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Application of new transform "tarig transform" to partial differential equations

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ABSTRACT

In this paper we derive the formulate for Tarig transform of partial derivatives and apply them to solve five types of initial value problems. Our purpose here is to show that the applicability of this interesting new transform and its effecting to solve such problems.

Introduction

The differential equations have played a central role in every aspect of applied mathematics for every long time and with the advent of the computer, their importance has increased father.

Thus investigation and analysis of differential equations cruising in applications led to many deep mathematical problems; therefore, there are so many different techniques in order to solve differential equations.

In order to solve the differential equations, the integral transforms were extensively used and thus there are several words on the theory and applications of integral transforms such as the Laplace, Fourier, Mellin, Hankel and Sumudu, to name but a few. Recently, Tarig M. Elzaki introduced a new integral transform, named Tarig transform, and further applied it to find the solution of ordinary and partial differential equations.

Definition and Derivations Tarig Transform of Derivatives

Tarig transform is defined as:

$$T\left[f(t)\right]=\frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u^2}}dt=F(u)\ ,\ t>0\ ,u\neq 0 \qquad (1)$$

To obtain Tarig transform of Partial derivatives we use integration by parts as follows:

$$T\left[\frac{\partial f}{\partial t}(x,t)\right]=\frac{1}{u}\int_0^\infty \frac{\partial f}{\partial t}e^{-\frac{t}{u^2}}dt$$

Integrating by parts to find:

$$T\left[\frac{\partial f}{\partial t}\right]=\frac{1}{u}\left\{\left[f(x,t)\,e^{-\frac{t}{u^2}}\right]_0^\infty+\frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u^2}}f(x,t)\,dt\right. \qquad (2)$$

$$=\frac{1}{u^2}T\left[f(x,t)\right]-\frac{1}{u}f(x,0)$$

We assume that f is piece wise continuous and is of exponential order.

Now

$$T\left[\frac{\partial f}{\partial x}\right]=\int_0^\infty \frac{1}{u}e^{-\frac{t}{u^2}}\frac{\partial f(x,t)}{\partial x}dt=\frac{\partial}{\partial x}\int_0^\infty \frac{1}{u}e^{-\frac{t}{u^2}}f(x,t)dt$$

$$=\frac{d}{dx}\left[T\left(f(x,t)\right)\right]=\frac{d}{dx}\left[F(u)\right] \qquad (3)$$

Also we can find that:

$$T\left[\frac{\partial^2 f}{\partial x^2}\right]=\frac{d^2}{dx^2}\left[F(u)\right] \qquad (4)$$

To find: $T\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]$ Let $\frac{\partial f}{\partial t}=g$, then,

$$T\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=T\left[\frac{\partial g(x,t)}{\partial t}\right]=\frac{T\left[g(x,t)\right]}{u^2}-\frac{1}{u}g(x,0)$$

By equation (2) we have,

$$T\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\frac{1}{u^4}T\left[f(x,t)\right]-\frac{1}{u^3}f(x,0)-\frac{1}{u}\frac{\partial}{\partial t}f(x,0) \qquad (5)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

Solution of Partial Differential Equations

In this paper we solve the linear first and second order partial differential equations, which are fundamental equations in mathematical physics' and occur in many branches of physics, a applied mathematics as well as in engineering.

Example 1:

Find the solution of the first order initial value problem,

$$\frac{\partial y}{\partial x}=2\frac{\partial y}{\partial t}+y \quad ,\quad y(x,0)=6e^{-3x} \qquad (6)$$

And y is bounded for $x>0$, $t>0$.

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Solution:

Let $F(u)$ be Tarig transform of, then, taking Tarig transform of (6) to get:

$$\frac{d}{dx}\left[F(u)\right]=2\left[\frac{1}{u^2}F(u)-\frac{1}{u}y(x,0)\right]+F(u)$$

Where $F(u)$ is Tarig transform of $y(t,x)$ using the initial condition to find that:

$$\frac{d}{dx}F(u)-\left(\frac{2}{u^2}+1\right)F(u)=-\frac{12}{u}e^{-3x}$$

This is the linear ordinary differential equation, the integrating factor is $e^{\frac{-(2+u^2)}{u^2}x}$ therefore:

$$F(u)=\frac{12}{u}\left[\frac{u^2}{2+4u^2}\right]e^{-3x}+ce^{\frac{2+u^2}{u^2}x}$$

$F(u)$ is bounded, then $c=0$ and

$$F(u)=6\left[\frac{u}{1+2u^2}\right]e^{-3x}$$

Taking the inverse Tarig transform to find:

$$y(x,t)=6e^{-3x}e^{-2t}=6e^{-3x-2t}$$

Example 2:

Consider the one dimensional unsteady heat conduction problem as follows:

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}\quad,\quad t>0\quad,\quad 0<x<1 \qquad (7)$$

With the boundary conditions:

$$U(x,0)=3\sin 2\pi x\ ,\ U(0,t)=U(1,t)=0 \qquad (8)$$

Solution:

Taking Tarig transform of eq (7) we have:

$$\frac{F(x,u)}{u^2}-\frac{1}{u}U(x,0)=\frac{d^2}{dx^2}F(x,u)$$

Or

$$u^2D^2F(x,u)-F(x,u)=-3u\sin 2\pi x \qquad (9)$$

The solutions of (9) are,

$$F_c(x,u)=c_1e^{\frac{x}{u}}+c_2e^{\frac{-x}{u}}.,$$

$$F_p(x,u)=\frac{-3u\sin 2\pi x}{u^2D^2-1}=\frac{3u\sin 2\pi x}{1+4\pi^2u^2}$$

Then:

$$F(x,u)=c_1e^{\frac{x}{u}}+c_2e^{\frac{-x}{u}}+\frac{3u\sin 2\pi x}{1+(2\pi)^2u^2}$$

Taking Tarig transform of the boundary conditions,

$$T\left[U(0,t)\right]=F(0,u)=0,\ T\left[U(1,t)\right]=F(1,u)=0,$$

Then we have: $c_1+c_2=0$ and $ce^{\frac{1}{u}}+c_1e^{-\frac{1}{u}}=0$

These means that $c_1=c_2=0$, then the solution of (9) is,

$$F(x,u)=\frac{3u\sin 2\pi x}{1+4\pi^2u^2}$$

Then:

$$U(x,t)=F^{-1}\left[\frac{3u\sin 2\pi x}{1+4\pi^2u^2}\right]=3e^{-4\pi^2t}\sin 2\pi x \qquad (*)$$

This problem has an interesting physical interpretation .If we consider a solid bounded by the infinite plane faces $x=0$ and $x=1$, the equation. $\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2}$ is the equation for heat conduction in this solid where $U=U(x,t)$ is the temperature at any plane face x at any time t and k is a constant called the diffusivity, which depends on the material of the solid, .The boundary conditions $U(0,t)=0$ and $U(1,t)=0$ Indicate that the temperatures at $x=0$ and $x=1$ are kept at temperature zero, while $U(x,0)=3\sin 2\pi x$ represents the initial temperature every where in $0<x<1$, result (*) is the temperature Everywhere in the solid at time $t>0$.

Example 3:

Consider the following wave equation,

$$\frac{\partial^2 w(x,t)}{\partial t^2}-4\frac{\partial^2 w(x,t)}{\partial x^2}=0\quad,\ 0\le x\le 1\ ,\ t>0 \qquad (10)$$

With the initial conditions:

$$w(x,0)=\sin \pi x\ ,\quad \frac{\partial w}{\partial t}(x,0)=0 \qquad (11)$$

And the boundary conditions,

$$w(0,t)=w(1,t)=0 \qquad (12)$$

Solution:

Applying Tarig transform to eq(10) we have:

$$\frac{\bar{w}(x,u)}{u^4}-\frac{w(x,0)}{u^3}-\frac{w_t(x,0)}{u}-4\bar{w}_{xx}(x,u)=0 \qquad (13)$$

Where $\bar{w}(x,u)$ is Tarig transform of $w(x,t)$. substituting eq(11) into eq(13) we get:

$$\bar{w}(x,u)-4u^4\bar{w}_{xx}(x,u)=u\sin \pi x \qquad (14)$$

Like example (2), using the boundary conditions (12) we get that: $w_c(x,u)=0$

Then the solution of (14) is,

$$\bar{w}(x,u)=\frac{u\sin \pi x}{1-4u^4D^2}=\frac{u\sin \pi x}{1+(2\pi)^2u^4},$$

Where $D^2=\frac{d^2}{dx^2}$

Then:

$$w(x,t)=\sin\pi x\,F^{-1}\left[\frac{u}{1+(2\pi)^2u^4}\right]=\sin\pi x\ \cos2\pi x$$

Example 4:

Consider the following Laplace equation,

$$U_{xx}(x,y)+U_{yy}(x,y)=0\quad,\ t>0 \tag{15}$$

With the initial conditions:

$$U(x,0)=0\ ,\ U_y(x,0)=\cos x \tag{16}$$

And the boundary conditions:

$$U(0,t)=U(1,t)=0\quad,\ 0\le x\le1 \tag{17}$$

Solution:

By using Tarig transform into eq(15) yields:

$$\bar{U}_{xx}(x,u)+\frac{\bar{U}(x,u)}{u^4}-\frac{U(x,0)}{u^3}-\frac{U_y(x,0)}{u}=0 \tag{18}$$

Using the initial conditions (16) to find:

$$u^4\bar{U}_{xx}(x,u)+\bar{U}(x,u)=u^3\cos x \tag{19}$$

The solutions of equation (19) under the boundary conditions (17) are,

$$\bar{U}_c(x,u)=0\quad and\quad \bar{U}_p=\frac{u^3\cos x}{1-u^4}$$

Then the general solution of (15) is,

$$U(x,y)=\cos x\ F^{-1}\left[\frac{u^3}{1-u^4}\right]=\cos x\ \sinh y$$

Example 5:

Consider the following Telegrapher's equation,

$$U_{tt}(x,t)+2\alpha U_t(x,t)=\alpha^2U_{xx}(x,t)\ ,\ 0<x<1\ ,\ t>0 \tag{20}$$

Where α is positive constant.

With the initial conditions:

$$U(x,0)=\cos x\quad,\ U_t(x,0)=0 \tag{21}$$

And the boundary conditions:

$$U(0,t)=0\ ,\quad U(1,t)=0 \tag{22}$$

Solution:

Using Tarig transform into eq(20) and making use of the initial conditions (21) we have:

$$\alpha^2u^4\bar{U}_{xx}(x,u)-\left(1+2\alpha u^2\right)\bar{U}(x,u)=-u\cos x-2\alpha u^3\cos x$$

Taking Tarig transform of the boundary conditions (22) we get:

$$\bar{U}(0,u)=0\quad,\quad\bar{U}(1,u)=0$$

Use these conditions to obtain the general solution of eq (23) in the form:

$$\bar{U}(x,u)=\frac{u+2\alpha u^3\cos x}{1+2\alpha u^2-\alpha^2u^4D^2}\ ,\ \left(D^2=\frac{d^2}{dx^2}\right)$$

$$\bar{U}(x,u)=u\cos x\left[\frac{1+2\alpha u^2}{\left(1+\alpha u^2\right)^2}\right]=\cos x\left[\frac{u}{1+\alpha u^2}+\frac{\alpha u^3}{\left(1+\alpha u^2\right)^2}\right]$$

Then:

$$U(x,t)=\cos x\,F^{-1}\left[\frac{u}{1+\alpha u^2}+\frac{\alpha u^3}{\left(1+\alpha u^2\right)^2}\right]=\cos x\left[e^{-\alpha t}+\alpha t e^{-\alpha t}\right]$$

And, $$U(x,t)=(1+\alpha t)e^{-\alpha t}\cos x$$

Conclusion

Application of tarig transform to solution of special different partial differential equation has been demonstrated.

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Appendix

Tarig Transform of Some Functions

S.N0.	$f(t)$	$F(u)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$
10	$H(t-a)$	$ue^{\frac{-a}{u^2}}$
11	$\delta(t-a)$	$\frac{1}{u}e^{\frac{-a}{u^2}}$
12	te^{at}	$\frac{u^3}{\left(1-au^2\right)^2}$