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Some product cordial graphs

S. K. Vaidya[1] and C. M. Barasara[2]

[1]Department of Mathematics, Saurashtra University, Rajkot - 360005.

[2]Atmiya Institute of Technology and Science, Rajkot - 360005.

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ABSTRACT

In this paper we prove that the graphs obtained by duplication of an edge in C_n, mutual duplication of pair of edges and mutual duplication of pair of vertices between two copies of cycle C_n admit product cordial labeling. Moreover let G and G' be two graphs such that their order and/or size differ by at most 1 then we prove that the new graph obtained by joining G and G' by a path P_k of arbitrary length is a product cordial graph.

Introduction

We begin with simple, finite, connected and undirected graph $G(p,q)$ with order p and size q. Throughout this work V and E respectively denote the vertex set and edge set of G. For all standard terminology and notations we follow Gross and Yellen [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 : If the vertices of the graph are assigned values subject to certain conditions then is known as *graph labeling*.

Most of the graph labeling problems have following three common characteristics:

1. a set of numbers for assignment of vertex labels;
2. a rule that assigns a label to each edge;
3. some condition(s) these labels must satisfy.

Labeled graphs have applications in many diversified fields. A detailed study on variety of applications of graph labeling is reported in Bloom and Golomb [2].

For extensive survey on graph labeling and bibliographic reference we refer to Gallian [3]. Enough literature is available in printed as well as in electronic form on different types of graph labeling. According to Beineke and Hegde [4], the concept of graph labeling is a frontier between number theory and structure of graphs.

Definition 1.2 : A mapping $f : V(G) \to \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f.

The induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let

$v_f(0) =$ number of vertices of G having label 0 under f

$v_f(1) =$ number of vertices of G having label 1 under f

$e_f(0) =$ number of edges of G having label 0 under f^*

$e_f(1) =$ number of edges of G having label 1 under f^*

Definition 1.3 : A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [5]. In the same paper he investigated several results on this newly introduced concept. Some labelings with variations in cordial theme have also been introduced such as prime cordial labeling, A-cordial labeling, E-cordial labeling, H-cordial labeling, Product cordial labeling etc. The present paper is aimed to investigate some results on product cordial labeling in which absolute difference is replaced by product of the vertex labels.

Definition 1.4 : A binary vertex labeling of graph G with induced edge labeling $f^* : E \to \{0,1\}$ defined by $f^*(e = uv) = f(u) f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. A graph G is called *product cordial* if it admits product cordial labeling.

The product cordial labeling was introduced by Sundaram et al. [6] and they proved that trees, unicyclic graphs of odd order, triangular snakes, dragons, helms and union of two path graphs are product cordial. They also established that a graph with p vertices and q edges with $p \ge 4$ is product cordial then $q < \frac{p^2 - 1}{4}$. The graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are product cordial is proved in Vaidya and Dani [7] while the product cordial labeling for some cycle related graphs is reported in Vaidya and Kanani [8]. In the same paper they have investigated product cordial labeling for the shadow graph of cycle C_n. Vaidya and Barasara [9] have proved that the cycle with one chord, the cycle with twin chords, the friendship graph

Tele:

E-mail addresses: samirkvaidya@yahoo.co.in, chirag.barasara@gmail.com

and the middle graph of path are product cordial graphs. Product cordial labeling in the context of tenser product of C_n with C_n, P_n with P_n and C_n with P_n is discussed in Vaidya and Vyas [10].

There are three types of problems that can be considered in this area.

1. How product cordiality is affected under various graph operations;
2. Construct new families of product cordial graphs by finding suitable labeling;
3. Given a graph theoretic property P, characterise the class of graphs with property P that are product cordial.

This paper is focused on problems of first type.

Definition 1.5 : Consider a cycle C_n and let $e_k = v_k v_{k+1}$ be an edge in it with $e_{k-1} = v_{k-1}v_k$ and $e_{k+1} = v_{k+1}v_{k+2}$ be its incident edges and $e'_k = v'_k v'_{k+1}$ be a new edge. The *duplication of the edge* e_k by the edge e'_k produced a new graph G in such a way that $N(v_k) \cap N(v'_k) = \{v_{k-1}\}$ and $N(v_{k+1}) \cap N(v'_{k+1}) = \{v_{k+1}\}$.

Definition 1.6 : Consider two copies of cycle C_n and let $e_k = v_k v_{k+1}$ be an edge in the first copy of C_n with $e_{k-1} = v_{k-1}v_k$ and $e_{k+1} = v_{k+1}v_{k+2}$ be its incident edges. Similarly let $e'_k = u_k u_{k+1}$ be an edge in the second copy of C_n with $e'_{k-1} = u_{k-1}u_k$ and $e'_{k+1} = u_{k+1}u_{k+2}$ be its incident edges. The *mutual duplication of a pair of edges* e_k, e'_k between two copies of cycle C_n produces a new graph G in such a way that $N(v_k) \cap N(u_k) = \{v_{k-1}, u_{k-1}\}$ and $N(v_{k+1}) \cap N(u_{k+1}) = \{v_{k+2}, u_{k+2}\}$.

Definition 1.7 : Consider two copies of cycle C_n. Then the *mutual duplication of a pair of vertices* v_k and v'_k respectively from each copy of cycle C_n produces a new graph G such that $N(v_k) = N(v'_k)$.

Main Results

Theorem - 2.1 : The graph obtained by duplication of an arbitrary edge e_k in cycle C_n is product cordial graph except for $n = 4,5,6,7,8$.

Proof: Let $e_1, e_2, ..., e_n$ be the edges of cycle C_n. Without loss of generality we duplicate the edge e_1 thus added vertices are v'_1 and v'_2. Now the resultant graph will have $n+2$ vertices and $n+3$ edges.

To define $f : V \to \{0,1\}$ we consider following two cases.

Case 1: When $n = 4,5,6,7,8$.

Then the graph under consideration violates the condition as shown in *Table 1*.

Case 2: When $n \neq 4,5,6,7,8$.

Sub Case 1: When $n = 3$ the graph and its product cordial labeling is shown in *Figure 1*.

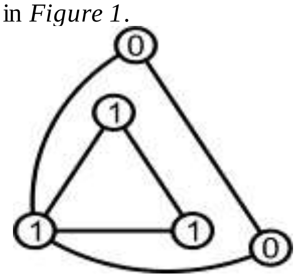

Figure 1

Sub Case 2: When n is odd.

$f(v_i) = 0, \quad 4 \le i \le \left\lfloor \frac{n+2}{2} \right\rfloor + 3$

$f(v_i) = 1, \quad \text{Otherwise}$

$f(v'_i) = 1, \quad i = 1,2$

In view of the above labeling pattern we have,

$v_f(0) + 1 = v_f(1) = \left\lceil \frac{n+2}{2} \right\rceil$

$e_f(0) = e_f(1) = \frac{n+3}{2}$

Sub Case 3: When n is odd.

$f(v_i) = 0, \quad 4 \le i \le \frac{n+8}{2}$

$f(v_i) = 1, \quad \text{Otherwise}$

$f(v'_i) = 1, \quad i = 1,2$

In view of the above labeling pattern we have,

$v_f(0) = v_f(1) = \frac{n+2}{2}$

$e_f(0) - 1 = e_f(1) = \frac{n+2}{2}$

Hence from the case 1 and case 2 we have the required result.

Illustration 2.2 : The graph obtained by duplication of an edge in C_9 and its product cordial labeling is shown in *Figure 2*.

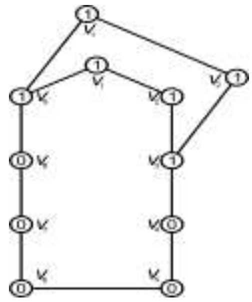

Figure 2

Theorem 2.3 : The graph obtained by mutual edge duplication in cycle C_n is product cordial graph except for $n = 3,4,5,6,7$.

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the set of vertices of first copy of cycle C_n and $\{u_1, u_2, ..., u_n\}$ be the set of vertices of second copy of cycle C_n. Without loss of generality we duplicate the edges $v_1 v_n$ and $u_1 u_n$. The resultant graph will have $2n$ vertices and $2n+4$ edges.

To define $f : V \to \{0,1\}$ we consider following two cases.

Case 1: When $n = 3,4,5,6,7$.

Then the graph under consideration violates the condition as shown in *Table 2*.

Case 2: When $n \neq 3,4,5,6,7$.

$f(v_i) = 0, \quad 3 \le i \le n-2$

$f(v_i) = 1, \quad \text{Otherwise}$

$f(u_i) = 0, \quad 3 \le i \le 6$

$f(u_i) = 1, \quad \text{Otherwise}$

In view of the above labeling pattern we have,

$v_f(0) = v_f(1) = n$

$e_f(0) = e_f(1) = n+2$

Hence from case 1 and case 2 we have the required result.

Illustation 2.4 : The graph obtained by mutual edge duplication of cycle C_8 and its product cordial labeling is shown in *Figure 3*.

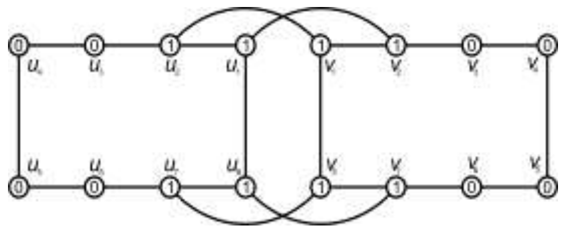

Figure 3

Theorem 2.5 : The graph obtained by mutual vertex duplication in cycle C_n is product cordial graph except for $n=3,4,5$.

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the set of vertices of first copy of cycle C_n and $\{u_1, u_2, ..., u_n\}$ be the set of vertices of second copy of cycle C_n. Without loss of generality we duplicate the vertices v_1 and u_1. The resultant graph will have $2n$ vertices and $2n+4$ edges.

To define $f: V \to \{0,1\}$ we consider following two cases.

Case 1: When $n=3,4,5$.

Then the graph under consideration violates the condition as shown in *Table 3*.

Case 2: When $n \neq 3,4,5$.

$f(u_i)=0, \quad 3 \le i \le n-1$

$f(u_i)=1, \quad$ Otherwise

$f(v_i)=0, \quad 3 \le i \le 5$

$f(v_i)=1, \quad$ Otherwise

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)=n$

$e_f(0)=e_f(1)=n+2$

Hence from case 1 and case 2 we have the required result.

Illustration 2.6 : The graph obtained by mutual vertex duplication of cycle C_7 and its product cordial labeling is shown in *Figure 4*.

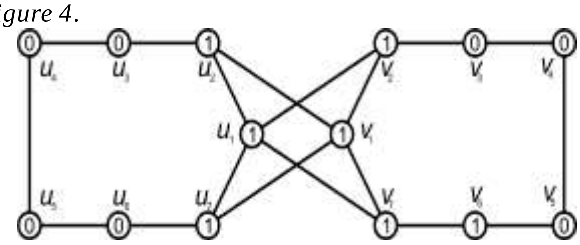

Figure 4

Theorem 2.7 : The graph obtained by joining

1. $G(p,q)$ and $G'(p,q)$
2. $G(p,q)$ and $G'(p-1,q)$
3. $G(p,q)$ and $G'(p,q-1)$
4. $G(p,q)$ and $G'(p-1,q-1)$
5. $G(p-1,q)$ and $G'(p,q-1)$

by a path P_k of arbitrary length admits prodct cordial labeling.

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the set of vertices of path P_k, $\{x_1, x_2, ..., x_n\}$ be the set of vertices of graph G and $\{y_1, y_2, ..., y_n\}$ be the set of vertices of G'. Without loss of generality $x_1 = v_1$ and $y_1 = v_n$. We assume that the resultant graph as G''.

To define $f: V \to \{0,1\}$ we consider following ten cases.

Case 1: When G'' is obtained by joining $G(p,q)$ and $G'(p,q)$ with path P_k and $k \equiv 0 \pmod 2$.

The graph G'' has $2p+k-2$ vertices and $2q+k-1$ edges.

$f(x_i)=0, \quad$ for all i

$f(y_i)=1, \quad$ for all i

$f(v_i)=0, \quad 2 \le i \le \frac{k}{2}$

$f(x_i)=1, \quad \frac{k}{2}+1 \le i \le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)=p+\frac{k}{2}-1$

$e_f(0)=e_f(1)+1=q+\frac{k}{2}$

Case 2: When G'' is obtained by joining $G(p,q)$ and $G'(p,q)$ with path P_k and $k \equiv 1 \pmod 2$.

The graph G'' has $2p+k-2$ vertices and $2q+k-1$ edges.

$f(x_i)=0, \quad$ for all i

$f(y_i)=1, \quad$ for all i

$f(v_i)=0, \quad 2 \le i \le \left\lfloor \frac{k}{2} \right\rfloor$

$f(x_i)=0, \quad \left\lfloor \frac{k}{2} \right\rfloor +1 \le i \le k-1$

In view of the above labeling pattern we have,

$v_f(0)+1=v_f(1)=p+\frac{k-1}{2}$

$e_f(0)=e_f(1)=q+\frac{k-1}{2}$

Case 3: When G'' is obtained by joining $G(p,q)$ and $G'(p-1,q)$ with path P_k and $k \equiv 0 \pmod 2$.

The graph G'' has $2p+k-3$ vertices and $2q+k-1$ edges.

$f(x_i)=0, \quad$ for all i

$f(y_i)=1, \quad$ for all i

$f(v_i)=0, \quad 2 \le i \le \frac{k}{2}$

$f(x_i)=1, \quad \frac{k}{2}+1 \le i \le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)+1=p+\frac{k}{2}-1$

$e_f(0)=e_f(1)+1=q+\frac{k}{2}$

Case 4: When G'' is obtained by joining $G(p,q)$ and $G'(p-1,q)$ with path P_k and $k \equiv 1 \pmod 2$.

The graph G'' has $2p+k-3$ vertices and $2q+k-1$ edges.

$f(x_i)=0,\quad$ for all i

$f(y_i)=1,\quad$ for all i

$f(v_i)=0,\quad 2\le i\le\left\lfloor\frac{k}{2}\right\rfloor$

$f(x_i)=1,\quad \left\lfloor\frac{k}{2}\right\rfloor+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)=p+\frac{k-3}{2}$

$e_f(0)=e_f(1)+1=q+\frac{k-1}{2}$

Case 5: When G'' is obtained by joining $G(p,q)$ and $G'(p,q-1)$ with path P_k and $k\equiv 0\,(\text{mod }2)$.

The graph G'' has $2p+k-2$ vertices and $2q+k-2$ edges.

$f(x_i)=1,\quad$ for all i

$f(y_i)=0,\quad$ for all i

$f(v_i)=1,\quad 2\le i\le\frac{k}{2}$

$f(x_i)=0,\quad \frac{k}{2}+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)=p+\frac{k}{2}-1$

$e_f(0)=e_f(1)=q+\frac{k}{2}-1$

Case 6: When G'' is obtained by joining $G(p,q)$ and $G'(p,q-1)$ with path P_k and $k\equiv 1\,(\text{mod }2)$.

The graph G'' has $2p+k-2$ vertices and $2q+k-2$ edges.

$f(x_i)=1,\quad$ for all i

$f(y_i)=0,\quad$ for all i

$f(v_i)=1,\quad 2\le i\le\left\lceil\frac{k}{2}\right\rceil$

$f(x_i)=0,\quad \left\lceil\frac{k}{2}\right\rceil+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)-1=p+\frac{k-3}{2}$

$e_f(0)+1=e_f(1)=q+\frac{k-1}{2}$

Case 7: When G'' is obtained by joining $G(p,q)$ and $G'(p-1,q-1)$ with path P_k and $k\equiv 0\,(\text{mod }2)$.

The graph G'' has $2p+k-3$ vertices and $2q+k-2$ edges.

$f(x_i)=1,\quad$ for all i

$f(y_i)=0,\quad$ for all i

$f(v_i)=1,\quad 2\le i\le\frac{k}{2}$

$f(x_i)=0,\quad \frac{k}{2}+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)-1=p+\frac{k}{2}-2$

$e_f(0)=e_f(1)=q+\frac{k}{2}-1$

Case 8: When G'' is obtained by joining $G(p,q)$ and $G'(p-1,q-1)$ with path P_k and $k\equiv 1\,(\text{mod }2)$.

The graph G'' has $2p+k-3$ vertices and $2q+k-2$ edges.

$f(x_i)=1,\quad$ for all i

$f(y_i)=0,\quad$ for all i

$f(v_i)=1,\quad 2\le i\le\left\lfloor\frac{k}{2}\right\rfloor$

$f(x_i)=0,\quad \left\lfloor\frac{k}{2}\right\rfloor+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)=p+\frac{k-3}{2}$

$e_f(0)=e_f(1)+1=q+\frac{k-1}{2}$

Case 9: When G'' is obtained by joining $G(p,q-1)$ and $G'(p-1,q)$ with path P_k and $k\equiv 0\,(\text{mod }2)$.

The graph G'' has $2p+k-3$ vertices and $2q+k-2$ edges.

$f(x_i)=0,\quad$ for all i

$f(y_i)=1,\quad$ for all i

$f(v_i)=0,\quad 2\le i\le\frac{k}{2}$

$f(x_i)=1,\quad \frac{k}{2}+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)+1=p+\frac{k}{2}-1$

$e_f(0)=e_f(1)=q+\frac{k}{2}-1$

Case 10: When G'' is obtained by joining $G(p,q-1)$ and $G'(p-1,q)$ with path P_k and $k\equiv 1\,(\text{mod }2)$.

The graph G'' has $2p+k-3$ vertices and $2q+k-2$ edges.

$f(x_i)=0,\quad$ for all i

$f(y_i)=1,\quad$ for all i

$f(v_i)=0,\quad 2\le i\le\left\lfloor\frac{k}{2}\right\rfloor$

$f(x_i)=1,\quad \left\lfloor\frac{k}{2}\right\rfloor+1\le i\le k-1$

In view of the above labeling pattern we have,

$v_f(0)=v_f(1)=p+\frac{k-3}{2}$

$e_f(0)=e_f(1)-1=q+\frac{k-3}{2}$

Thus from the above discussion we have the required result.

Illustation 2.8 : The graph obtained by joining $G=C_5$ and $G'=W_3$ with path P_5 and its product cordial labeling is shown in *Figure 5*.

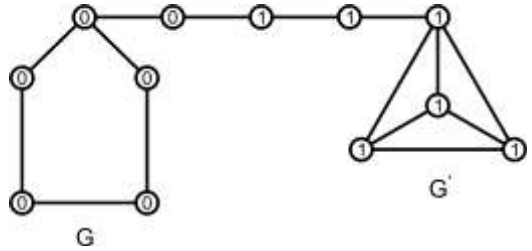

Figure 5

Concluding Remarks

Labeled graph is the topic of current interest due to its diversified applications. We investigate four new results in the context of some graph operations. To derive similar results for other graph families and for different labeling problems is an open area of research.

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Table 1

n	p	q	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$	$\lvert e_f(0)-e_f(1)\rvert$
4	6	7	3	3	5	2	3
5	7	8	3	4	5	3	2
6	8	9	4	4	6	3	3
7	9	10	4	5	6	4	2
8	10	11	5	5	7	4	3

Table 2

n	p	q	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$	$\lvert e_f(0)-e_f(1)\rvert$
3	6	10	3	3	7	3	4
4	8	12	4	4	8	4	4
5	10	14	5	5	9	5	4
6	12	16	6	6	10	6	4
7	14	18	7	7	10	8	2

Table 3

n	p	q	$v_f(0)$	$v_f(1)$	$e_f(0)$	$e_f(1)$	$\lvert e_f(0)-e_f(1)\rvert$
3	6	10	3	3	7	3	4
4	8	12	4	4	8	4	4
5	10	14	5	5	8	6	2