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A linear model for asymptotic growth curve

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ABSTRACT

In the present investigation, a linear model has been proposed which has been found very useful in growth studies wherever phenomenon exhibit asymptotic behavior. It possesses the identical properties those of Stevens's asymptotic regression growth model i.e. it belongs to the family of convex-concave curves and has neither maxima nor minima nor a point of inflexion. Its appropriateness and utility has also been examined with the help of data sets obtained from different areas of growth studies showing asymptotic nature. It has also been compared with Stevens's asymptotic regression model. An improvement has been shown over Stevens's asymptotic regression model.

Introduction

The asymptotic regression non-linear model with deterministic component

$$Y = \alpha - \beta\rho^X \qquad 0 < \rho < 1 \qquad (1)$$

belongs to the family of convex/concave curves and neither has maxima nor minima nor a point of inflexion. It is classified as intrinsically non-linear model as it cannot be transformed into a form linear in parameters. The model (1) has been very popular among researchers of different disciplines using nonlinear models. It has been extensively used in growth study wherever phenomenon exhibits asymptotic behaviour. In fertilizer application studies it is known as Mitscherlish's law and psychology, it represents learning curve and is known as spillman function in production function studies. Stevens (1951) has made a significant contribution for estimation of parameters of the model (1) and it is after that this model is also popularly known as Stevens asymptotic growth model.

The famous least squares method cannot be directly applied to estimate its parameters. Ratkowsky (1983) remarked that the parameter ρ appears non-linearly in model (1) and also create hurdle in direct application of least squares method of estimation for estimating its parameters. If ρ is known, (say $\hat{\rho}$) one can use the least squares technique to estimate α and β. Several methods has been suggested for estimation of ρ so that α and β can be obtained by regressing Y on $\hat{\rho}$

The Proposed Model

We have proposed a linear model which can be used as an alternative to Stevens's asymptotic model for explaining asymptotic growth behavior of phenomena.

$$Y = a + \frac{b}{X} + \frac{c}{X^2} \qquad (2)$$

Expression (2) is a linear model in which its parameters are appearing linearly and there is no problem in assuming an additive error term in model (2).

We can postulate a statistical model for (2) as:

$$Y_i = a + \frac{b}{X_i} + \frac{c}{X_i^2} + U_i \qquad X_i > 0 \;\; i = 1, 2, ...n \qquad (3)$$

The random variable U_i's are assumed to be independently and identically distributed random variable with mean zero and fixed variance σ^2 . The constant a , b c and σ^2 are unknown parameters of the model (3).

The model (3) has it's asymptote at a and belongs to the family of convex and concave curve. The model (3) has no point of inflection, it has no maxima or minima. It can be verified by following derivations. The second order differential of (3) with respect to X is

$$\frac{d^2Y}{dX^2} = \frac{2b}{X^3} + \frac{6c}{X^4} = 0$$

So that $$X = \frac{-3c}{b}$$

We are only considering positive values of X , therefore function (3) has no point of inflection if parameters b and c have same sign and $X > 0$.

For the maxima and minima of the function, we have

$$X = -\frac{2c}{b} \qquad (4)$$

Since $X > 0$, it implies that maxima and minima will exist only for negative value of X provided b and c are of same sign. Therefore the model will have no maxima or minima, for positive values of X .

To derive condition for the function (3) to be convex and concave, we have

$$\frac{d^2Y}{dX^2} = \frac{2}{X^3}\left(b + \frac{3c}{X}\right)$$

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Now the function will be convex if

$$\frac{d^2Y}{dX^2} > 0$$

$$\Rightarrow \quad b > -\frac{3c}{X}$$

Similarly it will be concave for

$$b < -\frac{3c}{X}$$

Thus for positive values of X, with restriction on its parameters, the model (3) will neither have maxima nor minima nor a point of inflection.

Estimation of parameters of proposed model and analysis of residuals

The parameters of the proposed model can be directly obtained by the application of the famous least squares method of estimation. For detailed description of estimation of parameters the reference can be made of Draper & Smith (1998). The model (3) admits an additive error term if error terms are independently and identically distributed random variable following normal distributions, the maximum likelihood estimators of the parameters can also be obtained.

The appropriateness of the proposed model has been verified with the help of careful examination of coefficient of determination - R^2, mean residuals sum of squares- s^2, mean absolute error (MAE), akaike information criterion (AIC) and standardized residuals.

The analysis of residuals has also been performed for checking model adequacy. Residuals are assumed to have zero mean, constant variance, no correlation and normal distribution or they are very near to zero mean almost a constant variance and negligible correlation and very near to normal distribution. Standardized residuals are calculated to verify assumption about residuals.

The residuals plot between residuals e and independent variable X is also an important toot to verify residuals assumptions. If this plot shows a horizontal band, it means that there is no model deficiency. Normal probability plots of residuals are used to verify departure from normality assumption. It is a useful tool for the checking the violation of normality assumption. For a data set, if residuals show a straight line or almost a straight line in normal probability plot of residuals, it confirms normality assumptions for them. Daniel and Wood (1980) have presented normal probability plot of residuals for sample sizes 8 to 384. Their study is very useful for understanding deviation from straight line which can be considered as acceptable in normal probability plot. However these plots are not shown in the paper as it will make it voluminous.

Fitting of model (3) to data sets

The model (3) has been fitted to different data sets exhibiting asymptotic behavior. Without loss of generality it can be assumed that $X = 1$ instead of $X = 0$.

The table1 describes values of parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{c}$ of respective data sets. It also gives the values of R^2, s^2, MAE and AIC. The results of analysis of data confirm that proposed linear model behaves as an asymptotic regression model (1). It fits extremely well to data sets and adequately explains the phenomenon. For all data sets, it has been observed that the values of R^2, which reflect the amount of regression explained is sufficiently high. The mean residual sum of squares- s^2 and mean absolute errors are also small in all cases. The values of standardized residuals are observed with in permissible limits of -3 to +3 for all data sets. All these clearly indicate that the proposed model adequately explain data sets and is a good asymptotic regression model.

The results of analysis of residuals are also highly satisfactory which confirm that residuals satisfy the assumptions of zero mean, no correlation, fixed variance and normal distribution. The residual plot of residual versus explanatory variables show a horizontal band in all cases. The normal probability plots of residuals show almost a straight line in all cases which confirm the assumption of normal distribution of residuals.

The tables of comparison showing values of R^2, s^2, MAE and AIC values of models (1) and (3) reveals that values of all these measures of model adequacy checking are comparable and in some data sets, model (3) shows superior results as comparable to model (1) .

The results of parameter estimates also confirm that model (3) belongs to family of convex / concave curves, having no maxima or minima or a point of inflection. All these discussions confirm that model (3) is a good asymptotic regression model.

Conclusion:

It has been observed theoretically as well as fitting on data sets that the proposed linear model of the form (3) can be considered as a growth model explaining asymptotic behaviour of the phenomenon.

The analysis of residuals further confirm that residuals are uncorrelated, have zero mean, fixed variance and follow normal distribution. It has been theoretically shown that proposed linear model (3) belongs to the family of convex / concave curves as listed in Ratkowsky (1989). It has been further shown that it has neither maxima nor minima or a point of inflection at the selected order of approximation and restrictions on parameters. All discussions lead to conclusion that there is no model inadequacy.

It has been thus established theoretically as well as on data sets that the model (3) is very useful, as an asymptotic regression growth model (1). Its applications to data sets are very convenient as famous least squares method of estimation is directly applicable and parameter estimates possess good statistical properties. The predictions, constructions of confidence intervals and test of significance procedures etc can be carried out very conveniently, using proposed model (3).

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Table1: Parameters Estimates of Model (3)

$\hat{a}$	0.7391	0.6344	0.6749	0.6602	0.6286	81.1901	250.8080	2.4067
$\hat{b}$	2.8155	4.3551	2.4312	1.0561	0.9639	-66.6239	-105.5830	-1.2468
$\hat{c}$	1.4200	-3.4057	0.5830	-0.1516	-1.1857	29.7717	-40.3766	0.3433

Table 2: Comparison of R^2, s^2, MAE & AIC for Model (3) and (1)

		R^2	s^2	MAE	AIC
1*	Model (3)	0.9881	0.0017	0.0266	0.0023
	Model (1)	0.9838	0.0025	0.0280	0.0033
2*	Model (3)	0.9836	0.0021	0.0308	0.0028
	Model (1)	0.9893	0.0015	0.0258	0.0019
3*	Model (3)	0.9797	0.0019	0.0277	0.0025
	Model (1)	0.9889	0.0011	0.0217	0.0014
4*	Model (3)	0.9276	0.0010	0.0209	0.0013
	Model (1)	0.9053	0.0014	0.0242	0.0018
5*	Model (3)	0.8306	0.0007	0.0164	0.0010
	Model (1)	0.8401	0.0005	0.0160	0.0006
6**	Model (3)	.0.9916	1.6554	0.6288	2.1984
	Model (1)	0.9910	1.7844	0.6799	2.3697
7**	Model (3)	0.9972	14.163	1.7451	18.809
	Model (1)	0.9938	31.547	2.9914	36.549
8****	Model (3)	0.9861	0.0004	0.0129	0.0005
	Model (1)	0.9842	0.0006	0.0138	0.0008