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On the Tarig transform and system of partial differential equations

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ABSTRACT

In this work a new integral transform, namely Tarig transform was applied to solve linear system of partial differential equations with constant coefficients. We derive the formulate for Tarig transform of partial derivatives and apply them to solve initial value problems. Our purpose here is to show the applicability of this interesting new transform and its effecting to solve such problems.

Introduction

The system of differential equations have played a central role in every aspect of applied mathematics for every long time and with the advent of the computer, their importance has increased father.

Thus investigation and analysis of differential equations cruising in applications led to many deep mathematical problems; therefore, there are so many different techniques in order to solve differential equations.

In order to solve the system of differential equations, the integral transforms were extensively used and thus there are several words on the theory and applications of integral transforms such as the Laplace, Fourier, Mellin, Hankel , Sumudu, and Elzaki transforms to name but a few. Recently, Tarig M. Elzaki introduced a new integral transform, named Tarig transform, and further applied it to solve ordinary and partial differential equations and system of partial differential equations

Definition and Derivations Tarig Transform of Derivatives

Tarig transform is defined as:

$$T\left[f(t)\right]=\frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u^2}}dt=F(u)\ ,\quad t>0\,,u\neq 0$$

To obtain the Tarig transform of Partial derivatives we use integration by parts as follows:

$$T\left[\frac{\partial f}{\partial t}(x,t)\right]=\frac{1}{u}\int_0^\infty \frac{\partial f}{\partial t}e^{-\frac{t}{u^2}}dt$$ Integrating by parts to find:

$$T\left[\frac{\partial f}{\partial t}\right]=\frac{1}{u}\left\{\left[f(x,t)\,e^{-\frac{t}{u^2}}\right]_0^\infty+\frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u^2}}f(x,t)\,dt\right. \qquad \text{(i)}$$

$$=\frac{1}{u^2}T\left[f(x,t)\right]-\frac{1}{u}f(x,0)$$

we assume that f is piecewise continuous and is of exponential order.

Now,

$$T\left[\frac{\partial f}{\partial x}\right]=\int_0^\infty \frac{1}{u}e^{-\frac{t}{u^2}}\frac{\partial f(x,t)}{\partial x}dt=\frac{\partial}{\partial x}\int_0^\infty \frac{1}{u}e^{-\frac{t}{u^2}}f(x,t)dt$$

$$=\frac{d}{dx}\left[T\left(f(x,t)\right)\right]=\frac{d}{dx}\left[F(u)\right]$$

Also we can find that: $T\left[\frac{\partial^2 f}{\partial x^2}\right]=\frac{d^2}{dx^2}\left[F(u)\right]$

To find: $T\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]$ Let $\frac{\partial f}{\partial t}=g$, then,

$$T\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=T\left[\frac{\partial g(x,t)}{\partial t}\right]=\frac{T\left[g(x,t)\right]}{u^2}-\frac{1}{u}g(x,0)$$

By equation (i) we have,

$$T\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\frac{1}{u^4}T\left[f(x,t)\right]-\frac{1}{u^3}f(x,0)-\frac{1}{u}\frac{\partial}{\partial t}f(x,0)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

The Solution System of Partial Differential Equation

In this paper we solve the linear first and second order system of partial differential equations, which are fundamental equations in mathematical physics' and occur in many branches of physics, applied mathematics as well as in engineering.

Example 1:

Consider the general system of the first order partial differential equation:

$$\begin{cases}\alpha_1 u_t(x,t)+\alpha_2 v_x(x,t)=a_1(x,t)\\ \beta_1 u_x(x,t)+\beta_2 v_t(x,t)=a_2(x,t)\end{cases} \qquad (1)$$

Where α_i and β_i , $i=1,2$ are constants. With the initial Conditions:

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$$u(x,0)=b_1(x)\ ,\ u(x,0)=b_2(x) \tag{2}$$

Solution:

Taking Tarig transform of equation (1) we obtain,

$$\begin{cases}\alpha_1\dfrac{\bar{u}(x,u)}{u^2}-\dfrac{\alpha_1 u(x,0)}{u}+\alpha_2\bar{v}_x(x,u)=\bar{a}_1(x,u)\\ \beta_1 u_x(x,u)+\dfrac{\beta_2\bar{v}(x,u)}{u^2}-\dfrac{\beta_2 v(x,0)}{u}=\bar{a}_2(x,u)\end{cases} \tag{3}$$

Substituting eq(2) into eq(3) yields:

$$\begin{cases}\alpha_1\bar{u}(x,u)+\alpha_2\bar{v}_x(x,u)=u^2\bar{a}_1(x,u)+u\alpha_1 b_1(x)\\ \beta_1 u^2\bar{u}_x(x,u)+\beta_2\bar{v}(x,u)=u^2\bar{a}_2(x,u)+u\,\beta_2 b_2(x)\end{cases} \tag{4}$$

Or

$$\begin{cases}\alpha_1\beta_2\bar{u}+\alpha_2\beta_2\bar{v}_x=\beta_2\bar{a}_1u^2+\beta_2\alpha_1 u\,b_1(x)\\ \alpha_2\beta_1u^4\bar{u}_{xx}+\alpha_2\beta_2u^2\bar{v}_x=\alpha_2(\bar{a}_2)_x u^4+\alpha_2\beta_2u^3b_2(x)\end{cases} \tag{5}$$

The solution of eq(5) is,

$$\bar{u}=\frac{\beta_2 u\,\bar{a}_1+\beta_2\alpha_1 u b_1(x)-\alpha_2(\bar{a}_2)_x u^4-\alpha u^3\beta_2 b_2(x)}{\alpha_1\beta_2-\alpha_2\beta_1u^4D^2}=F(x,u).$$

Where $D=\dfrac{d}{dx}$

$$u(x,t)=F^{-1}\left[F(x,u)\right]=g(x,t) \tag{6}$$

Substituting $u(x,t)$ into eq (1) we get:

$$v_x(x,t)=\frac{1}{\alpha_2}\left[a_1(x,t)-\alpha_1 g_t(x,t)\right]=H(x,t)$$

$$\Rightarrow v(x,t)=m(t)+\int\left[H(x,t)dx\right]=N(x,t)$$

Example 2:

Consider the system given by the following first order initial value problem,

$$\begin{cases}u_x(x,t)+v_t(x,t)=3x\\ 2u_t(x,t)-3v_x(x,t)=t\end{cases} \tag{7}$$

With the initial conditions:

$$u(x,0)=x^2\ \ ,\ v(x,0)=0 \tag{8}$$

Solution:

By using Tarig transform into eq(7) we have:

$$\begin{cases}\bar{u}_x(x,u)+\dfrac{\bar{v}(x,u)}{u^2}-\dfrac{v(x,0)}{u}=3xu\\ 2\,\dfrac{\bar{u}(x,u)}{u^2}-\dfrac{2u(x,0)}{u}-3\bar{v}_x(x,u)=u^3\end{cases} \tag{9}$$

Substituting eq (8) into eq (9) we have:

$$\begin{cases}u^2\bar{u}_x+\bar{v}=3xu^3\\ 2\bar{u}-3u^2\bar{v}_x=u^5+2ux^2\end{cases} \tag{10}$$

We can written eq(10) in the form,

$$\begin{cases}3u^4\bar{u}_{xx}+3u^2\bar{v}_x=9u^5\\ 2\bar{u}-3u^2\bar{v}_x=u^5+2ux^2\end{cases} \tag{11}$$

Or

$$\begin{cases}3u^4\bar{u}_{xx}+2\bar{u}=10u^5+2ux^2\\ \bar{u}(x,u)=\dfrac{10u^5}{2+3u^4D^2}+\dfrac{2ux^2}{2+3u^4D^2}\end{cases}$$

Where $D^2=\dfrac{d^2}{dx^2}$

$$\bar{u}(x,u)=5u^5+u\left[1+\frac{3u^4}{2}D^2\right]^{-1}x^2=2u^5+ux^2$$

$$\Rightarrow u(x,t)=F^{-1}\left[2u^5+ux^2\right]=t^2+x^2$$

Substituting $u(x,t)$ into eq (7) we get:

$$v_t(x,t)=3x-u_x=x\ \ ,\quad then\quad v(x,t)=xt+f(x)$$

By using $v(x,0)=0$ we get $f(x)=0$, *and* $u(x,t)=xt$

Example 3:

Consider the following system,

$$\begin{cases}z_t+w_x=xe^t+e^{x+t}\\ z_x-w_t=e^t-e^{x+t}\end{cases} \tag{12}$$

With initial conditions

$$z(x,0)=x\quad ,\ w(x,0)=e^x \tag{13}$$

Solution:

By using Tarig transform into eq (12) we get:

$$\begin{cases}\dfrac{\bar{z}(x,u)}{u^2}-\dfrac{z(x,0)}{u}+\bar{w}_x(x,u)=\dfrac{xu}{1-u^2}+\dfrac{e^xu}{1-u^2}\\ \bar{z}_x(x,u)-\dfrac{\bar{w}(x,u)}{u^2}+\dfrac{w(x,0)}{u}=\dfrac{u}{1-u^2}-\dfrac{e^xu}{1-u^2}\end{cases} \tag{14}$$

Substituting eq (13) into eq(14) we have:

$$\begin{cases}\bar{z}(x,u)+u^2\bar{w}_x(x,u)=\dfrac{ux}{1-u^2}+\dfrac{e^xu^3}{1-u^2}\\ u^2\bar{z}_x-\bar{w}(x,u)=\dfrac{u^3}{1-u^2}-\dfrac{e^xu}{1-u^2}\end{cases}$$

Or

$$\begin{cases}\bar{z}(x,u)+u^2\bar{w}_x(x,u)=\dfrac{ux}{1-u^2}+\dfrac{e^xu^3}{1-u^2}\\ u^4\bar{z}_{xx}(x,u)-u^2\bar{w}_x(x,u)=-\dfrac{e^xu^3}{1-u^2}\end{cases}$$

Or

$$u^4\bar{z}_{xx}(x,u)+\bar{z}(x,u)=\frac{ux}{1-u^2}$$

The solution of this equation with the boundary conditions is,

$$\bar{z}(x,u)=\left(\frac{u}{1-u^2}\right)\left(\frac{x}{1+u^4D^2}\right)=\frac{ux}{1-u^2}\ ,\quad D^2=\frac{d^2}{dx^2}$$

$$z(x,t)=xF^{-1}\left[\frac{u}{1-u^2}\right]=xe^t$$

Substituting $z(x,t)$ into eq (12) we get:

$$w_t=z_x-e^t+e^{x-t}=e^{x+t}\ \Rightarrow\ w(x,t)=e^{x-t}+f(x)$$

By using $w(x,0)=e^x$, we get: $f(x)=0$ and $w(x,t)=e^{x+t}$

Example 4:

Consider the following system,

$$\frac{\partial^2 z(x,t)}{\partial t^2}-\frac{\partial w(x,t)}{\partial x}=2x^2-e^t$$
$$\frac{\partial w(x,t)}{\partial t}+\frac{\partial^2 z(x,t)}{\partial x^2}=2t^2+xe^t \tag{15}$$

With the initial conditions:

$$z(x,0)=0\ ,\quad z_t(x,0)=0\ ,\ w(x,0)=x \tag{16}$$

Solution:

By applying Tarig transform to eq(15) we get:

$$\begin{cases}\dfrac{\bar{z}(x,u)}{u^4}-\dfrac{z(x,0)}{u^3}-\dfrac{z_t(x,0)}{u}-\bar{w}_x(x,u)=2x^2u-\dfrac{u}{1-u^2}\\[2ex] \dfrac{\bar{w}(x,u)}{u^2}-\dfrac{w(x,0)}{u}+\bar{z}_{xx}(x,u)=4u^5+\dfrac{xu}{1-u^2}\end{cases} \tag{17}$$

Substituting eq (16) in to eq (17) we have:

$$\begin{cases}\bar{z}(x,u)-u^4\bar{w}_x(x,u)=2x^2u^5-\dfrac{u^5}{1-u^2}\\[2ex] \bar{w}(x,u)+u^2\bar{z}_{xx}(x,u)=4u^7+\dfrac{xu}{1-u^2}\end{cases}$$

Or

$$\bar{z}(x,u)-u^4\bar{w}_x(x,u)=2x^2u^5-\frac{u^5}{1-u^2}$$

$$u^4\bar{w}_x(x,u)+u^6\bar{z}_{xxx}(x,u)=\frac{u^5}{1-u^2}$$

And

$$\bar{z}(x,u)+u^6\bar{z}_{xxx}(x,u)=2x^2u^5$$

By assume that the complete solution is zero, the solution of last equation is,

$$\bar{z}=2u^5\frac{x^2}{1+u^6D^3}\ \Rightarrow\ \bar{z}(x,u)=2u^5x^2\ \ then,\ \ z(x,t)=x^2F^{-1}\left[2u^5\right]=x^2t^2$$

Substituting $z(x,t)$ into eq (15) we get:

$$w_t=2t^2+xe^t-\frac{\partial^2 z}{\partial x^2}=xe^t\ \Rightarrow\ w(x,t)=xe^t+f(x)$$

Substituting $w(x,0)=x$ we get $f(x)=0$ and $w(x,t)=xe^t$

Example 5:

Consider the constant coefficients system of partial differential equation in the form:

$$\begin{cases}u_{xx}+v_y=-2u\\ v_{xx}+u_y=0\end{cases} \tag{18}$$

With the initial conditions:

$$u(x,0)=\sin x\ ,\quad v(x,0)=\cos x \tag{19}$$

Solution:

By using Tarig transform in to eq (19) yields:

$$\begin{cases}\bar{u}_{xx}(x,u)+\dfrac{\bar{v}(x,u)}{u^2}-\dfrac{v(x,0)}{u}=-2\bar{u}(x,u)\\[2ex] \bar{v}_{xx}(x,u)+\dfrac{\bar{u}}{u^2}(x,u)-\dfrac{u(x,0)}{u}=0\end{cases} \tag{20}$$

Substituting eq(19) into eq (20) we have:

$$\begin{cases}u^2\bar{u}_{xx}+\bar{v}=2u^2\bar{u}+u\cos x\\ u^2\bar{v}_{xx}+\bar{u}=u\sin x\end{cases}$$

Or

$$u^4\bar{u}_{xxxx}+u^2\bar{v}_{xx}=-2u^4\bar{u}_{xx}-u^3\cos x$$
$$u^2\bar{v}_{xx}+\bar{u}=u\sin x$$
$$u^4\bar{u}_{xxxx}+2u^4\bar{u}_{xx}-\bar{u}=-u^3\cos x-u\sin x$$

By assume that the complete solution is zero, the solution of last equation is,

$$\bar{u}(x,u)=\frac{-u^3\cos x-u\sin x}{u^4D^4+2u^4D^2-1}=\frac{-u^3\cos-u\sin x}{-1-u^4}$$

$$\bar{u}(x,u)=(\cos x)\left(\frac{u^3}{1+u^4}\right)+(\sin x)\left(\frac{u}{1+u^4}\right)$$

$$u(x,t)=\cos xF^{-1}\left[\frac{u^3}{1+u^4}\right]+(\sin x)F^{-1}\left[\frac{u}{1+u^4}\right]$$

$$u(x,t)=\cos x\sin y+\sin x\cos y=\sin(x+y)$$

Substituting $u(x,y)$ into eq(18) we have,:

$$v_y=-2u-u_{xx}=-\sin(x+y)\ \Rightarrow\ v(x,y)=\cos(x+y)+f(x)$$

By using, $v(x,0)=\cos x$, we get: $f(x)=0$ and $u(x,y)=\cos(x+y)$

Conclusion

Application of tarig transform to solution of system of different partial differential equation has been demonstrated.

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Appendix

Tarig Transform of Some Functions

S.N0.	$f(t)$	$F(u)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$