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Self-adjointness of product of two high-order singular differential operators

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ABSTRACT

The self-adjointness of product of two singular differential operators generated by a th-order symmetric differential expression is discussed. By means of the construction theory of real parameter on differential operators and matrix computation, the necessary and sufficient conditions which make the product being the self-adjoint operators in are obtained.

Introduction

Let l be a formal symmetric ordinary differential expression of order $2n$ on the interval (a,b). We assume that the product l^2 can be well formed. Let L be a differential operator generated by l in $L^2((a,b))$ with some boundary conditions ([1]).

It is known that the product l^2 is a formally symmetric differential expression of order $4n$ on (a,b) with equal deficiency indices, and the minimal operator associated with l^2 in $L^2((a,b))$ is a positive symmetric differential operator (cf. [2, 5]).

There have been many research results on the subject of the products or powers of differential operators, especially on deficiency indices of powers of l and commutativity of differential expressions (see [2-4]).

On the problem of determining the self-adjointness of the product of two differential operators, it has been considered in [6] when l is a limit-circle Sturm-Liouville operator on (a,b). By means of the construction theory of differential operator and matrix computation, a necessary and sufficient condition for the self-adjointness of the product two high-order differential operators with limit-circle case was given (see [7]).

Moreover, in [8], the self-adjointness of the product of three high-order differential operators with limit-point case was considered.

Here we consider the self-adjointness of the product of two $2n$ th-order ordinary differential operators with middle deficiency indices.

By using the techniques of the construction theory of real parameter on differential operators and matrix computation, the necessary and sufficient conditions which make the product $L = L_2 L_1$ being the self-adjoint operators in (a,b) are obtained, which promote and deepen the previous conclusions.

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Preliminaries

We consider the formal symmetric ordinary differential expression of order $2n$ on the interval (a,b), given by

$$l(y) := \sum_{j=0}^{n} (-1)^j (p_{n-j} y^{(j)})^{(j)}, x \in (a,b), -\infty \le a < b \le \infty. \qquad (1)$$

We assume that the real coefficients $p_j(t)$ satisfy the following basic conditions:

$$p_0(x) > 0 \text{ and } p_{n-j}(x) \in C^{(2n-j)}(a,b), j = 0,1,2,\ldots,n. \qquad (2)$$

The basic conditions ensure that the formal square l^2 of l, defined by

$$l^2 y = l(ly) \qquad (3)$$

exists as a differential expression. Furthermore, l and l^2 are regular on $[a,b)$ and singular on endpoint b.

Note that in this paper we will write a matrix A with m rows and n columns as $A = (a_{ij})_{m\times n}$, where a_{ij} is the element of A appearing in the i th row and j th column. Let A^T and A^* denote the transpose and complex conjugation transpose of A respectively.

In the following, one can easily see that the facts and notations we introduce for l are applicable for l^2 similarly.

Let n_+ and n_- denote the deficiency indices of the formally symmetric differential expression l on $[a,b)$ associated with the upper and lower half-planes respectively. Write $\operatorname{def}(l) = (n_+, n_-)$. Here, n_+ and n_- are not necessarily equal.

Definition 1 [11] Under the basic conditions (2), we say that l^2 is partially separated in $L^2((a,b))$ if $y \in L^2((a,b))$, $y^{(4n-1)} \in AC_{loc}((a,b))$ and $l^2(y) \in L^2((a,b))$ together imply that $l(y) \in L^2((a,b))$.

Lemma 1 Let (2) hold. Then l^2 is partially separated in $L^2((a,b))$ if and only if

$$\operatorname{def}(l)=(n_+ + n_-, n_+ + n_-). \qquad (4)$$

Proof See [11, 180-181].

We define the operator $L_M(l)$ by $L_M(l)y = ly$, $y \in D_M(l)$, where

$D_M(l)=\{y\in L^2((a,b)): y^{(2n-1)}\in AC_{loc}((a,b)), l(y)\in L^2((a,b))\}$.

The operator $L_M(l)$ is called the maximal operator of l on $[a,b]$ and its domain $D_M(l)$ is called the maximal domain of l on $[a,b]$. It is known that $D_M(l)$ is dense in $L^2((a,b))$. Thus the adjoint of $L_M(l)$ is well defined. Let $L_0(l)=L_M(l)$, the operator $L_0(l)$ is called the minimal operator of l on $[a,b]$ and $D_0(l)$ is called the minimal domain of l on $[a,b]$.

For any $y,z\in D_M(l)$, we have the Green's formula,

$$\int_a^b (l(y)\bar z - y\overline{l(z)})dt = [y,z]_{2n}(b)-[y,z]_{2n}(a),$$

where

$$[y,z]_{2n}(x)=R(\bar z)Q_{2n}(x)C(y),$$

$$C(y)=\begin{pmatrix} y(x)\\ \vdots \\ y^{(2n-1)}(x)\end{pmatrix},\quad R(z)=(z(x),\cdots,z^{(2n-1)}(x)), \qquad (5)$$

and $Q_{2n}(x)=[q_{jk}]$ $(j,k=1,2,\cdots,2n)$. $[\cdot,\cdot]_{2n}$ is called the Lagrange bilinear form corresponding to $l(y)$ on $[a,b]$.

Lemma 2[9] $y\in D_0(l)$ holds if and only if

(i) $y(a)=y'(a)=\cdots=y^{(2n-1)}(a)=0$,

(ii) $\forall z\in D_m(l)$, $[y,z]_{2n}(b)=0$.

Let z_j $(j=1,2,\cdots,2n)$ be a set of functions in $D_M(l)$ which satisfy the following conditions: $z_j^{(k-1)}(a)=\delta_{jk}$, $z_j^{(k-1)}(a_1)=0$, $z_j(x)=0$,

$$a<a_1\le x<b, (j,k=1,2,\cdots,2n). \qquad (6)$$

Lemma 3[12] If $\operatorname{def}(l)=(m,m)$, then for any $\lambda_0\in\mathbb{R}$, the number of solutions in $L^2((a,b))$ of $ly=\lambda_0 y$ is not more than m.

Lemma4[10] Let $\lambda_0\in(\mu_1,\mu_2)$ and $\theta_1,\theta_2,\cdots,\theta_m\in L^2((a,b))$ be linearly independent square integrable solutions of $ly=\lambda_0 y$. Then there must exist $\theta_1,\theta_2,\cdots,\theta_{2m-2n}$, satisfy the following conditions:

$\operatorname{rank}([\theta_i,\theta_j](a))_{1\le i,j\le 2m-2n}=2m-2n$ and

$$D_M(l)=D_0(l)\dotplus \operatorname{span}\{z_1,z_2,\cdots,z_{2n}\}\dotplus\operatorname{span}\{\theta_1,\theta_2,\cdots,\theta_{2m-2n}\},$$

where the symbol $\dotplus$ denotes a direct sum, $\operatorname{span}\{z_1,z_2,\cdots,z_{2n}\}$ denotes the linear span of $z_1,z_2,\cdots,z_{2n}$ and $\operatorname{span}\{\theta_1,\theta_2,\cdots,\theta_{2m-2n}\}$ denotes the linear span of $\theta_1,\theta_2,\cdots,\theta_{2m-2n}$.

Definition 2 Under the assumption that Lemma 3, $\theta_1,\theta_2,\cdots,\theta_{2m-2n}$ are called the second L^2- solutions of $ly=\lambda_0 y$.

Let

$B=([\theta_i,\theta_j](a))^T_{1\le i,j\le 2m-2n}$,

where $\theta_1,\theta_2,\cdots,\theta_{2m-2n}$ are the second L^2- solutions of $ly=\lambda_0 y$. Then we have the following lemma:

Lemma 5[10] Let $l(y)$ be a closed symmetric operator with deficiency indices (m,m). Then linear manifold $D\subset D_M(l)$ is the self-adjoint extension domain of L_0 if and only if there exist numerical matrixes $M_{m\times 2n}$ and $N_{m\times(2m-2n)}$, satisfy:

(1) $\operatorname{rank}(M\oplus N)=m$;

(2) $MQ^{-1}(a)M^*+NBN^*=0$;

(3) $D=\left\{y\in D(L_M): M\begin{pmatrix} y(a)\\ \vdots\\ y^{(2n-1)}\end{pmatrix} - N\begin{pmatrix}[y,\theta_1](b)\\ \vdots \\ [y,\theta_{2m-2n}](b)\end{pmatrix}=0\right\}$.

Self-adjointness of product of two operators on (a,b)

In this section, we always assume that the basic conditions (2) hold and l^2 is partially separated in $L^2([a,b])$. Let

$$ly=y^{(2n)}+q(t)y, \qquad (7)$$

where $n\in\mathbb{N}$, $t\in(a,b)$ and $q(t)$ is real function. We easily see from (7) that

$$Q_{2n}^{-1}(t)=\begin{pmatrix} 0 & 0 & \cdots & 0 & -1\\ 0 & 0 & \cdots & 1 & 0\\ \vdots & \vdots & 0 & \vdots & \vdots\\ 0 & -1 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\end{pmatrix}. \qquad (8)$$

Then $Q_{2n}^2(a)=-I_{2n\times 2n}$, $Q_{2n}^{-1}(a)=-Q_{2n}(a)$.

Here assume that $\operatorname{def}(l)=(m,m)$. By Lemma 4, the second L^2- solutions $\theta_1,\theta_2,\cdots,\theta_{2m-2n}$ of $ly=0$ satisfy

$$(C_{2n}(\theta_1)(a),C_{2n}(\theta_2)(a),\cdots,C_{2n}(\theta_{2m-2n})(a))=W(\theta(a))=\begin{pmatrix} O_{(2n-m)\times(2m-2n)}\\ I_{(2m-2n)\times(2m-2n)}\\ O_{(2n-m)\times(2m-2n)}\end{pmatrix},$$

where $C_{2n}(\theta_k)(a)=(\theta_k(a),\theta_k'(a),\cdots,\theta_k^{(2n-1)}(a))^T$ and $W(\theta(t))$ denotes the Wronski matrix of $\{\theta_i(t); i=1,2,\cdots,2m-2n\}$. Let $B_{2m-2n}=([\theta_i,\theta_j]_{2n})^T$, then

$$B_{2m-2n}(b)=B_{2m-2n}(a)=([\theta_i,\theta_j]_{2n}(a))^T=(W(\theta(a)))^T Q_{2n}(a)W(\theta(a))$$
,

i.e,

$$B_{2m-2n}(b)=Q_{2m-2n}(a). \tag{9}$$

Here,

$$Q_{2m-2n}(a)=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & -1 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 1 & \cdots & 0 & 0\\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let

$$\begin{cases} L_i(y)=y^{(2n)}+q(t)y, t\in(a,b),\\ \quad y\in D(L_i)\subset D_M(l), i=1,2, \end{cases}$$

where

$$D(L_1)=\left\{y\in D_M(l): A\begin{pmatrix} y(a)\\ \vdots\\ y^{(2n-1)}\end{pmatrix}-B\begin{pmatrix}[y,\theta_1]_{2n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{2n}(b)\end{pmatrix}=0\right\}, \tag{10}$$

$$D(L_2)=\left\{y\in D_M(l): C\begin{pmatrix} y(a)\\ \vdots\\ y^{(2n-1)}\end{pmatrix}-D\begin{pmatrix}[y,\theta_1]_{2n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{2n}(b)\end{pmatrix}=0\right\}, \tag{11}$$

A,C are $m\times 2n$ order numerical matrixes, B,D are $m\times(2m-2n)$ order numerical matrixes and $\mathrm{Rank}(A\oplus B)=\mathrm{Rank}(C\oplus D)=2m$. Let

$$l^2(y)=y^{(4n)}+(qy)^{(2n)}+qy^{(2n)}+q^2y=y^{(4n)}+\sum_{k=1}^{2n-1}C_{2n}^k q^{(2n-k)}y^{(k)}+2qy^{(2n)}+(q^2+q^{(2n)})y. \tag{12}$$

By calculation, we can get

$$Q_{4n}(t)=\begin{pmatrix} H_1(t) & Q_{2n}(t)\\ Q_{2n}(t) & 0\end{pmatrix},$$

$$Q_{4n}^{-1}(t)=\begin{pmatrix} 0 & Q_{2n}^{-1}(t)\\ Q_{2n}^{-1}(t) & H_2(t)\end{pmatrix}, \tag{13}$$

where $H_2(t)=-Q_{2n}^{-1}(t)H_1(t)Q_{2n}^{-1}(t)$,

$$H_1(t)=\begin{pmatrix} 0 & (C_{2n-1}^1+1)q^{(2n-2)}(t) & \cdots & (C_{2n-1}^{2n-2}-1)q'(t) & 2q(t)\\ -(C_{2n-1}^1+1)q^{(2n-2)}(t) & 0 & \cdots & -2q(t) & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -(C_{2n-1}^{2n-2}-1)q'(t) & 2q(t) & \cdots & 0 & 0\\ -2q(t) & 0 & \cdots & 0 & 0\end{pmatrix}$$

.

Proposition 6 If $\operatorname{def}(l)=(m,m)$, then $\operatorname{def}(l^2)=(m',m'')$ and $2m\le m',m''\le m+2n$.

Proof If $\operatorname{def}(l)=(m,m)$, then $\psi_1,\psi_2,\cdots,\psi_{2n-m}$ are the first L^2-solutions of $ly=0$ and are also the first L^2-solutions of $l^2(y)=0$. In fact, for $y\in D_M(l)$, $[\psi_i,y](b)=0$, $D(l^2)\subset D(l)\subset D_M(l)$. So $l^2(y)$ have middle defect indices (m',m''). It follows from $ly=iy$, $ly=-iy$, $l^2(y)=-y$ has at least $2m$ linearly independent square integrable solutions on $[a,b]$, i.e., $2m\le m',m''\le m+2n$.

Corollary 7 If $l(y)$ is limit-circle case, then $l^k(y)(k=1,2,\cdots)$ is also limit-circle case.

Lemma 8 If $\operatorname{def}(l)=(m,m)$, then $\operatorname{def}(l^2)=(2m,2m)$.

Proof If $\operatorname{def}(l)=(m,m)$. Under the basic conditions (2), it is known that l^2 is a formally symmetric differential expression of order $4n$ on $[a,b]$. Furthermore, it follows from (4) that $\operatorname{def}(l^2)=(2m,2m)$.

In the following, similar to the notations of Section 1, let $L_0(l^2)$ and $L_M(l^2)$ be the minimal and maximal operators generated by l^2 in $L^2((a,b))$, $D_0(l^2)$ and $D_M(l^2)$ be the domains of $L_0(l^2)$ and $L_M(l^2)$, respectively. Let $[\cdot,\cdot]_{4n}(t)$ be the Lagrange bilinear form corresponding to l^2 on $[a,b]$.

Lemma 9 For any $y,z\in D_M(l^2)$, we have

$[y,z]_{4n}(t)=[l(y),z]_{2n}(t)+[y,l(z)]_{2n}(t)$, $t\in(a,b)$.

Let $\varphi_1,\varphi_2,\cdots,\varphi_{2m-2n}$ are the second L^2- solutions of l^2 and satisfy

$$(C_{4n}(\varphi_1)(a),C_{4n}(\varphi_2)(a),\cdots,C_{4n}(\varphi_{2m-2n})(a))=\begin{pmatrix}0_{(4n-m)\times(2m-2n)}\\ I_{(2m-2n)\times(2m-2n)}\\ 0_{(2n-m)\times(2m-2n)}\end{pmatrix}$$

From Lemma 4, $\varphi_i\in L^2((a,b))$ $(i=1,2,\cdots,2m-2n)$. Furthermore $l(\theta_i)=0$, $l^2(\theta_i)=l(l\theta_i)=0$, then $\theta_i(i=1,2,\cdots,2m-2n)$ is also the solution of l^2. The matrix of $\theta_1,\theta_2,\cdots,\theta_{2m-2n},\varphi_1,\varphi_2,\cdots,\varphi_{2m-2n}$ is defined by

$$(C_{4n}(\theta_1)(a),\cdots,C_{4n}(\theta_{2m-2n}),C_{4n}(\varphi_1)(a),\cdots,C_{4n}(\varphi_{2m-2n})(a))=\begin{pmatrix}\psi_{11} & 0_{2n\times(2m-2n)}\\ \psi_{21} & \psi_{11}\end{pmatrix} \tag{14}$$

where $\psi_{11}=\begin{pmatrix}0_{(2n-m)\times(2m-2n)}\\ I_{(2m-2n)\times(2m-2n)}\\ 0_{(2n-m)\times(2m-2n)}\end{pmatrix}$. So $\theta_i,\varphi_i(i=1,2,\cdots,2m-2n)$ are the second L^2-solutions of $l^2y=0$ and

$$B_{4m-4n}(b)=B_{4m-4n}(a)=([\xi_i,\xi_j]_{4n}(a))^T=(W(\theta(a)))^T Q_{4n}(a)W(\theta(a))$$

$$=\begin{pmatrix}\psi_{11}^T & \psi_{21}^T\\ 0_{2n\times(2m-2n)}^T & \psi_{11}^T\end{pmatrix}\begin{pmatrix}0 & Q_{2n}\\ Q_{2n} & 0\end{pmatrix}\begin{pmatrix}\psi_{11} & 0_{2n\times(2m-2n)}\\ \psi_{21} & \psi_{11}\end{pmatrix}=\begin{pmatrix}0 & Q_{2m-2n}(a)\\ Q_{2m-2n}(a) & 0\end{pmatrix} \tag{15}$$

with $[\xi_i,\xi_j]_{4n}(a)=[\xi_i,\xi_j]_{4n}(t)$,

$$\xi_i=\begin{cases}\theta_i, i=1,2,\cdots,2m-2n,\\ \varphi_{i-2m+2n}, i=2m-2n+1,\cdots,4m-4n.\end{cases}$$

Let $L=L_2L_1$. According to (7), (10), (11), (12), $L(y)$ can be expressed as follows:

$$\begin{cases}L(y)=y^{(4n)}+\sum_{k=1}^{2n-1}C_{2n}^{k}q^{(2n-k)}y^{(k)}+2qy^{(2n)}+(q^2+q^{(2n)})y,\\ A\begin{pmatrix}y(a)\\ \vdots\\ y^{(2n-1)}(a)\end{pmatrix}-B\begin{pmatrix}[y,\theta_1]_{2n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{2n}(b)\end{pmatrix}=0,\\ C\begin{pmatrix}l(y)(a)\\ \vdots\\ l(y)^{(2n-1)}(a)\end{pmatrix}-D\begin{pmatrix}[l(y),\theta_1]_{2n}(b)\\ \vdots\\ [l(y),\theta_{2m-2n}]_{2n}(b)\end{pmatrix}=0.\end{cases}$$

For $\varphi_i\in D_M(l^2)\subset D_M(l)$, from Lemma 4, it is easily seen that

$$\varphi_i=y_{0i}+\sum_{s=1}^{2n}d_{is}z_s+\sum_{j=1}^{2m-2n}a_{ij}\theta_j\ ,i=1,2,\cdots,2m-2n, \qquad (16)$$

where $y_{0i}\in D_0(l)$, d_{is}, a_{ij} are real constants.

In addition, for $\forall y\in D_0(l)$,

$$y=y_0+\sum_{i=1}^{2n}\bar{d}_iz_i+\sum_{i=1}^{2m-2n}(\bar{c}_i\theta_i+c_i^*\varphi_i)\,, \qquad (17)$$

where $y_0\in D_0(l^2)$, $\bar{d}_i,\bar{c}_i,c_i^*$ are also real constants. From Lemma 2, (8), (15) and (17), we obtain

$$\begin{pmatrix}[y,\theta_1]_{4n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{4n}(b)\\ [y,\varphi_1]_{4n}(b)\\ \vdots\\ [y,\varphi_{2m-2n}]_{4n}(b)\end{pmatrix}=B_{4m-4n}(b)\begin{pmatrix}\bar{c}_1\\ \vdots\\ \bar{c}_{2m-2n}\\ c_1^*\\ \vdots\\ c_{2m-2n}^*\end{pmatrix}=\begin{pmatrix}0 & Q_{2m-2n}(a)\\ Q_{2m-2n}(a) & 0\end{pmatrix}\begin{pmatrix}\bar{c}_1\\ \vdots\\ \bar{c}_{2m-2n}\\ c_1^*\\ \vdots\\ c_{2m-2n}^*\end{pmatrix}$$

Thus

$$\begin{pmatrix}\bar{c}_1\\ \vdots\\ \bar{c}_{2m-2n}\\ c_1^*\\ \vdots\\ c_{2m-2n}^*\end{pmatrix}=\begin{pmatrix}0 & Q_{2m-2n}^{-1}(a)\\ Q_{2m-2n}^{-1}(a) & 0\end{pmatrix}\begin{pmatrix}[y,\theta_1]_{4n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{4n}(b)\\ [y,\varphi_1]_{4n}(b)\\ \vdots\\ [y,\varphi_{2m-2n}]_{4n}(b)\end{pmatrix} \qquad (18)$$

By (16), (17), y can be defined by

$$y=y_0+\sum_{i=1}^{2n}\bar{d}_iz_i+\sum_{i=1}^{2m-2n}\bar{c}_i\theta_i+\sum_{i=1}^{2m-2n}c_i^*(y_{0i}+\sum_{j=1}^{2n}d_{ij}z_j)+\sum_{i=1}^{2m-2n}\sum_{j=1}^{2m-2n}c_i^*a_{ij}\theta_j\,.$$

From Lemma 2, (8), (9) and (18), we have

$$\begin{pmatrix}[y,\theta_1]_{2n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{2n}(b)\end{pmatrix}=B_{2m-2n}(b)\begin{pmatrix}\bar{c}_1\\ \vdots\\ \bar{c}_{2m-2n}\end{pmatrix}+B_{2m-2n}(b)A^T\begin{pmatrix}c_1^*\\ \vdots\\ c_{2m-2n}^*\end{pmatrix}$$

$$=\begin{pmatrix}Q_{2m-2n}(a)A^TQ_{2m-2n}^T(a) & I_{(2m-2n)\times(2m-2n)}\end{pmatrix}\begin{pmatrix}[y,\theta_1]_{4n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{4n}(b)\\ [y,\varphi_1]_{4n}(b)\\ \vdots\\ [y,\varphi_{2m-2n}]_{4n}(b)\end{pmatrix}$$

where

$$A=\begin{pmatrix}a_{11} & \cdots & a_{1(2m-2n)}\\ \vdots & \ddots & \vdots\\ a_{(2m-2n)1} & \cdots & a_{(2m-2n)(2m-2n)}\end{pmatrix}.$$

It follows from Lemma 9, we can conclude that

$$[l(y),\theta_i]_{2n}(b)=[y,\theta_i]_{4n}(b)-[y,l(\theta_i)]_{2n}(b)=[y,\theta_i]_{4n}(b)$$

.

Hence

$$\begin{pmatrix}[y,\theta_1]_{2n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{2n}(b)\\ [l(y),\theta_1]_{2n}(b)\\ \vdots\\ [l(y),\theta_{2m-2n}]_{2n}(b)\end{pmatrix}=\begin{pmatrix}[y,\theta_1]_{2n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{2n}(b)\\ [y,\theta_1]_{4n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{4n}(b)\end{pmatrix}$$

$$=\begin{pmatrix}Q_{2m-2n}(a)A^TQ_{2m-2n}^{-1}(a) & I_{(2m-2n)\times(2m-2n)}\\ I_{(2m-2n)\times(2m-2n)} & 0\end{pmatrix}\begin{pmatrix}[y,\theta_1]_{4n}(b)\\ \vdots\\ [y,\theta_{2m-2n}]_{4n}(b)\\ [y,\varphi_1]_{4n}(b)\\ \vdots\\ [y,\varphi_{2m-2n}]_{4n}(b)\end{pmatrix}.$$

So by (10) and (11), we have

$$N=\begin{pmatrix}B & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}Q_{2m-2n}(a)A^TQ_{2m-2n}^{-1}(a) & I_{(2m-2n)\times(2m-2n)}\\ I_{(2m-2n)\times(2m-2n)} & 0\end{pmatrix}=\begin{pmatrix}BQ_{2m-2n}(a)A^TQ_{2m-2n}^{-1}(a) & B\\ D & 0\end{pmatrix}, \qquad (19)$$

By (7), we obtain

$$\begin{pmatrix}l(y)(a)\\ l'(y)(a)\\ l''(y)(a)\\ \vdots\\ l^{(2n-1)}(y)(a)\end{pmatrix}=$$

$$\begin{pmatrix} q(a) & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ q'(a) & q(a) & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ q''(a) & 2q'(a) & q(a) & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{(2n-1)}(a) & C_{2n-1}^{1}q^{(2n-2)}(a) & C_{2n-1}^{2} & \cdots & q(a) & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\begin{pmatrix} y(a) \\ y'(a) \\ y''(a) \\ \vdots \\ y^{(4n-1)}(a) \end{pmatrix}$$

$$=\begin{pmatrix} H_3(a) & I_{2n\times 2n} \end{pmatrix}\begin{pmatrix} y(a) \\ y'(a) \\ y''(a) \\ \vdots \\ y^{(4n-1)}(a) \end{pmatrix}. \qquad (20)$$

Thus

$$\begin{pmatrix} y(a) \\ \vdots \\ y^{(2n-1)}(a) \\ l(y)(a) \\ \vdots \\ l^{(2n-1)}(y)(a) \end{pmatrix}=\begin{pmatrix} I_{2n\times 2n} & 0 \\ H_3(a) & I_{2n\times 2n} \end{pmatrix}\begin{pmatrix} y(a) \\ y'(a) \\ y''(a) \\ \vdots \\ y^{(4n-1)}(a) \end{pmatrix}.$$

Furthermore, by (10) and (11), we have

$$M=\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} I_{2n\times 2n} & 0 \\ H_3(a) & I_{2n\times 2n} \end{pmatrix}=\begin{pmatrix} A & 0 \\ CH_3(a) & C \end{pmatrix}. \qquad (21)$$

Using Lemma 5, (19) and (21), $L=L_2L_1$ can be denoted by

$$\begin{cases} L(y)=y^{(4n)}+\sum_{k=1}^{2n-1}C_{2n}^{k}q^{(2n-k)}y^{(k)}+2qy^{(2n)}+(q^2+q^{(2n)})y, \\ M\begin{pmatrix} y(a) \\ \vdots \\ y^{(4n-1)}(a) \end{pmatrix}-N\begin{pmatrix} [y,\xi_1]_{4n} \\ \vdots \\ [y,\xi_{2m-2n}]_{4n} \\ [y,\xi_{2m-2n+1}]_{4n} \\ \vdots \\ [y,\xi_{4m-4n}]_{4n} \end{pmatrix}=0. \end{cases}$$

Lemma 10 (i) $Q_{2n}^{-1}(t)H_3^{*}(t)+H_3(t)Q_{2n}^{-1}(t)+H_2(t)=0$;

(ii) $AQ_{2m-2n}(a)+Q_{2m-2n}(a)A^{T}=0$.

Proof (i) By (8), (13) and (20), we can prove this conclusion after the simple calculation.

(ii) From Lemma 9, (6), (16) and (17), we see that

$$[\varphi_i,\varphi_j]_{4n}(b)=[l(\varphi_i),\varphi_j]_{2n}(b)+[\varphi_i,l(\varphi_j)]_{2n}(b)$$

$$=[l(\varphi_i),\sum_{k=1}^{2m-2n}a_{jk}\theta_k]_{2n}(b)+[\sum_{k=1}^{2m-2n}a_{ik}\theta_k,l(\varphi_j)]_{2n}(b)=\sum_{k=1}^{2m-2n}a_{jk}[l(\varphi_i),\varphi_j]_{2n}(b)+\sum_{k=1}^{2m-2n}a_{ik}[\varphi_i,l(\varphi_j)]_{2n}(b)$$

$$=\sum_{k=1}^{2m-2n}a_{jk}[l(\varphi_i),\varphi_j]_{4n}(b)+\sum_{k=1}^{2m-2n}a_{ik}[l(\varphi_i,\varphi_j]_{4n}(b)=\sum_{k=1}^{2m-2n}a_{ik}[l(\varphi_i),\varphi_j]_{4n}(a)+\sum_{k=1}^{2m-2n}a_{ik}[l(\varphi_i),\varphi_j]_{4n}(a)$$

Compare it to (15), we get

$$0=[\varphi_i,\varphi_j]_{4n}(b)=AQ_{2m-2n}(a)+Q_{2m-2n}(a)A^{T}.$$

Theorem 11 $\mathrm{Rank}(M\oplus N)=2m$.

Proof

Since $M=\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} I_{2n\times 2n} & 0 \\ H_3(a) & I_{2n\times 2n} \end{pmatrix}=M_1M_2$,

$$N=\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Q_{2m-2n}(a)A^{T}Q_{2m-2n}^{-1}(a) & I_{(2m-2n)\times(2m-2n)} \\ I_{(2m-2n)\times(2m-2n)} & 0 \end{pmatrix}=N_1N_2$$

,

we have

$$M\oplus N=\begin{pmatrix} M & N \end{pmatrix}=\begin{pmatrix} M_1 & N_1 \end{pmatrix}\begin{bmatrix} M_2 & 0 \\ 0 & N_2 \end{bmatrix}=PQ.$$

And $\det Q=\det M_2\cdot\det N_2=1\times(-1)\neq 0$, i.e., Q is invertible. So $\mathrm{Rank}PQ=\mathrm{Rank}P=2m$, i.e., $\mathrm{Rank}(M\oplus N)=2m$.

Theorem 12 $L=L_2L_1$ is self-adjoint operator if and only if $AQ_{2n}^{-1}(a)C^{*}+BB_{2m-2n}(b)D^{*}=0$.

Proof By (13), (21) and Lemma 10 (i), we have

$$MQ_{4n}^{-1}(a)M^{*}=\begin{pmatrix} A & 0 \\ CH_3 & C \end{pmatrix}\begin{pmatrix} 0 & Q_{2n}^{-1}(a) \\ Q_{2n}^{-1}(a) & H_2(a) \end{pmatrix}\begin{pmatrix} A^{*} & H_3^{*}C^{*} \\ 0 & C^{*} \end{pmatrix}$$

$$\begin{pmatrix} 0 & AQ_{2n}^{-1}(a) \\ CQ_{2n}^{-1}(a) & CH_3Q_{2n}^{-1}(a)+CH_2(a) \end{pmatrix}\begin{pmatrix} A^{*} & H_3^{*}C^{*} \\ 0 & C^{*} \end{pmatrix}=\begin{pmatrix} 0 & AQ_{2n}^{-1}(a)C^{*} \\ CQ_{2n}^{-1}(a)A^{*} & 0 \end{pmatrix}$$

.Similarly, from (15), (19) and Lemma 10 (ii), we obtain

$$NB_{4m-4n}(b)N^{*}=\begin{pmatrix} 0 & BQ_{2m-2n}(a)D^{*} \\ DQ_{2m-2n}(a)B^{*} & 0 \end{pmatrix}.$$

Thus, $MQ_{4n}^{-1}(a)M^{*}+NB_{4m-4n}(b)N^{*}=0$ if and only if $AQ_{2n}^{-1}(a)C^{*}+BQ_{2m-2n}(a)D^{*}=0$.

$Q_{2m-2n}(a)=B_{2m-2n}(b)$,so

$AQ_{2n}^{-1}(a)C^{*}+BB_{2m-2n}(b)D^{*}=0$.

Due to $B_{2m-2n}(b)=Q_{2m-2n}(a)=-Q_{2m-2n}^{-1}(a)$, Theorem 12 also has the following form:

Theorem 13 $L=L_2L_1$ is self-adjoint operator if and only if $AQ_{2n}^{-1}(a)C^{*}-BQ_{2m-2n}^{-1}(a)D^{*}=0$

($AQ_{2n}^{-1}(a)C^{*}+BQ_{2m-2n}(a)D^{*}=0$).

If $L_1=L_2$, then $L=L_1^2$ and $A=C$, $B=D$. By Lemma 5 and Theorem 13, we can conclude the following conclusion:

Corollary 14 $L=L_1^2$ is self-adjoint operator if and only if L_1 is self-adjoint operator.

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