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Controllability results for impulsive differential systems with infinite delay

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ABSTRACT

This paper is concerned with the controllability of impulsive functional differential equations with infinite delay in a Banach space. Sufficient conditions for controllability are obtained by using the measures of noncompactness and Monch fixed point therem under the assumption of noncompactness of the evolution system.

Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years, because, all the structures of its emergence have deep physical background and realistic mathematical models. The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparision with the duration of the process. It has seen considerable development in the last decade, see the monographs of of Bainov and Simeonov [2], Lakshmikantham et al. [13] and Samoilenko and Perestyuk [22] and the papers of [1, 4, 7, 8, 18, 24, 25], where the numerous properties of their solutions are studied and detailed bibliographies are given. In many areas of science there has been an increasing interest in the investigation of functional differential equations, incorporating memory or aftereffect, i.e., there is an effect of infinite delay on state equations. We refer the reader to Kolmanovskii and Myshkis [11, 12], Wu [23] and the references there in for a wealth of reference materials on the subject.

The notion of controllability is of great importance in mathematical control theory. Many basic problems of control theory pole-assignment, structural engineering and optimal control may be solved under the assumption that the system is controllable. The concept of controllability plays an crucial role in both finite and infinite dimensional spaces, that is systems represented by ordinary differential equations and partial differential equations, respectively. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [5, 6, 9, 14, 15, 17, 19, 21]. In [9], the author studied the controllability of impulsive functional differential systems of the form

$$x'(t) = A(t)x(t) + f\big(t, x(t)\big) + (Bu)(t), \text{ a.e. on } [0, b],$$
$$\Delta x|_{t=t_i} = I_i(x(t_i)), \ i = 1,2,\dots,s,$$
$$x(0) + M(x) = x_0,$$

where $A(t)$ is a family of linear operators which generates an evolution operator $U{:}\,\Delta = \{(t,s) \in [0,b] \times [0,b]{:}\, 0 \le s \le t \le b\} \to L(X)$, here X is a Banach space, $L(X)$ is the space of all bounded linear operators in X; $f{:}\,[0,b] \times X \to X$; $0 < t_1 < \cdots < t_s < t_{s+1} = b$; $I_i{:}\,X \to X$, $i = 1,2,\dots,s$, are impulsive functions; $M{:}\,PC([0,b],X) \to X$; B is a bounded linear operator from a Banach space V to X and the control function $u(\cdot)$ is given in $L^2([0,b],V)$. The results are obtained by using the measures of noncompactness and Monch fixed point theorem.

Inspired by the above mentioned works [5, 9, 25], in this paper we investigate the sufficient conditions for the controllability of impulsive differential system with infinite delay of the form

$$x'(t) = A(t)x(t) + f(t,x_t) + (Bu)(t), \ t \in J = [0,b], \ t \ne t_k, \ k = 1,2,\dots,m,$$
$$\Delta x|_{t=t_k} = I_k(x(t_k)), \ k = 1,2,\dots,m,$$
$$x_0 = \phi \in \mathcal{B}_h,$$

where $\mathcal{B}_h$ is the abstract phase space which will be defined later; $A(t)$ is a family of linear operators which generates an evolution system $\{U(t,s){:}\, 0 \le s \le t \le b\}$; The state variable $x(\cdot)$ takes the values in the real Banach space X with norm $\|\cdot\|$; The control function $u(\cdot)$ is given in $L^2(J,V)$ a Banach space of admissible control functions with V as a Banach space; B is a bounded linear operator from V into X; $f{:}\,J \times \mathcal{B}_h \to X$ is given function; $I_k{:}\,X \to X$; $k = 1,2,\dots,m$, are impulsive functions; $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$; $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Denote $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1,2,\dots,m$, and define the following space:

Let $\mathcal{PC}((-\infty,b],X) = \{x{:}\,(-\infty,b] \to X$ such that $x(t)$ is continuous at $t \ne t_k$ and left continuous at $t = t_k$ and the right limit $x(t_k^+)$ exists for $k = 1,2,\dots,m\}$. It is easy to verify that $\mathcal{PC}((-\infty,b],X)$ is a Banach space with the norm $\|x\|_{\mathcal{PC}} = \sup\{\|x(t)\|{:}\, t \in (-\infty,b]\}$.

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The histories $x_t:(-\infty,0]\to X$ defined by $x_t(\theta)=x(t+\theta)$, $-\infty<\theta\le 0$, which belong to $\mathcal{B}_h$. Our approach here is based on semigroup theory, measures of noncompactness and Monch fixed point theorem.

Preliminaries

At first, we define the abstract phase space $\mathcal{B}_h$ as given in [5].

Assume that $h:(-\infty,0]\to(0,+\infty)$ is a continuous function with $l=\int_{-\infty}^{0}h(s)ds<+\infty$. For any $a>0$, we define

$$\mathcal{B}=\{\psi:[-a,0]\to X \text{ such that } \psi(t) \text{ is bounded and measurable}\}$$

and equip the space $\mathcal{B}$ with the norm

$$\|\psi\|_{[-a,0]}=\sup_{s\in[-a,0]}\|\psi(s)\|,\quad \forall\psi\in\mathcal{B}.$$

Let us define

$$\mathcal{B}_h=\{\psi:(-\infty,0]\to X \text{ such that for any } c>0,\ \psi|_{[-c,0]}\in\mathcal{B} \text{ and } \int_{-\infty}^{0}h(s)\|\psi\|_{[s,0]}ds<+\infty\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}_h}=\int_{-\infty}^{0}h(s)\|\psi\|_{[s,0]}ds,\quad \forall\psi\in\mathcal{B}_h,$$

then it is easy to see that $(\mathcal{B}_h,\|\cdot\|_{\mathcal{B}_h})$ is a Banach space. Now we consider the space,

$$\mathcal{B}_h'=\{x\in\mathcal{PC}((-\infty,b],X) \text{ such that } x_0=\phi\in\mathcal{B}_h\}.$$

Set $\|\cdot\|_{\mathcal{B}_h'}$ be a seminorm in $\mathcal{B}_h'$ defined by,

$$\|x\|_{\mathcal{B}_h'}=\|\phi\|_{\mathcal{B}_h}+\sup\{\|x(s)\|:s\in[0,b]\},\quad x\in\mathcal{B}_h'.$$

Next we recall some basic definitions and lemmas which are used throughout this paper.

Definition 2.1. *A function $x\in\mathcal{PC}((-\infty,b],X)$ is said to be a mild solution of the system $(1.1)-(1.3)$ if, $x_0=\phi$ on $(-\infty,0]$; $\Delta x|_{t=t_k}=I_k(x(t_k))$, $k=1,2\ldots,m$; and the following integral equation is satisfied.*

$$x(t)=U(t,0)\phi(0)+\int_0^t U(t,s)[Bu(s)+f(s,x_s)]ds+\sum_{0<t_k<t}U(t,t_k)I_k(x(t_k)),\quad t\in J.$$

Definition 2.2. *The system $(1.1)-(1.3)$ is said to be controllable on the interval J if, for every initial function $\phi\in\mathcal{B}_h$ and $x_1\le X$, there exists a control $u\in L^2(J,V)$ such that the mild solution $x(\cdot)$ of $(1.1)-(1.3)$ satisfies $x(b)=x_1$.*

Definition 2.3. *Let E^+ be the positive cone of an order Banach space $(E,\le)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in E^+ is called a measure of noncompactness(MNC) on X if $\Phi(\overline{co}\Omega)=\Phi(\Omega)$ for all bounded subsets $\Omega\subseteq X$, where $\overline{co}\Omega$ stands for the closed convex hull of Ω.*

The MNC Φ is said:

Monotone if for all bounded subsets Ω_1, Ω_2 of X we have: $(\Omega_1\subseteq\Omega_2)\Rightarrow(\Phi(\Omega_1)\le\Phi(\Omega_2))$;

Nonsingular if $\Phi(\{a\}\cup\Omega)=\Phi(\Omega)$ for every $a\in X$, $\Omega\subset X$;

Regular if $\Phi(\Omega)=0$ if and only if Ω is relatively compact in X.

One of the most examples of MNC is the noncompactness measure of Hausdorff β defined on each bounded subset Ω of X by

$\beta(\Omega)=\inf\{\varepsilon>0;\ \Omega$ can be covered by a finite number of balls of radii smaller than $\varepsilon\}$.

It is well known that MNC β enjoys the above properties and other properties see [3, 10]:

For all bounded subsets Ω,Ω_1,Ω_2 of X,

$\beta(\Omega_1+\Omega_2)\le\beta(\Omega_1)+\beta(\Omega_2)$, where $\Omega_1+\Omega_2=\{x+y:x\in\Omega_1,y\in\Omega_2\}$;

$\beta(\Omega_1\cup\Omega_2)\le max\{\beta(\Omega_1),\beta(\Omega_2)\}$;

$\beta(\lambda\Omega)\le|\lambda|\beta(\Omega)$ for any $\lambda\in\mathbb{R}$;

If the map $Q:D(Q)\subseteq X\to Z$ is Lipschitz continuous with constant k, then $\beta_Z(Q\Omega)\le k\beta(\Omega)$ for any bounded subset $\Omega\subseteq D(Q)$, where Z is a Banach space.

Definition 2.4. *A two parameter family of bounded linear operators $U(t,s)$, $0\le s\le t\le b$ on X is called an evolution system if the following two conditions are satisfied:*

(i) $U(s,s)=I$, $U(t,r)U(r,s)=U(t,s)$ for $0\le s\le r\le t\le b$;

(ii) $(t,s)\to U(t,s)$ is strongly continuous for $0\le s\le t\le b$.

Since the evolution system $U(t,s)$ is strongly continuous on the compact operator set $J\times J$, then there exists $M_1>0$ such that $\|U(t,s)\|\le M_1$ for any $(t,s)\in J\times J$. More details about evolution system can be found in Pazy [20].

Definition 2.5. *A countable set $\{f_n\}_{n=1}^{\infty}\subset L^1([0,b],X)$ is said to be semicompact if the sequence $\{f_n\}_{n=1}^{\infty}$ is relatively compact in X for almost all $t\in[0,b]$ and if there is a function $\mu\in L^1([0,b],\mathbb{R}^+)$ satisfying $sup_{n\ge1}\|f_n(t)\|\le\mu(t)$ for a.e. $t\in[0,b]$.*

Lemma 2.1. *([5]) Assume that $x\in\mathcal{B}_h'$; then for $t\in J$, $x_t\in\mathcal{B}_h$. Moreover,*

$$l\|x(t)\|\le\|x_t\|_{\mathcal{B}_h}\le\|\phi\|_{\mathcal{B}_h}+l\sup_{s\in[0,t]}\|x(s)\|,$$

where $l=\int_{-\infty}^{0}h(t)dt<+\infty$.

Lemma 2.2. *([3]) If $W\subset C([a,b],X)$ is bounded and equicontinous, then $\beta(W(t))$ is continuous for $t\in[a,b]$ and*

$$\beta(W)=\sup\{\beta(W(t)),\ t\in[a,b]\},\quad \text{where}\quad W(t)=\{x(t):x\in W\}\subseteq X.$$

Lemma 2.3. *([25]) If $W\subset\mathcal{PC}([a,b],X)$ is bounded and piecewise equicontinuous on $[a,b]$ then $\beta(W(t))$ is piecewise continuous for $t\in[a,b]$ and*

$$\beta(W)=\sup\{\beta(W(t)),\ t\in[a,b]\}.$$

Lemma 2.4. *([17]) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,b],\mathbb{R}^+)$. Assume that there exist $\mu,\eta\in L^1([0,b],\mathbb{R}^+)$ satisfying $sup_{n\ge1}\|f_n(t)\|\le\mu(t)$ and $\beta(\{f_n(t)\}_{n=1}^{\infty})\le\eta(t)$ a.e. $t\in[0,b]$, then for all $t\in[0,b]$, we have*

$$\beta(\{\int_0^t U(t,s)f_n(s)ds:n\ge1\})\le 2M_1\int_0^t\eta(s)ds.$$

Lemma 2.5. *([17]) Let $(Gf)(t)=\int_0^t U(t,s)f(s)ds$, If $\{f_n\}_{n=1}^{\infty}\subset L^1([0,b],X)$ is semicompact, then the set $\{Gf_n\}_{n=1}^{\infty}$ is relatively compact in $C([0,b],X)$ and moreover if $f_n\to f_0$, then for all $t\in[0,b]$,*

$$(Gf_n)(t)\to(Gf_0)(t),\ as\ n\to\infty.$$

The following fixed-point theorem, a nonlinear alternative of Monch type, plays a key role in our proof of controllability of the system $(1.1)-(1.3)$.

Lemma 2.6. *([16, Theorem 2.2]) Let D be a closed convex subset of a Banach space X and $0\in D$. Assume that $F:D\to X$ is a continuous map which satisfies Monch's condition, that is $(M\subseteq D$ is countable, $M\subseteq\overline{co}(\{0\}\cup F(M))\Rightarrow\overline{M}$ is compact $)$. Then F has a fixed point in D.*

Controllability Results

In this section, we present and prove the controllability results for the problem $(1.1)-(1.3)$. In order to prove the main theorem of this section, we list the following hypotheses:

(H1). $A(t)$ is a family of linear operators, $A(t):D(A)\to X$, $D(A)$ not depending on t and dense subset of X, generating an equicontinuous evolution system $\{U(t,s):0\le s\le t\le b\}$, i.e., $(t,s)\to\{U(t,s)x:x\in T\}$ is equicontinous for $t>0$ and for all bounded subsets T and

$$M_1=\sup\{\|U(t,s)\|:(t,s)\in J\times J\}.$$

(H2). The function $f:J\times\mathcal{B}_h\to X$ satisfies:

For a.e. $t\in J$, the function $f(t,\cdot):\mathcal{B}_h\to X$ is continuous and for all $\phi\in\mathcal{B}_h$, the function $f(\cdot,\phi):J\to X$ is strongly measurable.

For every positive integer r, there exists $\alpha_r \in L^1([0,b];\mathbb{R}^+)$ such that

$$\sup_{\|\phi\|_{\mathcal{B}_h}\le r} \| f(t,\phi) \| \le \alpha_r(t) \text{ for a.e. } t \in J,$$

and

$$\liminf_{r\to\infty} \int_0^b \frac{\alpha_r(t)}{r} dt = \sigma < \infty.$$

There exists integrable function $\eta: [0,b] \to [0,\infty)$ such that

$$\beta(f(t,D)) \le \eta(t) \sup_{-\infty<\theta\le 0} \beta(D(\theta)) \text{ for a.e. } t \in J \text{ and } D \subset \mathcal{B}_h,$$

where $D(\theta) = \{v(\theta): v \in D\}$.

(H3). The linear operator $W: L^2(J,V) \to X$ is defined by

$$W = \int_0^b U(t,s)Bu(s)ds \quad \text{such that}$$

W has an invertible operator W^{-1} which take values in $L^2(J,V)/\ker W$, and there exist positive constants M_2 and M_3 such that

$$\| B \| \le M_2, \quad \| W^{-1} \| \le M_3.$$

There is $K_W \in L^1(J,\mathbb{R}^+)$ such that, for every bounded set $Q \subset X$,

$$\beta(W^{-1}Q)(t) \le K_W(t)\beta(Q).$$

(H4). $I_k: X \to X$, $k = 1,2\ldots,m$, be a continuous operator such that: There are nondecreasing functions $L_k: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\| I_k(x) \| \le L_k(\| x \|) \quad k = 1,2\ldots,m, \quad x \in X,$$

and

$$\liminf_{\rho\to\infty} \frac{L_k(\rho)}{\rho} = \lambda_k < \infty, \quad k = 1,2\ldots,m.$$

There exist constants $C_k \ge 0$ such that,

$$\beta(I_k(S)) \le C_k\beta(S), \quad k = 1,2\ldots,m,$$

for every bounded subset S of X.

(H5). The following estimation holds true:

$$N = [(M_1 + 2M_1^2M_2 \| K_W \|_{L^1})\textstyle\sum_{k=1}^m C_k + (2M_1 + 4M_1^2M_2 \| K_W \|_{L^1}) \| \eta \|_{L^1}] < 1.$$

Theorem 3.1. *Assume that the hyphotheses $(H1)-(H5)$ are satisfied. Then the impulsive differential system $(1.1)-(1.3)$ is controllable on J provided that,*

$$M_1(1 + M_1M_2M_3b^{\frac{1}{2}})(\sigma l + \textstyle\sum_{k=1}^m \lambda_k) < 1. \tag{3.1}$$

Proof. Using the hypothesis $(H3)(i)$, for every $x \in \mathcal{PC}((-\infty,b],X)$, define the control

$$u_x(t) = W^{-1}[x_1 - U(b,0)\phi(0) - \int_0^b U(b,s)f(s,x_s)ds - \textstyle\sum_{0<t_k<b} U(b,t_k)I_k(x(t_k))](t).$$

We shall now show that when using this control the operator defined by

$$(Fx)(t) = \begin{cases} \varphi(t), & t \in (-\infty,0], \\ U(t,0)\varphi(0) + \int_0^t U(t,s)[f(s,x_s) + (Bu_x)(s)]ds \\ \quad + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k)), & t \in J, \end{cases}$$

has a fixed point. This fixed point is then a solution of $(1.1)-(1.3)$. Clearly $x(b) = (Fx)(b) = x_1$, which implies the system $(1.1)-(1.3)$ is controllable.We rewrite the problem $(1.1)-(1.3)$ as follows:

For $\phi \in \mathcal{B}_h'$, we define $\hat{\phi}$ by

$$\hat{\phi}(t) = \begin{cases} U(t,0)\phi(0), & t \in J, \\ \phi(t), & t \in (-\infty,0]. \end{cases}$$

Then $\hat{\phi} \in \mathcal{B}_h'$. Let $x(t) = y(t) + \hat{\phi}(t)$, $t \in (-\infty,b]$. It is easy to see that y satisfies $y_0 = 0$ and

$$y(t) = \int_0^t U(t,s)[f(s,y_s + \hat{\phi}_s) + Bu_y(s)]ds + \textstyle\sum_{0<t_k<t} U(t,t_k)I_k(y(t_k) + \hat{\phi}(t_k)),$$

where

$$u_y(s) = W^{-1}[x_1 - U(b,0)\phi(0) - \int_0^b U(b,s)f(s,y_s + \hat{\phi}_s)ds - \textstyle\sum_{k=1}^m U(b,t_k)I_k(y(t_k) + \hat{\phi}(t_k))](s),$$

if and only if x satisfies

$$x(t) = U(t,0)\phi(0) + \int_0^t U(t,s)[f(s,x_s) + Bu_x(s)]ds + \textstyle\sum_{0<t_k<t} U(t,t_k)I_k(x(t_k)),$$

and $x(t) = \phi(t)$, $t \in (-\infty,0]$. Define $\mathcal{B}_h'' = \{y \in \mathcal{B}_h': y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}_h''$,

$$\| y \|_{\mathcal{B}_h'} = \| y_0 \|_{\mathcal{B}_h} + \sup\{\| y(s) \|: s \in [0,b]\} = \sup\{\| y(s) \|: s \in [0,b]\},$$

and thus $(\mathcal{B}_h'', \|\cdot\|_{\mathcal{B}_h})$ is a Banach space. Set $B_q = \{y \in \mathcal{B}_h'': \| y \|_{\mathcal{B}_h}' \le q\}$ for some $q > 0$. Clearly B_q is a nonempty, closed, convex and bounded set in $\mathcal{B}_h''$. Then for any $y \in B_q$, from Lemma 2.1, we have

$$\| y_t + \hat{\phi}_t \|_{\mathcal{B}_h} \le \| y_t \|_{\mathcal{B}_h} + \| \hat{\phi}_t \|_{\mathcal{B}_h}$$

$$\le \| y_0 \|_{\mathcal{B}_h} + l \sup_{s\in[0,t]} \| y(s) \| + \| \hat{\phi}_0 \|_{\mathcal{B}_h} + l \sup_{s\in[0,t]} \| \hat{\phi}(s) \|$$

$$\le l(q + M_1 \| \phi(0) \|) + \| \phi \|_{\mathcal{B}_h} = q'. \tag{3.2}$$

In the view of Lemma 2.1, for each $t \in J$, $\| y(t) + \hat{\phi}(t) \| \le l^{-1} \| y_t + \hat{\phi}_t \|_{\mathcal{B}_h}$. For each $t \in J$, $y \in B_q$, we have by (3.2) and $(H4)(i)$,

$$\sup_{t\in J} \| y(t) + \hat{\phi}(t) \| \le l^{-1} \| y_t + \hat{\phi}_t \|_{\mathcal{B}_h} \le l^{-1}q',$$

$$\begin{aligned} \| I_k(y(t_k) + \hat{\phi}(t_k)) \| &\le L_k(\| y(t_k) + \hat{\phi}(t_k) \|) \\ &\le L_k(\sup_{t\in J} \| y(t) + \hat{\phi}(t) \|) \\ &\le L_k(l^{-1}q'), \quad k = 1,2,\ldots,m. \end{aligned} \tag{3.3}$$

Let $G: \mathcal{B}_h'' \to \mathcal{B}_h''$ be an operator defined by

$$(Gy)(t) = \begin{cases} 0, & t \in (-\infty,0], \\ \int_0^t U(t,s)[f(s,y_s + \hat{\phi}_s) + Bu_y(s)]ds \\ \quad + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k) + \hat{\phi}(t_k)), & t \in J. \end{cases} \tag{3.4}$$

Obviously the operator F has a fixed point is equivalent to G has one. So it turns out to prove G has a fixed point.

Let $G = G_1 + G_2$,

where

$$(G_1y)(t) = \textstyle\sum_{0<t_k<t} U(t,t_k)I_k(y(t_k) + \hat{\phi}(t_k)), \tag{3.5}$$

$$(G_2y)(t) = \int_0^t U(t,s)[f(s,y_s + \hat{\phi}_s) + Bu_y(s)]ds. \tag{3.6}$$

Step 1: There exists a positive number $q \ge 1$ such that $G(B_q) \subseteq B_q$.

Suppose the contrary. Then for each positive integer q, there exists a function $y^q(\cdot) \in B_q$ but $G(y^q) \notin B_q$. i.e., $\| G(y^q)(t) \| > q$ for some $t \in J$.

We have from $(H1)-(H4)$,

$$\begin{aligned} q &< \| (Gy^q)(t) \| \\ &\le M_1 \int_0^b \| f(s,y_s^q + \hat{\phi}_s) + Bu_{y^q}(s) \| ds + M_1 \textstyle\sum_{k=1}^m L_k(\| y^q(t_k) + \hat{\phi}(t_k) \|) \\ &\le M_1 \int_0^b \alpha_{q'}(s)ds + M_1 \int_0^b \| Bu_{y^q}(s) \| ds + M_1 \textstyle\sum_{k=1}^m L_k(l^{-1}q') \\ &\le M_1 \int_0^b \alpha_{q'}(s)ds + M_1M_2b^{\frac{1}{2}} \| u_{y^q} \|_{L^2} + M_1 \textstyle\sum_{k=1}^m L_k(l^{-1}q'), \end{aligned} \tag{3.7}$$

where

$$\| u_{y^q} \|_{L^2} \le M_3[\| x_1 \| + M_1 \| \phi \|_{\mathcal{B}_h} + M_1 \int_0^b \alpha_{q'}(s)ds + M_1 \textstyle\sum_{k=1}^m L_k(l^{-1}q')]. \tag{3.8}$$

Hence by (3.7),

$$\begin{aligned} q &< M_1 \int_0^b \alpha_{q'}(s)ds + M_1M_2b^{\frac{1}{2}}M_3[\| x_1 \| + M_1 \| \phi \|_{\mathcal{B}_h} + M_1 \int_0^b \alpha_{q'}(s)ds + M_1 \textstyle\sum_{k=1}^m L_k(l^{-1}q')] \\ &\quad + M_1 \textstyle\sum_{k=1}^m L_k(l^{-1}q') \\ &\le (1 + M_1M_2M_3b^{\frac{1}{2}})M_1[\int_0^b \alpha_{q'}(s)ds + \textstyle\sum_{k=1}^m L_k(l^{-1}q')] + M, \end{aligned}$$

where $M = M_1M_2M_3b^{\frac{1}{2}}(\| x_1 \| + M_1 \| \phi \|_{\mathcal{B}_h})$ is independent of q. Dividing both sides by q and noting that $q' = l(q + M_1 \| \phi(0) \|) + \| \phi \|_{\mathcal{B}_h} \to \infty$ as $q \to \infty$, we obtain

$$\lim_{q\to+\infty}\inf\left(\frac{\int_0^b \alpha_{q'}(s)ds}{q}\right) = \lim_{q\to+\infty}\inf\left(\frac{\int_0^b \alpha_{q'}(s)ds}{q'}\cdot\frac{q'}{q}\right) = \sigma l,$$

$$\lim_{q\to+\infty}\inf\left(\frac{\sum_{k=1}^m L_k(l^{-1}q')}{q}\right) = \lim_{q\to+\infty}\inf\left(\frac{\sum_{k=1}^m L_k(l^{-1}q')}{l^{-1}q'}\cdot\frac{l^{-1}q'}{q}\right) = \sum_{k=1}^m \lambda_k.$$

Thus we have

$$1 \le M_1(1+M_1M_2M_3b^{\frac{1}{2}})(\sigma l + \textstyle\sum_{k=1}^m \lambda_k).$$

This contradicts (3.1). Hence for some positive number q, $G(B_q)\subseteq B_q$.

Step 2: $G:\mathcal{B}_h'' \to \mathcal{B}_h''$ is continuous.

Let $\{y^{(n)}(t)\}_{n=1}^{\infty} \subseteq \mathcal{B}_h''$ with $y^{(n)} \to y$ in $\mathcal{B}_h''$. In the view of 3.2, we have

$$\| y_t^{(n)} + \hat{\phi}_t \|_{\mathcal{B}_h} \le q', \quad t\in J.$$

Then there is a number $q>0$ such that $\| y^{(n)}(t)\| \le q$ for all n and a.e. $t\in J$, so $y^{(n)}\in B_q$ and $y\in B_q$. By $(H_2)(i)$, $f(t,y_t^{(n)}+\hat{\phi}_t)\to f(t,y_t+\hat{\phi}_t)$ for each $t\in J$. By $(H_2)(ii)$, $\| f(t,y_t^{(n)}+\hat{\phi}_t)-f(t,y_t+\hat{\phi}_t)\| < 2\alpha_{q'}(t)$ and by (H_4), $I_k(y^{(n)}(t_k)+\hat{\phi}(t_k)) \to I_k(y(t_k)+\hat{\phi}(t_k)),\ k=1,2,\dots,m.$

Then we have

$$\| G_1 y^{(n)} - G_1 y\|_{\mathcal{B}_h}' \le \sum_{k=1}^m M_1 \| I_k(y^{(n)}(t_k)+\hat{\phi}(t_k)) - I_k(y(t_k)+\hat{\phi}(t_k))\|. \quad (3.9)$$

and

$$\| G_2 y^{(n)} - G_2 y\|_{\mathcal{B}_h}'$$
$$\le M_1\int_0^b \| f(s,y_s^{(n)}+\hat{\phi}_s) - f(s,y_s+\hat{\phi}_s)\| ds$$
$$+M_1M_2\int_0^b \| u_{y^{(n)}}(s) - u_y(s)\| ds$$
$$\le M_1\int_0^b \| f(s,y_s^{(n)}+\hat{\phi}_s) - f(s,y_s+\hat{\phi}_s)\| ds + M_1M_2 b^{\frac{1}{2}} \| u_y^{(n)} - u_y\|_{L^2}, \quad (3.10)$$

where

$$\| u_y^{(n)} - u_y\|_{L^2} \le M_3[M_1\int_0^b \| f(s,y_s^{(n)}+\hat{\phi}_s) - f(s,y_s+\hat{\phi}_s)\| ds$$
$$+M_1\sum_{k=1}^m \| I_k(y^{(n)}(t_k)+\hat{\phi}(t_k)) - I_k(y(t_k)+\hat{\phi}(t_k))\|].$$

Observing $(3.9)-(3.11)$ and by dominated convergence theorem we have that,

$$\| Gy^{(n)} - Gy\|_{\mathcal{B}_h}' \le \| G_1y^{(n)} - G_1 y\|_{\mathcal{B}_h}' + \| G_2y^{(n)} - G_2y\|_{\mathcal{B}_h}' \to 0, \quad \text{as } n\to+\infty. \quad (3.11)$$

That is G is continuous.

Step 3: G is equicontinuous on every J_k, $k=1,2,\dots,m$. That is $G(B_q)$ is piecewise equicontinuous on J.

Indeed for $t_1,t_2\in J_k$, $t_1<t_2$ and $y\in B_q$, we deduce that

$$\| (Gy)(t_2) - (Gy)(t_1)\|$$
$$\le \int_0^{t_1} \| U(t_2,s) - U(t_1,s)\| \| f(s,y_s+\hat{\phi}_s) + Bu_y(s)\| ds$$
$$+\int_{t_1}^{t_2} \| U(t_2,s)\| f(s,y_s+\hat{\phi}_s) + Bu_y(s)\| ds$$
$$\le \int_0^{t_1} \| U(t_2,s)-U(t_1,s)\| \alpha_{q'}(s)ds + \int_0^{t_1}\| U(t_2,s)-U(t_1,s)\| M_2M_3[\| x_1\| + M_1\|\phi(0)\|$$
$$+M_1\int_0^b \alpha_{q'}\,ds + M_1\sum_{k=1}^m L_k(l^{-1}q')]ds + \int_{t_1}^{t_2} \| U(t_2,s)\| \alpha_{q'}(s)ds$$
$$+\int_{t_1}^{t_2} \| U(t_2,s)\| M_2M_3[\| x_1\| + M_1\|\phi(0)\| + M_1\int_0^b \alpha_{q'}\,ds + M_1\sum_{k=1}^m L_k(l^{-1}q')]ds. \quad (3.12)$$

By the equicontinuity of $U(\cdot,s)$ and the absolute continuity of the Lebesgue integral, we can see that the right hand side of (3.12) tends to zero and independent of y as $t_2\to t_1$. Hence $G(B_q)$ is equicontinuous on $J_k(k=1,2,\dots,m)$.

Step 4: The Monch's condition holds.

Suppose $W\subseteq B_q$ is countable and $W\subseteq \overline{co}(\{0\}\cup G(W))$. We shall show that $\beta(W)=0$, where β is the Hausdorff MNC.

Without loss of generality, we may assume that $W=\{y^{(n)}\}_{n=1}^{\infty}$. Since G maps B_q into an equicontinuous family, $G(W)$ is equicontinuous on J_k. Hence $W\subseteq \overline{co}(\{0\}\cup G(W))$ is also equicontinuous on every J_k.

By $(H_4)(ii)$, we have

$$\beta(\{G_1 y^{(n)}(t)\}_{n=1}^{\infty})$$
$$=\beta(\{\textstyle\sum_{0<t_k<t} U(t,t_k)I_k(y^{(n)}(t_k)+\hat{\phi}(t_k))\}_{n=1}^{\infty})$$
$$\le M_1\sum_{k=1}^m \beta(\{I_k(y^{(n)}(t_k)+\hat{\phi}(t_k))\}_{n=1}^{\infty})$$
$$\le M_1\sum_{k=1}^m C_k\beta(\{y^{(n)}(t_k)\}_{n=1}^{\infty}). \quad (3.13)$$

By Lemma 2.4 and from $(H2)(iii)$, $(H3)(ii)$ and $(H4)(ii)$, we have that

$$\beta_V(\{u_{y^{(n)}}(s)\}_{n=1}^{\infty})$$
$$\le K_W(s)[\beta(\{\int_0^b U(b,s)f(s,y_s^{(n)}+\hat{\phi}_s)ds\}_{n=1}^{\infty})ds$$
$$+\beta(\{\textstyle\sum_{k=1}^m U(b,t_k)I_k(y^{(n)}(t_k)+\hat{\phi}(t_k))\}_{n=1}^{\infty})]$$
$$\le K_W(s)[2M_1\int_0^b \eta(s)\sup_{-\infty<\theta\le 0}\beta(\{y^{(n)}(s+\theta)+\hat{\phi}(s+\theta)\}_{n=1}^{\infty})ds$$

(3.9)

$$+M_1\sum_{k=1}^m C_k\beta(\{y^{(n)}(t_k)\}_{n=1}^{\infty})]$$
$$\le K_W(s)[2M_1\int_0^b \eta(s)\sup_{0\le\tau\le s}\beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty})ds + M_1\sum_{k=1}^m C_k\beta(\{y^{(n)}(t_k)\}_{n=1}^{\infty})]. \quad (3.14)$$

This implies that

$$\beta(\{G_2 y^{(n)}(t)\}_{n=1}^{\infty}) \quad (3.10)$$
$$\le \beta(\{\int_0^t U(t,s)f(s,y_s^{(n)}+\hat{\phi}_s)ds\}_{n=1}^{\infty}) + \beta(\{\int_0^t U(t,s)Bu_{y^{(n)}}(s)ds\}_{n=1}^{\infty})$$
$$\le 2M_1\int_0^b \eta(s)\sup_{-\infty<\theta\le 0}\beta(\{y^{(n)}(s+\theta)+\hat{\phi}(s+\theta)\}_{n=1}^{\infty})ds + 2M_1M_2\int_0^b \beta_V(\{u_{y^{(n)}}(s)\}_{n=1}^{\infty})ds$$
$$\le 2M_1\int_0^b \eta(s)\sup_{0\le\tau\le s}\beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty})ds + 4M_1^2M_2(\int_0^b K_W(s)ds) \quad (3.11)$$
$$\times(\int_0^b \eta(s)\sup_{0\le\tau\le s}\beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty})ds)$$
$$+2M_1^2M_2\int_0^b K_W(s)ds(\textstyle\sum_{k=1}^m C_k\beta(\{y^{(n)}(t_k)\}_{n=1}^{\infty})), \text{ for each } t\in J. \quad (3.15)$$

From (3.13) an d (3.15) we obtain that

$$\beta(\{Gy^{(n)}(t)\}_{n=1}^{\infty})$$
$$\le \beta(\{G_1y^{(n)}(t)\}_{n=1}^{\infty}) + \beta(\{G_2y^{(n)}(t)\}_{n=1}^{\infty})$$
$$\le M_1\sum_{k=1}^m C_k\beta(\{y^{(n)}(t_k)\}_{n=1}^{\infty}) + (2M_1 + 4M_1^2M_2\int_0^b K_W(s)ds)$$
$$\times\int_0^b \eta(s)\sup_{0\le\tau\le s}\beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty})ds$$
$$+2M_1^2M_2\int_0^b K_W(s)ds(\textstyle\sum_{k=1}^m C_k\beta(\{y^{(n)}(t_k)\}_{n=1}^{\infty})), \quad (3.16)$$

for each $t\in J$.

Since W and $G(W)$ are equicontinuous on every J_k, according to Lemma 2.3, the inequality (3.16) implies that, (3.12)

$$\beta(\{Gy^{(n)}\}_{n=1}^{\infty})$$
$$\le [M_1\sum_{k=1}^m C_k + (2M_1+4M_1^2M_2\|K_W\|_{L^1})\|\eta\|_{L^1}]\beta(\{y^{(n)}\}_{n=1}^{\infty})$$
$$+[2M_1^2M_2\|K_W\|_{L^1}\sum_{k=1}^m C_k]\beta(\{y^{(n)}\}_{n=1}^{\infty})$$
$$=[(M_1+2M_1^2M_2\|K_W\|_{L^1})\sum_{k=1}^m C_k + (2M_1+4M_1^2M_2\|K_W\|_{L^1})\|\eta\|_{L^1}]\beta(\{y^{(n)}\}_{n=1}^{\infty})$$
$$=N\beta(\{y^{(n)}\}_{n=1}^{\infty}).$$

That is $\beta(GW) \le N\beta(W)$, where N is defined in $(H5)$. Thus from the Monch's codition, we get that

$$\beta(W) \le \beta(\overline{co}(\{0\} \cup G(W)) = \beta(G(W)) \le N\beta(W),$$

since $N < 1$, which implies that $\beta(W) = 0$. So we have that W is relatively compact in $\mathcal{B}_h''$. In the view of Lemma 2.6, i.e., Monch's fixed point theorem, we conclude that G has a fixed point y in W. Then $x = y + \hat{\phi}$ is a fixed point of F and thus the system $(1.1) - (1.3)$ is controllable on $[0, b]$.

Remark 3.1. *Note that if f is compact or Lipschitz continuous, then $(H2)(iii)$ is automatically satisfied. In the following, by using another MNC, we will prove the result of the Theorem 3.1 in the case there is no equicontinuity of the evolution system $U(t,s)$ and hypothesis $(H5)$. Here we assume that the impulsive operators I_k are compact. So, instead of $(H4)$, we give the hypothesis $(H4)'$:*

$(H4)'$ $I_k: X \to X,\ k = 1,2 \dots, m$, be a continuous compact operator such that, there are nondecreasing functions $L_k: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\| I_k(x) \| \le L_k(\| x \|) \quad k = 1,2 \dots, m, \quad x \in X,$$

and

$$\lim_{\rho \to \infty} \inf \frac{L_k(\rho)}{\rho} = \lambda_k < \infty, \quad k = 1,2 \dots, m.$$

Theorem 3.2. *Let $\{A(t)\}_{t \in [0,b]}$ be a family of linear operators that generates a strongly continuous evolution system $\{U(t,s): (t,s) \in J \times J\}$. Assume that the hypothesis $(H2), (H3)$ and $(H4')$ are satisfied. Then the impulsive differential system $(1.1) - (1.3)$ is controllable on J.*

Proof. In the view of Theorem 3.1, we should only prove that the function $G: \mathcal{B}_h'' \to \mathcal{B}_h''$ given by the formula (3.4) satisfies the Monch's condition.

For this purpose, let $W \subseteq B_q$ be countable and $W \subseteq \overline{co}(\{0\} \subset G(W))$. We shall prove that W is relatively compact.

We will denote by Φ the following MNC in $\mathcal{B}_h''$ defined by (see[10]),

$$\Phi(\Omega) = \max_{E \in \Delta(\Omega)} (\alpha(E), mod_c(E)) \quad (3.17)$$

for all bounded subsets of Ω of $\mathcal{B}_h''$, where $\Delta(\Omega)$ is the set of countable subsets of Ω, α is the real MNC defined by,

$$\alpha(E) = \sup_{t \in [0,b]} e^{-Lt} \beta(E(t)),$$

with $E(t) = \{x(t): x \in E\}$, L is a constant that we shall choose appropriately.

$\mathrm{mod}_c(E)$ is the modulus of equicontinuity of the function set E given by the formula

$$\mathrm{mod}_c(E) = \lim_{\delta \to 0} \sup_{x \in E} \max_{0 \le k \le m} \max_{t_1, t_2 \in J_k, \|t_1 - t_2\| < \delta} \| x(t_1) - x(t_2) \|.$$

It was proved in [10] that Φ is well defined. (i.e., there is $E_0 \in \Delta(\Omega)$ which achieves the maximum in (3.17)) and is a monotone, nonsingular and regular MNC.

Let us choose a constant $L > 0$, such that

$$p = (2M_1 + 4M_1^2 M_2 \| K_W \|_{L^1}) \sup_{t \in [0,b]} \int_0^t \eta(s) e^{-L(t-s)} ds < 1, \quad (3.18)$$

where $M_1 = \sup\{\| U(t,s) \|: (t,s) \in J \times J\}$ and η is the integrable function in the hypothesis $(H2)$. Let $G_y = G_1 y + G_2 y$ as defined in theorem (3.1). From the regularity of Φ, it is enough to prove that $\Phi(W) = (0,0)$. Since $\Phi(G(W))$ is a maximum, let $\{z^{(n)}\}_{n=1}^{\infty} \subseteq G(W)$ be the denumerable set which achieves its maximum. Then there exists a set $\{y^{(n)}\}_{n=1}^{\infty} \subseteq W$ such that

$$z^{(n)}(t) = (Gy^{(n)})(t) = (G_1 y^{(n)})(t) + (G_2 y^{(n)})(t), \text{for all } n \ge 1,\ t \in [0,b]. \quad 3.19$$

Now we give an estimation for $\alpha(\{z^{(n)}\}_{n=1}^{\infty})$. Since $I_k(\cdot)$ is compact, we get

$$\beta(\{(G_1 y^{(n)})(t)\}_{n=1}^{\infty}) = 0, \text{ for } t \in [0,b]. \quad 3.20$$

From $(3.14), (3.15)$, noticing that $C_k = 0$, as I_k is compact, we have that

$$\begin{aligned}
&\beta(\{(G_2 y^{(n)})(t)\}_{n=1}^{\infty}) \\
&\le 2M_1 \int_0^t \eta(s) \sup_{0 \le \tau \le s} \beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty}) ds \\
&+ 4M_1^2 M_2 \| K_W \|_{L^1} \int_0^t \eta(s) \sup_{0 \le \tau \le s} \beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty}) ds \\
&\le (2M_1 + 4M_1^2 M_2 \| K_W \|_{L^1}) \int_0^t \eta(s) \sup_{0 \le \tau \le s} \beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty}) ds \\
&\le (2M_1 + 4M_1^2 M_2 \| K_W \|_{L^1}) \int_0^t \eta(s) e^{Ls} \sup_{t \in [0,b]} (e^{-Lt} \beta(\{y^{(n)}(t)\}_{n=1}^{\infty})) ds \\
&= (2M_1 + 4M_1^2 M_2 \| K_W \|_{L^1}) \alpha(\{y^{(n)}\}_{n=1}^{\infty}) \int_0^t \eta(s) e^{Ls} ds, \text{for } t \in [0,b]. \quad 3.21
\end{aligned}$$

From (3.20) and (3.21), it follows that

$$\begin{aligned}
\alpha(\{z^{(n)}\}_{n=1}^{\infty}) &= \sup_{t \in [0,b]} e^{-Lt} \beta(\{(G_1 y^{(n)})(t) + (G_2 y^{(n)})(t)\}_{n=1}^{\infty}) \\
&\le \sup_{t \in [0,b]} e^{-Lt} (2M_1 + 4M_1^2 M_2 \| K_W \|_{L^1}) \alpha(\{y^{(n)}\}_{n=1}^{\infty}) \int_0^t \eta(s) e^{Ls} ds \\
&= \alpha(\{y^{(n)}\}_{n=1}^{\infty})(2M_1 + 4M_1^2 M_2 \| K_W \|_{L^1}) \sup_{t \in [0,b]} \int_0^t \eta(s) e^{-L(t-s)} ds \\
&= \alpha(\{y^{(n)}\}_{n=1}^{\infty}) p.
\end{aligned}$$

Therefore, we have that

$$\alpha(\{y^{(n)}\}_{n=1}^{\infty}) \le \alpha(W) \le \alpha(\overline{co}(\{0\} \cup G(W))) = \alpha(\{z^{(n)}\}_{n=1}^{\infty}) \le \alpha(\{y^{(n)}\}_{n=1}^{\infty}) p.$$

From (3.18), we obtain that

$$\alpha(\{y^{(n)}\}_{n=1}^{\infty}) = \alpha(W) = \alpha(\{z^{(n)}\}_{n=1}^{\infty}) = 0.$$

From the definition of α, we have

$$\beta(\{y^{(n)}(t)\}_{n=1}^{\infty}) = \beta(\{z^{(n)}(t)\}_{n=1}^{\infty}) = 0, \text{for every } t \in [0,b]. \quad 3.22$$

(3.17)

From (3.14) and (3.22), noticing that $C_k = 0$ in (3.14), we get that

$$\begin{aligned}
&\beta(\{f(t, y_t^{(n)} + \hat{\phi}_t) + (Bu_{y^{(n)}})(t)\}_{n=1}^{\infty}) \\
&\le \eta(t) \sup_{-\infty < \theta \le 0} \beta(\{y^{(n)}(t+\theta) + \hat{\phi}(t+\theta)\}_{n=1}^{\infty}) \\
&+ 2M_1 M_2 K_W(s) \int_0^b \eta(s) \sup_{0 \le \tau \le s} \beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty}) ds \\
&\le \eta(t) \sup_{0 \le \tau \le t} \beta(\{y^{(n)}(\tau)\}_{n=1}^{\infty}) \\
&+ 2M_1 M_2 K_W(s) \int_0^b \eta(s) \sup_{0 \le \tau \le s} \beta(\{y_n(\tau)\}_{n=1}^{\infty}) ds = 0,
\end{aligned}$$

That is, $\{f(t, y_t^{(n)} + \hat{\phi}_t) + Bu_{y^{(n)}}(t)\}_{n=1}^{\infty}$ is relatively compact for almost all $t \in [0,b]$ in X. Moreover, from the fact that $\{y^{(n)}\}_{n=1}^{\infty} \subseteq B_q$, by $(H2)(ii)$ and (3.8), it is easy to see that $\{f(t, y_t^{(n)} + \hat{\phi}_t) + Bu_{y^{(n)}}(t)\}_{n=1}^{\infty}$ is uniformly integrable for a.e. $t \in [0,b]$. So $\{f(\cdot, y^{(n)} + \hat{\phi}) + Bu_{y^{(n)}}\}_{n=1}^{\infty}$ is semicompact according to the Definition 2.5. By applying Lemma 2.5, we have that $G_2(\{y^{(n)}\}_{n=1}^{\infty})$ is relatively compact in $\mathcal{B}_h''$.

On the other hand, by the strong continuity of $U(t,s)$ and the compactness of I_k, we can easily verify that $G_1(\{y^{(n)}\}_{n=1}^{\infty})$ is relatively compact. Then by (3.19), $\{z^{(n)}\}_{n=1}^{\infty}$ is also relatively compct in $\mathcal{B}_h''$. Since Φ is a monotone, nonsingular, regular MNC, from Monch's condition, we have that

$$\Phi(W) \leq \Phi(\overline{co}(\{0\} \cup G(W))) = \Phi(\{z_n\}_{n=1}^{\infty}) = (0,0).$$

Therefore, W is relatively compact in $\mathcal{B}_h''$. This completes the the proof.

References

[1] A. Anguraj and M. Mallika Arjunan, Existence and uniqueness of mild and classical solutions of impulsive evolution equations, *Electronic Journal of Differential Equations*, 111 (2005), 1-8.

[2] D.D. Bainov and P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical Group, England, 1993.

[3] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, in: Lecture Notes in Pure and Applied Matyenath, vol. 60, Marcel Dekker, New York, 1980.

[4] Y. K. Chang, A. Anguraj and M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, *Nonlinear Analysis: Hybrid Systems*, 2(2008), 209-218.

[5] Y. K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, *Chaos, Solitons and Fractals*, 33(2007), 1601-1609.

[6] L. Chen and G. Li, Approximate controllability of impulsive differential equations with nonlocal conditions, *International Journal of Nonlinear Science*, 10(2010), 438-446.

[7] C. Cuevas, E. Hernandez and M. Rabelo, The existence of solutions for impulsive neutral functional differrential equations, *Computers and Mathematics with Applications*, 58(2009), 744-757.

[8] E. Hernandez M., M. Rabello and H. Henriaquez, Existence of solutions for impulsive partial neutral functional differential equations, *Journal of Mathematical Analysis and Applications*, 331(2007), 1135-1158.

[9] S. Ji, G. Li and M. Wang, Controllability of impulsive differential systems with nonlocal conditions, *Applied Mathematics amd Computation*, 217(2011), 6981-6989.

[10] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter, 2001.

[11] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer, Dordrecht, 1992.

[12] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer, Dordrecht, 1999.

[13] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[14] M. Li, M. Wang and F. Zhang, Controllability of impulsive functional differential systems in Banach spaces, *Chaos, Solitons and Fractals*, 29(2006), 175-181.

[15] B. Liu, Controllability of impulsive neutral functional differential inclusions with infinite delay, *Nonlinear Analysis*, 60(2005), 1533-1552.

[16] H. Monch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Analysis*, 4(1980), 985-999.

[17] V. Obukhovski and P. Zecca, Controllability for systems governed by semilinear differential inclusions in a Banach space with a noncompact semigroup, *Nonlinear Analysis*, 70(2009), 3424-3436.

[18] J. Y. Park, K. Balachandran and N. Annapoorani, Existence results for impulsive neutral functional integrodifferential equations with infinite delay, *Nonlinear Analysis*, 71(2009), 3152-3162.

[19] J. Y. Park, K. Balachandran and G. Arthi, Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces, *Nonlinear Analysis: Hybrid Systems*, 3(2009), 184-194.

[20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[21] B. Radhakrishnan and K. Balachandran, Controllability of impulsive neutral functional evolution integrodifferential systems with infinite delay, *Nonlinear Analysis: Hybrid Systems*, 5(2011), 655-670.

[22] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[23] J.Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, New York, 1996.

[24] B. Yan, Boundary value problems on the half-line with impulses and infinite delay, *Journal of Mathematical Analysis and Applications*, 259(2001), 94-114.

[25] R. Ye, Existence of solutions for impulsive partial neutral functional differential equation with infinite delay, *Nonlinear Analysis*, 73(2010), 155-162.