{du(\phi, t)}{dt}$

Then $z(\phi, k) = (k+1)u(\phi, k+1)$

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$(4)\, If\, z(\phi,t) = \dfrac{d^n u(\phi,t)}{dt^n}$

$Then\; z(\phi,k) = \dfrac{(k+n)!}{k!} u(\phi,k+n)$

$(5)\; If\quad z(\phi,t) = u(\phi,t)\, v(\phi,t)$

$Then\quad z(\phi,k) = \sum\limits_{m=0}^{k} u(\phi,m)\, v(\phi,k-m)$

$(6)\, If\; z(\phi,t) = u_1(\phi,t) u_2(\phi,t) \ldots u_n(\phi,t)\, Then$

$z(\phi,k) = \sum\limits_{k_{n-1}=0}^{k} \sum\limits_{k_{n-2}=0}^{k_{n-1}} \ldots\ldots \sum\limits_{k_2=0}^{k_3} \sum\limits_{k_1=0}^{k_2} u_1(\phi,k_1) u_2(\phi,k_2-k_1)\ldots u_{n-1}(\phi,k_{n-1}-k_{n-2}) u_n(\phi,k-k_{n-1})$

$(7)\quad If\quad z(\phi,t) = \varphi(x_1,x_2,\ldots,x_n) t^m$

$Then\qquad z(\phi,k) = \varphi\delta(m-k) = \begin{cases} 1 & k=m \\ 0 & k\neq m \end{cases}$

Note that c is Constant, n is a nonnegative integer
and $\phi = \phi(x_1,x_2,\ldots,x_n)$.

Example:-
Consider the following linear PDE with boundary conditions

$$x\frac{\partial w(x,t)}{\partial x} + \frac{\partial w(x,t)}{\partial t} = xt \qquad (4)$$

With the boundary conditions

$$w(x,0) = 0\;,\quad w(0,t) = 0 \qquad (5)$$

Using the projected differential transform method with respect to t of eq (4) yields .

$$x\frac{\partial w(x,h)}{\partial x} + (h+1)w(x,h+1) = xs(h-1) \qquad (6)$$

Substituting eq (5) in to eq (6) we have

$$w(x,1) = 0,\; w(x,2) = \frac{x}{2!},\; w(x,3) = \frac{-x}{3!},\; w(x,4) = \frac{x}{4!}$$

And so on in generally we find

$$w(x,h) = \frac{(-1)^h x}{h!}, h = 2,3,4,\text{L L}$$

Substituting $w(x,h)$ in to eq (2) we get:

$$w(x,t) = \sum_{h=o}^{\infty} w(x,h) t^h = \sum_{h=2}^{\infty} \frac{x(-1)^h}{h!} t^h = \sum_{h=2}^{h} \frac{x(-t)^h}{h!} =$$

$$(t-1)x + \sum_{n=0}^{\infty} \frac{x(-t)^h}{h!} = (t-1)x + xe^{-t}$$

Example:-
Consider the following nonlinear P.D.E with initial condition.

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} \qquad (7)$$

$i.c\; u(x,0) = \dfrac{x^2}{c}\quad x>0, c>0$ arbitrary constant Taking the

projected differential transform methods of eq (7) we get

$$(h+1)u(x,h+1) = \sum_{m=o}^{h} \frac{\partial u(x,m)}{\partial x}\cdot\frac{\partial}{\partial x}u(x,h-m) +$$

$$\sum_{m=o}^{h} u(x,m)\frac{\partial^2 u(x,h-m)}{\partial x^2} \qquad (8)$$

Using eq (8) and I.C we have

$$u(x,1) = \frac{6x^2}{c^2},\; (x,2) = \frac{36x^2}{c^2},\; u(x,4) = \frac{216x^2}{c^2}$$

And so on in general $\quad u(x,h) = \dfrac{(6)^h x^2}{c^{h+1}}$

Substituting $u(x,h)$ into equation (2) we get:

$$u(x,h) = \sum_{h=o}^{\infty} u(x,h) t^h = \sum_{h=o}^{\infty} \frac{(6)^h x^2 t^h}{c^{h+1}} =$$

$$\frac{x^2}{c}\sum_{h=o}^{\infty}\left(\frac{6t}{c}\right)^h = \frac{x^2}{c}\left(1-\frac{6t}{c}\right)^{-1} = \left(\frac{x^2}{c-6t}\right)$$

Example:-
Consider the following nonlinear P.D.E

$$\frac{\partial u}{\partial t} = 2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} \qquad (9)$$

With the initial condition

$$u(x,0) = \frac{x+h}{2\sqrt{c}} \qquad (10)$$

By applying the projected differential transform method of eq (9) we

$$(h+1)u(x,h+1) = 2\sum_{k=o}^{h}\sum_{m=o}^{k} u(x,m)\frac{\partial}{\partial x}u(x,k-m)\frac{\partial}{\partial u}(x,h-k)$$

$$+ \sum_{k=o}^{h}\sum_{m=o}^{k} u(x,m)u(x,k-m)\frac{\partial^2 u(x,h-k)}{\partial x^2} \qquad (11)$$

Using eqs (10) and (11) we get

$$u(x,1) = \frac{\frac{1}{2}}{1!c}\frac{(x+h)}{2\sqrt{c}},\; u(x,2) = \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\frac{1}{c^2}\left(\frac{x+h}{2\sqrt{c}}\right)$$

$$u(x,3) = \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!\, c^3}\cdot\left(\frac{x+h}{2\sqrt{c}}\right)$$

And so on in

general $u(x,h) = \dfrac{x+h}{2\sqrt{c}}\left(\prod\limits_{i=0}^{h-1} \dfrac{\left(\frac{1}{2}+i\right)}{h!\, c^h}\right)\quad, h=1,2,3,\text{L}$

Substituting $u(x,h)$ into equation (2) we have

$$u(x,t) = \sum_{h=o}^{\infty} u(x,h) t^h = \frac{x+h}{2\sqrt{c}} + \sum_{h=1}^{\infty}\left(\frac{x+h}{2\sqrt{c}}\right)\prod_{i=0}^{h-1}\frac{\left(\frac{1}{2}+i\right)}{h!\, c^h} t^h$$

$$= \frac{x+h}{2\sqrt{c}}\left[1+\sum_{h=1}^{\infty}\frac{\left(\prod_{i=0}^{h-1}\left(\frac{1}{2}+i\right)\right)\cdot\left(\frac{t}{c}\right)^{h}}{h!}\right]=\frac{x+h}{2\sqrt{c}}\left[1-\frac{t}{c}\right]^{\frac{-1}{2}}$$

Example:-

Consider the Klein – Gordon equation

$$u_{tt}-u_{xx}+\frac{\pi^2}{4}u+u^2=x^2\sin^2\frac{\pi}{2}t \qquad (12)$$

With initial conditions

$$u(x,0)=0 \quad ,u_t(t,0)=\frac{\pi}{2}x \qquad (13)$$

Taking the projected differential transform method of equation (12) we have

$$(h+1)(h+2)u(x,h+2)-\frac{\partial^2 u(x,h)}{\partial x^2}+\frac{\pi^2}{4}u(x,h)$$

$$+\sum_{m=o}^{h}u(x,m)u(x,h-m)=x^2\sum_{m=o}^{h}A(m)A(h-m) \qquad (14)$$

Where $A(t)=\sin\frac{\pi}{2}t$. The transform of $A(t)$ is

$$A(h)=\begin{vmatrix}\left(\frac{\pi}{2}\right)^{h}\frac{(-1)^{\frac{h-1}{2}}}{h!} & h \quad is \quad odd \\ 0 & h \quad is \quad even\end{vmatrix} \qquad (15)$$

Using eqs (13) , (14) and (15) we have

$$u(x,2)=0 \quad ,u(x,3)=-\left(\frac{\pi}{2}\right)^{3}\frac{x}{3!}$$

$$u(x,4)=0 \quad , \quad u(x,5)=\left(\frac{\pi}{2}\right)^{5}\frac{x}{5!} \quad \text{And so on in}$$

general

$$u(x,h)=\begin{vmatrix}0 & h \ is \quad even \\ (-1)^{\frac{h-1}{2}}\left(\frac{\pi}{2}\right)^{h}\frac{x}{h!} & h \ is \ odd\end{vmatrix}$$

Substituting $u(x,h)$ into equation (2) yields

$$u(x,t)=x\sum_{h=o}^{\infty}\frac{(-1)^{h}\left(\frac{\pi}{2}t\right)^{2h+1}}{(2h+1)!}=x\sin\frac{\pi}{2}t$$

Conclusion:-

Projected differential transform have been applied to linear and nonlienear partial differential equations .The results for all examples can be obtained in Tayler's series form, all the calculations in the method are very easy. In summary, using projected differential transformation to solve PDE consists of three main steps. First, transformacting PDE into algebraic equation, second, solving the equations, finally inverting the solution of algebraic equations to obtain a closed form series solution.

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