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A new characterization of A_{26} by their element orders

Guangju Zeng and Wenjun Xu
School of Science, Sichuan University of Science and Engineering Zigong, 643000, China.

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ABSTRACT

Given an arbitrary finite group G, denote by $\omega(G)$ the set of its element orders. The group G is said to be recognizable by the set $\omega(G)$ if the equality $\omega(G)=\omega(H)$ implies an isomorphism of G and H for each finite group H. For a prime $p\geq 5$, the alternating groups A_p, A_{p+1}, A_{p+2} are recognizable. But for A_{p+3} are has not known. In this paper, we will give an example for $p+3$ not a prime, namely, that A_{26} is characterizable.

Introduction

All groups in this paper are finite. The influence of element orders on the structure of finite groups was studied by some authors (see [1], [4], [8], [9], [11], [13], [16], [19] and [21]). A group G is recognizable by its element order set $\omega(G)$ if the equality $\omega(G)=\omega(H)$ implies that $G\cong H$. It is proved that an alternating group A_n of degree n, where $n=r, r+1, r+2$ and $r>5$ is a prime, is recognizable by the se of elements orders (see [5], [17]). Among the remaining alternating groups $A_{10}, A_{16}, A_{22}, A_{26}, A_{27}, A_{28}, A_{34}, \cdots$, A_{10} was nonrecognizable (see [7, Proposition 2]), A_{16} and A_{22} were recognizable (see [17, Theorem 2], and [12, Theorem]).

Denote by $\pi(G)$ the set of prime divisors of the order $|G|$ of G. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$. If $\rho(G)$ is some indepedent set with the maximal number of vertices in $GK(G)$ (the subset of vertices of a graph is called an independent set if its vertices in $GK(G)$ are pairwise nonadjacent), then $t(G)=\rho(G)$. Denote by $t(2,G)$ the maximal number of vertices in the independent sets of $GK(G)$ containing 2.

Further notations are standard (see [2])

Preliminaries

In this section, we given some basic results which we will use in the sequel.

Lemma 2.1 [14, Theorem, p397] Let G be a finite group satisfying the two conditions:

(a) there exist three primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$; i.e., $t(G\geq 3)$;

(b) there exists an odd prime in $\pi(G)$ pairwise nonadjacent in $GK(G)$ to the prime 2; i.e., $t(G\geq 2)$.

Then there is a finite nonabelian simple group S such that $S\leq \overline{G}=G/K\leq Aut(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S)\geq t(G)-1$, and one of the following statements holds:

(1) $S\cong A_7$ or $L_2(q)$ for some odd q, and $t(S)=t(2,S)=3$.

(2) For every prime $p\in\pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2,S)\geq t(2,G)$.

Lemma 2.2 ([20]) Let G be a finite simple group and $23\in\pi(G)\subseteq\{2,3,5,7,11,13,17,19,23\}$. Then G is isomorphic to one of the following groups: $L_2(23), U_3(23), M_{23}, M_{24}, Co_1, Co_2, Co_3, Fi_{23}$, or $A_i, i=23,24,\cdots,28$.

Main results

In this section, we will give the main results and its proof.

Theorem 3.1 Let G be a finite group such that $\omega(G)=\omega(A_{26})$, where A_{26} is the alternating group of degree 26. Then $G\cong A_{26}$

Proof. Since G does not contain any elements of order $13\cdot 17, 13\cdot 19, 17\cdot 19$, $\{13\cdot 17, 13\cdot 19, 17\cdot 19\}\cap\omega(G)=\Phi$, and so from Lemma 2.5 of [3], G is insoluble. Further more, $\rho(2,G)=\{2,23\}$ from [15, Theorem 7.1, and Table 2].

Hence by Corollary 2.6 of [3], the conditions of Lemma 2.1 are satisfied.

Then there is a finite nonabelian simple group S such that $S\leq\overline{G}=G/K\leq Aut(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S)\geq t(G)-1$, and one of the following statements holds:

(1) $S\cong A_7$ or $L_2(q)$ for some odd q, and $t(S)=t(2,S)=3$.

Tele:
E-mail addresses: 64215969@qq.com

(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2,S) \geq t(2,G)$.

If $S \cong A_7$, we have $A_7 \leq G/K \leq S_7$ and $\{1,13,17,19,23\} \subseteq \pi(K)$. Since K is soluble, then there exists a Hall $\{13,17,19,23\}$-subgroup. For the subset $\rho = \{13,17,19\}$ of $\pi(G)$, the three numbers 13, 17, 19 divide the product $|K| \cdot |\overline{G}/S|$, which contradicts Proposition 3 of [14].

Then $S \cong L_2(q)$, where $q = r^t$, r is an odd prime.

Since $\pi(L_2(q)) \subseteq \pi(G)$, We have from [20] that $S \cong L_2(7), L_2(11), L_2(13), L_2(17)$, $L_2(27), L_2(19), L_2(23)$. The same reason, S is not isomorphic to $L_2(7), L_2(11), L_2(13)$ $L_2(17), L_2(19), L_2(23)$, or $L_2(27)$

Thus (1) does not hold and so we only think (2).

Let $S_{23} \in Syl_2(G)$. Then there exists a $G_{23} \in Syl_{23}(G)$ such that $S_{23} \cong G_{23}$. From Lemma 2.2, we have that: $L_2(23), U_3(23), M_{23}, M_{24}, Co_1, Co_2, Co_3, Fi_{23}$, or $A_i, i = 23,24,\cdots,28$. We have from Proposition 3 of [14] that at least two of primes $13,17,19$ must divides the order of S, and so $S \cong Co_1, Fi_{23}, A_i$ where $i = 23,24,25,26,27,28$.

If $S \equiv Co_1$, from [2], $\omega(S) = \{1,\cdots,16,18,20,21,22,23,26,28,30,33,35,36,39,$ $40,42,60\}$. By Proposition 3 of [14], we have that at least two of the primes $7,11,13$ must divide the order of K. It is easy to get that $5 \mid |K|$ and $13 \nmid |K|$. Thses $5 \cdot 13 \in \omega(G)$ and $5 \cdot 13 \notin \omega(G/K)$ imply $5 \mid |K|$. And $4 \cdot 11 \in \omega(G)$ and $4 \cdot 11 \notin \omega(G/K)$. Thus we have $\{2,7\} \subseteq \pi(K)$, and the two numbers 11,13 are not prime divisors of $|K|$.

Then all subgroups of order 13 are conjugate in G, and the fact that a Hall $\{2,7\}$-subgroup of order 156 of K is nilpotent by Thompson Theorem (see [10, Theorem 10.5.4]. This implies that G contains an element of order 13 which centralizes elements of order 4 and 11 in K, Thus $4 \cdot 11 \cdot 13 \in \omega(G)$, a contradiction.

If $S \cong Fi_{23}$, from [2], $\omega(Fi_{23}) = \{1,\cdots,18,20,\cdots,24,26,27,28,30,35,36,39,42,60\}$ .By Proposition 3 of [14], we have that at least two of the primes 11,13,17 must divide the order of $|K|$. It is easy to get that $7 \mid |K|$ and $17 \nmid |K|$. These $5 \cdot 17 \in \omega(G)$ and $5 \cdot 17 \notin \omega(G/K)$ imply $5 \mid |K|$. And $5 \cdot 11 \in \omega(G)$ and $5 \cdot 11 \notin \omega(G/K)$. Thus we have $\{5,7\} \subseteq \pi(K)$, and the two numbers 11,13 are not prime divisors of $|K|$.

Then all subgroups of order 17 are conjugate in G, and the fact that a Hall {11,13}-subgroup of K is nilpotent by Thompson Theorem (see [10, Theorem 10.5.4]. This implies that G contains an element of order 17 which centralizes elements of order 11 and 13 in K, Thus $11 \cdot 13 \cdot 17 \in \omega(G)$, a contradiction.

Then S must be an alternating group and so $S \cong A_i, i = 23,24,25,26,27,28$.

If $K \neq 1$, we take K to be an elementary abelian r-group with $r \in \pi(K)$. Since S contains a Frobenius group of order 49 with a cyclic complement of order 7, we know that $r \neq 7$; for otherwise $49 \in \omega(G)$ (see [19, Lemma 4]), a contradiction. Since S contains a Frobenius group of order $23 \cdot 11$ with a cyclic complement of order 11, we have $r \neq 11$ or 19; for otherwise, $11^2 \in \omega(G)$ or $11 \cdot 19 \in \omega(G)$ (see [19, Lemma 4]), a contradiction. Since S contains a Frobenius group of order $17 \cdot 16$ with a cyclic complement of order 16, we have $r \neq 2$ or 13; for otherwise, $2^5 \in \omega(G)$ or $16 \cdot 13 \in \omega(G)$ (see [19, Lemma 4]), a contradiction. Similarly, Since S contains a Frobenius group of order $8 \cdot 7$ with a cyclic complement of order 7, we have $r \neq 7$; for otherwise, $7^2 \in \omega(G)$ (see [19, Lemma 4]), a contradiction. Thus $K = 1$ and $S \leq G \leq Aut(S)$.

Case 1. $S \cong A_{23}, A_{24}$, or A_{25}

In this case, $S \leq S_{25}$. But $153 \in \omega(G)$ and $153 \notin \omega(G)$, a contradiction.

Case 2. $S \cong S_{27}$ or S_{28}

In this case, from [5, lemma 2], we have that $\omega(A_{26}) \subset \omega(A_{27}) \subset \omega(A_{28})$. Then there exists an element of order 125 of A_{27}, A_{28} such that 125 belongs to both A_{27} and A_{28}, but $125 \notin \omega(G)$, a contradiction.

From Cases 1 and 2, we have $G \cong A_{26}$.

This completes the proof.

Remark 3.2 The alternating A_{26} is another example which can be recognizable. Also, the methods of this paper also can be used to A_{11}, A_{22}

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