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Q-homomorphism in q-fuzzy subgroups

A Vethamanickam, KR Balasubramanian and K.L.Muruganantha Prasad
Department of Mathematics, H.H.The Rajahs College,Pudukkottai-622001.

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ABSTRACT
In this paper, we study the Q-homomorphism in Q-fuzzy subgroup and prove some results on these.
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Introduction

After the introdution of fuzzy sets by L.A.Zadeh [18], several researchers explored on the generalization of the notion of fuzzy set. Azriel Rosenfeld [3] defined a Fuzzy groups. Anthony.J.M. and Sherwood.H[2] defined a fuzzy groups redefined. Choudhury.F.P. and Chakraborty.A.B. and Khare.S.S.[5] defined a fuzzy subgroups and fuzzy homomorphism. A.Solairaju and R.Nagarajan[14] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. We introduce the concept of Q-homomorphism in Q-fuzzy subgroups and established some results.

Preliminaries:

Definition: Let X be a non-empty set. A fuzzy subset A of X is a function $A : X \to [\,0, 1\,]$.

Definition: Let X be a non-empty set and Q be a non-empty set. A Q-fuzzy subset A of X is a function $A : XxQ \to [\,0, 1\,]$.

Example: Let X = { a, b, c } be a set and Q = { p }. Then A = { ⟨ (a, p), 0.4 ⟩, ⟨ (b, p), 0.2 ⟩, ⟨ (c, p), 0.5 ⟩ } is a Q-fuzzy subset of X.

Definition: The union of two Q-fuzzy subsets A and B of a set X is defined by $(A\cup B)(x, q) = \max \{ A(x, q), B(x, q) \}$, for all x in X and q in Q.

Definition: The intersection of two Q-fuzzy subsets A and B of a set X is defined by $(A\cap B)(x, q) = \min \{ A(x, q), B(x, q) \}$, for all x in X and q in Q.

Definition: If (G, .) and (G^{l}, .) are any two groups and Q be a non-empty set, then the function $f : GxQ \to G^{l}xQ$ is called a Q-homomorphism if f(xy, q) = f(x, q)f(y, q), for all x and y in G and q in Q.

Definition: If (G, .) and (G^{l}, .) are any two groups and Q be a non-empty set, then the function $f: GxQ\to G^{l}xQ$ is called a Q-antihomomorphism if f(xy, q) = f(y, q)f(x, q), for all x and y in G and q in Q.

Definition: Let (G, ·) be a group and Q be a set. A Q-fuzzy subset A of G is said to be a Q-fuzzy subgroup(QFSG) of G if the following conditions are satisfied:

(i) $A(xy, q) \geq \min \{A(x, q), A(y, q)\}$,

(ii) $A(x^{-1}, q) \geq A(x, q)$, for all x and y in G and q in Q.

Definition: Let (G, ·) and (G^{l}, ·) be any two groups and Q be a set. Let $f : GxQ \to G^{l}xQ$ be any function and A be a Q-fuzzy subgroup in G, V be a Q-fuzzy subgroup in f (GxQ) = $G^{l}xQ$, defined by $V(y, q) = \sup_{(x,q)\in f^{-1}(y,q)} A(x, q)$, for all x in G and y in G^{l} and q in Q. Then A is called a preimage of V under f and is denoted by $f^{-1}(V)$.

Definition: Let A and B be any two Q-fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by AxB, is defined as AxB ={ ⟨ ((x, y),q), AxB((x, y), q) ⟩ / for all x in G and y in H and q in Q }, where AxB((x, y), q) = min {A(x, q), B(y, q) }.

Definition: Let A and B be any two Q-fuzzy subgroups of a group (G, ·). Then A and B are said to be conjugate Q-fuzzy subgroups of G if for some g in G, A(x, q) = B($g^{-1}xg$, q), for every x in G and q in Q.

Definition: Let A be a Q-fuzzy subset in a set S, the strongest Q-fuzzy relation on S, that is a Q-fuzzy relation on A is V given by V((x, y), q) = min {A(x, q), A(y, q)}, for all x and y in S and q in Q.

Proposition: Let A be a Q-fuzzy subgroup of a group G. If A(x, q) < A(y, q), for some x and y in G and q in Q, then A(xy, q) = A(x, q) = A(yx, q), for all x and y in G and q in Q.

proof: Let A be a Q-fuzzy subgroup of a group G. Also we have A(x, q) < A(y, q), for some x and y in G and q in Q, A(xy, q) ≥ min { A(x, q), A(y, q) } = A(x, q) ; and A(x, q) = A(xyy^{-1}, q) ≥ min{A(xy, q) , A(y^{-1}, q) } ≥ min{ A(xy, q), A(y, q) }= A(xy, q). Therefore, A(xy, q) = A(x, q) , for all x and y in G and q in Q. And, A(yx, q) ≥ min { A(y, q) , A(x, q) }= A(x, q) ; and A(x, q) = A($y^{-1}yx$, q) ≥ min { A(y^{-1}, q), A(yx, q) }≥ min{A(y, q), A(yx, q) }= A(yx, q).

Therefore, A(yx, q) = A(x, q), for all x and y in G and q in Q. Hence A(xy, q) = A(x, q) = A(yx, q), for all x and y in G and q in Q.

Tele:
E-mail addresses: dr_vethamanickam@yahoo.co.in,
balamohitha@gmail.com, lkmprasad@gmail.com

Proposition: Let A be a Q-fuzzy subgroup of a group G. If A(x, q) > A(y, q) , for some x and y in G and q in Q, then A(xy, q) = A(y, q) = A(yx, q), for all x and y in G and q in Q.
Proof: It is trivial.

Proposition: Let A be a Q-fuzzy subgroup of a group G such that Im A = { α } , where α in [0, 1]. If A = B$\cup$C, where B and C are Q-fuzzy subgroups of G, then either B $\subseteq$ C or C $\subseteq$ B.

Proof: Let A = B$\cup$C = { $\langle$ (x , q), A(x, q) $\rangle$ / x in G and q in Q }, B = {$\langle$ (x, q), B(x, q) $\rangle$ / x in G and q in Q } and C = { $\langle$ (x, q), C(x, q) $\rangle$ / x in G and q in Q }. Assume that B(x, q) > C(x, q) and B(y, q) < C(y, q), for some x and y in G and q in Q. Then, α = A(x, q) = B$\cup$C(x, q)= max { B(x, q) , C(x, q) }= B(x, q) >C(x, q). Therefore, α>C(x, q). And, α = A(y, q) = B$\cup$C(y, q) = max{ B(y, q), C(y, q) }= C(y, q) > B(y, q). Therefore, α > B(y, q). So that, C(y, q) > C(x, q) and B(x, q) > B(y, q). Hence B(xy, q) = B(y, q) and C(xy, q) = C(x, q), by Proposition 1.1 and 1.2. But then, α = A(xy, q) = B$\cup$C(x, q) = max { B(xy, q), C(xy, q) }= max { B(y, q), C(x, q) }< α----------(1). It is a contradiction by (1).

Therefore, either B $\subseteq$ C or C $\subseteq$ B is true.

Proposition: If A and B are Q-fuzzy subgroups of the groups G and H, respectively, then AxB is a Q-fuzzy subgroup of GxH.

Proof: Let A and B be Q-fuzzy subgroups of the groups G and H respectively.

Let x_1 and x_2 be in G, y_1 and y_2 be in H. Then (x_1, y_1) and (x_2, y_2) are in GxH. Now, AxB [(x_1, y_1)(x_2, y_2), q] = AxB((x_1x_2, y_1y_2), q) = min { A(x_1x_2, q), B(y_1y_2, q) }≥ min{ min{A(x_1, q), A(x_2, q)}, min{B(y_1, q), B(y_2, q) } } = min{ min{A(x_1, q), B(y_1, q) }, min{A(x_2, q), B(y_2, q) } }= min{ AxB ((x_1, y_1), q), AxB ((x_2, y_2), q)}. Therefore, AxB [(x_1, y_1)(x_2, y_2), q] ≥ min{AxB ((x_1, y_1), q), AxB((x_2, y_2), q) }. And AxB [$(x_1, y_1)^{-1}$, q] = AxB((x_1^{-1}, y_1^{-1}), q) = min { A(x_1^{-1}, q), B(y_1^{-1}, q) }≥ min { A(x_1, q), B(y_1, q) }= AxB((x_1, y_1), q). Therefore, AxB [$(x_1, y_1)^{-1}$, q] ≥ AxB ((x_1, y_1), q). Hence AxB is a Q-fuzzy subgroup of GxH.

Proposition: Let a Q-fuzzy subgroup A of a group G be conjugate to a Q-fuzzy subgroup M of G and a Q-fuzzy subgroup B of a group H be conjugate to a Q-fuzzy subgroup N of H. Then a Q-fuzzy subgroup AxB of a group GxH is conjugate to a Q-fuzzy subgroup MxN of GxH.

Proof: Let A and B be Q-fuzzy subgroups of the groups G and H respectively.

Let x, x^{-1} and f be in G and y, y^{-1} and g be in H. Then (x, y) ,(x^{-1}, y^{-1}) and (f, g) are in GxH. Now, AxB ((f, g), q) = min { A(f, q), B(g, q) }= min { M(xf x^{-1}, q), N(yg y^{-1}, q) }= MxN ((xf x^{-1}, yg y^{-1}), q) = MxN[(x, y)(f, g) (x^{-1}, y^{-1}), q] = MxN[(x, y) (f, g)$(x, y)^{-1}$, q]. Therefore, AxB((f, g), q) = MxN[(x, y) (f, g)$(x, y)^{-1}$, q]. Hence a Q-fuzzy subgroup AxB of GxH is conjugate to a Q-fuzzy subgroup MxN of GxH.

Proposition: Let A and B be Q-fuzzy subsets of the groups G and H, respectively. Suppose that e and e^{I} are the identity element of G and H, respectively. If AxB is a Q-fuzzy subgroup of GxH, then at least one of the following two statements must hold.

(i) B(e^{I} , q) ≥ A(x, q), for all x in G and q in Q,

(ii) A(e, q) ≥ B(y, q), for all y in H and q in Q.

Proof: Let AxB is a Q-fuzzy subgroup of GxH. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in G and b in H such that A(a, q) > B(e^{I}, q) and B(b, q) > A(e, q), q in Q. We have, AxB((a, b), q)=min{A(a, q), B(b, q)}> min{A(e, q), B(e^{I} , q) = AxB((e, e^{I}), q).

Thus AxB is not a Q-fuzzy subgroup of GxH. Hence either B(e^{I} , q) ≥ A(x, q), for all x in G and q in Q or A(e, q) ≥ B(y, q), for all y in H and q in Q.

Proposition: Let A and B be Q-fuzzy subsets of the groups G and H, respectively and AxB is a Q-fuzzy subgroup of GxH. Then the following are true:

(i) if A(x, q) ≤ B(e^{I}, q), then A is a Q-fuzzy subgroup of G.

(ii) if B(x, q) ≤ A(e, q), then B is a Q-fuzzy subgroup of H.

(iii) either A is a Q-fuzzy subgroup of G or B is a Q-fuzzy subgroup of H.

Proof: Let AxB be a Q-fuzzy subgroup of GxH, x and y in G and q in Q. Then (x, e^{I}) and (y, e^{I}) are in GxH. Now, using the property A(x, q) ≤ B(e^{I}, q), for all x in G and q in Q, we get, A(xy^{-1}, q) = min { A(xy^{-1}, q), B($e^{I}e^{I}$, q) } = AxB (((xy^{-1}), ($e^{I}e^{I}$)), q) = AxB [(x, e^{I})(y^{-1}, e^{I}), q] ≥ min { AxB ((x, e^{I}), q), AxB ((y^{-1}, e^{I}), q) }= min{min{A(x, q), B(e^{I} ,q)}, min{A(y^{-1}, q), B(e^{I}, q)} }= min { A(x, q), A(y^{-1}, q) }≥ min { A(x, q), A(y, q) }. Therefore, A(xy^{-1}, q)≥ min{A(x, q), A(y, q)}, for all x,y in G and q in Q.Hence A is a Q-fuzzy subgroup of G.

Thus (i) is proved.

Now, using the property B(x, q) ≤ A(e, q), for all x in H and q in Q, we get, B(xy^{-1}, q) = min { B(xy^{-1}, q), A(ee, q) }= AxB (((ee) ,(xy^{-1})), q) = AxB [(e, x)(e, y^{-1}), q] ≥ min { AxB((e, x), q), AxB((e, y^{-1}), q) }= min{min{B(x, q), A(e, q) }, min{ B(y^{-1}, q), A(e, q) }} = min{ B(x, q), B(y^{-1}, q) }≥ min { B(x, q), B(y, q) }. Therefore, B(xy^{-1}, q) ≥ min { B(x, q), B(y, q) }, for all x and y in H and q in Q. Hence B is a Q-fuzzy subgroup of H.

Thus (ii) is proved. (iii) is clear.

Proposition: Let A be a Q-fuzzy subset of a group G and V be the strongest Q-fuzzy relation of G. Then A is a Q-fuzzy subgroup of G if and only if V is a Q-fuzzy subgroup of GxG.

Proof: Suppose that A is a Q-fuzzy subgroup of G. Then for any x = (x_1, x_2) and y = (y_1, y_2) are in GxG. We have, V ((x–y), q) = V [(x_1, x_2) – (y_1, y_2), q] = V ((x_1-y_1 , x_2-y_2), q) = min { A((x_1-y_1), q), A((x_2-y_2), q) }≥ min {min{A(x_1, q), A(y_1, q) }, min{A(x_2, q), A(y_2, q) } }= min{min{A(x_1, q), A(x_2, q) }, min{A(y_1, q), A(y_2, q) } } = min { V((x_1, x_2), q), V((y_1, y_2), q) } = min { V (x, q), V (y, q) }. Therefore, V((x – y), q) ≥ min {V (x, q), V(y, q) }, for all x and y in GxG and q in Q. This proves that V is a Q-fuzzy subgroup of GxG. Conversely, assume that V is a Q-fuzzy subgroup of GxG, then for any x = (x_1, x_2) and y = (y_1, y_2) are in GxG, we have min { A((x_1– y_1), q), A((x_2 – y_2), q) } = V((x_1– y_1 , x_2 – y_2), q) = V [(x_1, x_2) – (y_1, y_2), q] = V((x – y), q) ≥ min{ V(x, q), V (y, q) } = min { V((x_1, x_2), q), V ((y_1, y_2), q) }= min{min{A(x_1, q), A(x_2, q) }, min{A(y_1, q), A(y_2, q)} }.

If we put x_2 = y_2 = 0, we get, A((x_1– y_1), q) ≥ min { A(x_1, q), A(y_1, q), for all x_1 and y_1 in G and q in Q. Hence A is a Q-fuzzy subgroup of G.

Proposition: Let (G, $\cdot$) and (G^{I}, $\cdot$) be any two groups and Q be a non-empty set. The Q-homomorphic image of a Q-fuzzy subgroup of G is a Q-fuzzy subgroup of G^{I}.

Proof: Let (G, $\cdot$) and (G^{I}, $\cdot$) be any two groups and Q be a non-empty set and f : GxQ→G^{I}xQ be a Q-homomorphism. That is f(xy, q) = f(x, q)f(y, q) , for all x and y in G and q in Q. Let V=f(A), where A is a Q-fuzzy subgroup of G. We have to prove that V is a Q-fuzzy subgroup of G^{I}. Now, for f(x, q) and f(y, q) in G^{I}xQ, we have V(f(x, q)f(y, q)) = V(f(xy, q)) ≥ A(xy, q) ≥ min { A(x, q), A(y, q)} which implies that V(f(x, q)f(y, q)) ≥ min { V(f(x, q)), V(f(y, q)) }. For f(x, q) in G^{I}xQ, we have V($[f(x, q)]^{-1}$) = V(f(x^{-1}, q)) ≥ A(x^{-1}, q) ≥ A(x, q) which implies that $V([f(x, q)]^{-1}) \geq V(f(x, q))$. Hence V is a Q-fuzzy subgroup of a group G^l.

Proposition: Let $(G, \cdot)$ and $(G^l, \cdot)$ be any two groups and Q be a non-empty set. The Q-homomorphic pre-image of a Q-fuzzy subgroup of G^l is a Q-fuzzy subgroup of G.

Proof: Let $(G, \cdot)$ and $(G^l, \cdot)$ be any two groups and Q be a non-empty set and $f : GxQ \rightarrow G^l xQ$ be a Q-homomorphism. That is $f(xy, q) = f(x, q)f(y, q)$, for all x and y in G and q in Q. Let $V=f(A)$, where V is a Q-fuzzy subgroup of G^l. We have to prove that A is a Q-fuzzy subgroup of G. Let x and y in G and q in Q. Then, $A(xy, q) =V(f(xy, q)) = V(f(x, q)f(y, q)) \geq \min \{V(f(x, q)), V(f(y, q))\} = \min \{ A(x, q), A(y, q) \}$ which implies that $A(xy, q) \geq \min \{A(x, q), A(y, q) \}$, for x and y in G and q in Q. And $A(x^{-1}, q) = V(f(x^{-1}, q)) = V([f(x, q)]^{-1}) \geq V(f(x, q)) = A(x, q)$ which implies that $A(x^{-1}, q) \geq A(x, q)$, for x in G and q in Q. Hence A is a Q-fuzzy subgroup of a groupG.

Proposition: Let $(G, \cdot)$ and $(G^l, \cdot)$ be any two groups and Q be a non-empty set. The Q-antihomomorphic image of a Q-fuzzy subgroup of G is a Q-fuzzy subgroup of G^l.

Proof: Let $(G, \cdot)$ and $(G^l, \cdot)$ be any two groups and Q be a set and $f : GxQ \rightarrow G^l xQ$ be a Q-antihomomorphism. That is $f(xy, q) = f(y, q)f(x, q)$, for all x and y in G and q in Q. Let $V = f(A)$, where A is a Q-fuzzy subgroup of G. We have to prove that V is a Q-fuzzy subgroup of G^l. Now, let $f(x, q)$ and $f(y, q)$ in $G^l xQ$, we have $V(f(x, q)f(y, q)) = V(f(yx, q)) \geq A(yx, q) \geq \min \{ A(x, q), A(y, q) \}$which implies that $V(f(x, q)f(y, q)) \geq \min\{ V(f(x, q)), V(f(y, q)) \}$. For x in G and q in Q, $V([f(x, q)]^{-1}) = V(f(x^{-1}, q)) \geq A(x^{-1}, q) \geq A(x, q)$ which implies that $V([f(x, q)]^{-1}) \geq V(f(x, q))$, for x in G and q in Q. Hence V is a Q-fuzzy subgroup of G^l.

Proposition: Let $(G, \cdot)$ and $(G^l, \cdot)$ be any two groups and Q be a non-empty set. The Q-antihomomorphic pre-image of a Q-fuzzy subgroup of G^l is a Q-fuzzy subgroup of G.

Proof: Let $(G, \cdot)$ and $(G^l, \cdot)$ be any two groups and Q be a set and $f : GxQ \rightarrow G^l xQ$ be a Q-antihomomorphism. That is $f(xy, q) = f(y, q)f(x, q)$, for all x and y in G and q in Q. Let $V = f(A)$, where V is a Q-fuzzy subgroup of G^l. We have to prove that A is a Q-fuzzy subgroup of G. Let x and y in G and q in Q.

Now, $A(xy, q) = V(f(xy, q)) = V(f(y, q)f(x, q)) \geq \min\{V(f(x, q)), V(f(y, q)) \}= \min\{A(x, q), A(y, q)\}$ which implies that $A(xy, q) \geq \min\{A(x, q) , A(y, q)\}$. And, $A(x^{-1}, q) = V(f(x^{-1}, q)) = V([f(x, q)]^{-1}) \geq V(f(x, q)) = A(x, q)$ which implies that $A(x^{-1}, q) \geq A(x, q)$, for x in G and q in Q. Hence A is a Q-fuzzy subgroup of a group G.

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