



Exponential periodic attractor and exponential convergence of a class of functional differential equation with time-varying delays

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ABSTRACT

By using Mawhin continuation theorem and comparison theorem, the sufficient conditions ensuring the existence of exponential periodic attractor and exponential convergence of a class of functional differential equation with time varying delays are established. The results are very different from some previously known results [1,19,28]. Finally, applications and an example are given to illustrate the effectiveness of the results.

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Introduction

Recently, the delayed neural networks and bidirectional associative memory (BAM) neural networks have been paid more and more attention due to their wide applications in signal processing, image processing, pattern recognition and artificial intelligence, etc. Many results have been reported in literature [1-27]. The existence and stability (exponential stability, asymptotic stability, exponential convergence etc.) of equilibrium point and periodic solution are referred to [1, 21-24, 28] and [4, 10-16, 18, 19, 25-27], respectively. For example, M.J. Tan and Y. Tan [1] proposed the following general neural networks with viable coefficients and time-varying delays:

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t). \quad (1.1)$$

By using Banach fixed point theorem and spectral theory, the authors established the sufficient conditions for the existence and globally exponential stability of a unique equilibrium point of system (1.1). Liu et al. [19] further studied BAM neural networks with periodic coefficients and obtained several sufficient conditions guaranteeing the existence and globally exponential stability of periodic solution.

It is well known that the properties of stability and convergence are important in design and application in neural networks. In this paper, we consider a class of functional differential equation with periodic coefficients and time-varying delays as follows.

$$\begin{aligned} x_i'(t) = & -c_i(t)x_i(t) + f_i(x_1(t), x_2(t), \dots, x_n(t), x_1(t-\tau_{i1}(t)), \\ & \dots, x_n(t-\tau_{in}(t))) + I_i(t) \end{aligned} \quad (1.2)$$

with initial conditions $x_i(s) = \phi_i(s)$, $s \in [-\tau, 0]$, where $x_i(t)$ is the activation of the i th neuron at time t , continuous function $c_i(t)$ denotes the neuron charging time, $I_i(t)$ is the external input on the neuron, $\tau_{ij}(t)$ corresponds to the finite

speed of the axonal signal transmission, $\phi_i(s)$ is continuous ω -periodic functions. For, $i, j = 1, 2, \dots, n$,
 $\tau = \max_{1 \leq i, j \leq n} \tau_{ij}^+, \tau_{ij}^+ = \max_{t \in [0, \omega]} \tau_{ij}(t)$.

Actually, both delayed cellular neural networks and BAM neural networks can be contained in functional differential equation (1.2), which can be seen in [28]. Therefore, our model is more general.

The aim of this paper is, by using continuation theorem, some analysis techniques and constructing suitable Lyapunov functional, to derive the existence of exponential periodic attractor and exponential convergence of system (1.2). The results are based on Mawhin's continuation theorem, matrix theory, some new estimation techniques for the priori bounds of the solutions of $Lx = \lambda Nx$, $\lambda \in (0, 1)$ and comparison theorem. To the best of the author's knowledge, few results based on the method have been reported [29]. It is much of interesting.

The organization of the rest is as follows. In Section 2, some preliminaries are introduced. In Section 3, sufficient conditions ensuring the existence of periodic solution are established. In Section 4, the exponential periodic attractor and exponential convergence are investigated. In Section 5, applications and an example are given to show the usefulness of the main results. Finally, a simple conclusion is drawn in Section 6.

Preliminaries

In this section, some preliminaries are presented. First we denote the solution of (1.2) through

$$\phi = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T$$

as $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))^T$, and define

$$\|\phi\| = \max_{1 \leq i \leq n} \sup_{s \in [-\tau, 0]} |\phi_i(s)| \quad \text{for } \phi = (\phi_1, \phi_2, \dots, \phi_n)^T.$$

Definition 2.1 [30] A real matrix K is said to be a nonsingular M- matrix if K has the form $K = \alpha I - P, \alpha > 0, P \geq 0$, where $\alpha > \rho(P)$ (the spectral radius of the matrix P), I denotes the identity matrix.

Lemma 2.1 [31] Let Q be an $n \times n$ matrix with non- positive off- diagonal elements. Then Q is an M- matrix if and only if one of the following conditions holds:

- (i) There exists a vector $\xi > 0$ such that $Q\xi > 0$;
- (ii) There exists a vector $\xi > 0$ such that $\xi^T Q > 0$.

Now let us introduce the continuation theorem due to Gaines and Mawhin [32].

Let X and Y be two real Banach spaces, $L : DomL \cap X \rightarrow Y$ be a Fredholm operator of index zero, and $P : X \rightarrow X, Q : Y \rightarrow Y$ be continuous projectors such that $ImP = KerL, KerQ = ImL$ and $X = KerL \oplus KerP, Y = ImL \oplus ImQ$. Denote by $K_p : ImL \rightarrow KerP \cap DomL$ the inverse of L_p (the restriction of L on $KerP \cap DomL$), by $J : ImQ \rightarrow KerL$ the algebraic and topological isomorphism of ImQ onto $kerL$, due to the same dimensions of these two subspaces.

Lemma 2.2 [32] (continuation theorem) Let X and Y be two Banach spaces and $L : DomL \cap X \rightarrow Y$ be a Fredholm mapping of index zero. $\Omega \subset X$ be an open bounded set and $N : X \rightarrow Y$ be a continuous operator which is L- compact on $\bar{\Omega}$. Suppose that

- (a) for each $\lambda \in (0, 1), x \in \partial\Omega \cap DomL, Lx \neq \lambda Nx$;
- (b) for each $x \in \partial\Omega \cap KerL, QNx \neq 0$;
- (c) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap DomL$. Throughout the rest of this paper, we always assume that:

- (A₁) $c_i(t) > 0, I_i(t)$ are all continuous periodic functions defined on $R^+ = [0, \infty)$ with period ω for $i = 1, 2, \dots, n$.
- (A₂) Function $f_i : R \rightarrow R$ is continuous and there exist positive constants M_{ij}, N_{ij} such that

$$\begin{aligned} & |f_i(x_1, \dots, x_n, y_1, \dots, y_n) - f_i(u_1, \dots, u_n, v_1, \dots, v_n)| \\ & \leq \sum_{j=1}^n M_{ij}|x_j - u_j| + \sum_{j=1}^n N_{ij}|y_j - v_j| \end{aligned}$$

with $f_i(0) = 0$ for any

$$x = (x_1, \dots, x_n)^T, y = (y_1, \dots, y_n)^T, u = (u_1, \dots, u_n)^T, v = (v_1, \dots, v_n)^T \in R^n \text{ and } i = 1, 2, \dots, n.$$

For convenience, we shall use the following notations.

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t)dt, \quad f^+ = \sup_{t \in [0, \omega]} |f(t)|, \quad f^- = \inf_{t \in [0, \omega]} |f(t)|,$$

where $f(t)$ is an ω -periodic function.

Existence of periodic solution

In this section, by employing continuation theorem, we shall establish the sufficient conditions of the existence of periodic solution of system (1.2).

Theorem 3.1 Assume that (A₁), (A₂) hold. Further,

- (A₃) $\Gamma = C - D$ is a nonsingular M- matrix,
- (A₄) $\kappa > 0$, where

$$C = \text{diag}(c_1^-, c_2^-, \dots, c_n^-), D = (M_{ij} + N_{ij})_{n \times n}, \kappa = \min_{1 \leq i \leq n} \left\{ c_i^- - \sum_{j=1}^n (M_{ji} + N_{ji}) \right\}.$$

Then system (1.2) admits at least one ω -periodic solution.

Proof. Let $X = Y = \{x \in C(R, R^n), x(t + \omega) = x(t)\}$ endowed with the norm $\|x\| = \sum_{i=1}^n \max_{t \in [0, \omega]} |x_i(t)|$, then X

and Y are all Banach spaces. For

$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in X$, we define

$$L : DomL \cap X \rightarrow X, \quad x \rightarrow x' = (x'_1(t), x'_2(t), \dots, x'_n(t))^T,$$

and $N : X \rightarrow X, Nx = ((Nx)_1(t), \dots, (Nx)_n(t))^T$, where for $i = 1, 2, \dots, n$,

$$(Nx)_i(t) = A_i(t) = -c_i(t)x_i(t) + f_i(x_1(t), x_2(t), \dots, x_n(t), x_1(t - \tau_{i1}(t)), \dots, x_n(t - \tau_{in}(t))) + I_i(t),$$

$DomL = \{x \in C^1(R, R^n) : x(t + \omega) = x(t)\}$. It is not difficult to show that $KerL = R^n$.

$ImL = \{x(t) \in X : \int_0^\omega x_i(t)dt = 0, i = 1, 2, \dots, n.\}$ is

closed in X and $\dim KerL = \text{codim} ImL = n$. Then, L is a Fredholm mapping of index zero.

Define operators P and Q as follows:

$$Px = Qx = \frac{1}{\omega} \int_0^\omega x(t)dt.$$

It is easy to show that P and Q are continuous and satisfy $ImP = KerL, ImL = KerQ = Im(I - Q)$. Let

$L_p = L|_{DomL \cap KerP}$, then the generalized inverse $L_p^{-1} = K_p$ is given by

$$(K_p x)_i(t) = \int_0^t x_i(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t x_i(s)dsdt.$$

Therefore,

$$QNx = \left(\frac{1}{\omega} \int_0^\omega A_i(t)dt, \dots, \frac{1}{\omega} \int_0^\omega A_n(t)dt \right)^T \text{ and}$$

$$\begin{aligned} K_p(I - Q)Nx &= \left(\int_0^t A_1(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t A_1(s)dsdt, \dots, \int_0^t A_n(s)ds \right. \\ & \left. - \frac{1}{\omega} \int_0^\omega \int_0^t A_n(s)dsdt \right)^T - \left(\frac{t}{\omega} - \frac{1}{2} \right) \left(\int_0^\omega A_1(s)ds, \dots, \int_0^\omega A_n(s)ds \right)^T. \end{aligned}$$

Thus, QN and $K_p(I - Q)N$ are all continuous. For any open bounded set $\Omega \subset Z$, it follows from the expression of QN that $QN(\bar{\Omega})$ is bounded. Noting that N is a completely continuous mapping and the expression of $K_p(I - Q)N$, using Arzela-Ascoli theorem, one obtains that $\overline{K_p(I - Q)N(\bar{\Omega})}$ is relatively compact. Hence, N is L-compact on $\bar{\Omega}$.

Now, what we need to do is just to search for an appropriate open bounded subset Ω for the application of the Mawhin's continuation theorem. Corresponding to operator equation $Lx = \lambda Nx, \lambda \in (0, 1)$, we have

$$\begin{aligned} x'_i(t) &= \lambda(-c_i(t)x_i(t) + f_i(x_1(t), \dots, x_n(t), x_1(t - \tau_{i1}(t)), \\ & \dots, x_n(t - \tau_{in}(t))) + I_i(t)) \end{aligned} \tag{3.1}$$

Suppose that $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in X$ is an arbitrary solution of (3.1) for some $\lambda \in (0, 1)$. Since any

$x_i(t)$ ($i = 1, 2, \dots, n$) is continuously differential, then there exists $t_i \in [0, \omega]$ such that $|x_i(t_i)| = \max_{t \in [0, \omega]} |x_i(t)|$ and $x'_i(t_i) = 0$ for $i = 1, 2, \dots, n$. It leads to

$$\begin{aligned} & |c_i(t_i)x_i(t_i)| = |f_i(x_1(t_i), \dots, x_n(t_i), x_1(t_i - \tau_{i1}(t_i)), \dots, x_n(t_i - \tau_{in}(t_i))) \\ & \quad + I_i(t_i)| \\ \leq & |f_i(x_1(t_i), \dots, x_n(t_i), x_1(t_i - \tau_{i1}(t_i)), \dots, x_n(t_i - \tau_{in}(t_i)))| + |I_i(t_i)|. \end{aligned} \tag{3.2}$$

From (3.2) and (A_2) , we obtain

$$\begin{aligned} c_i^- |x_i(t_i)| & \leq \sum_{j=1}^n M_{ij} |x_j(t_i)| + \sum_{j=1}^n N_{ij} |x_j(t_i - \tau_{ij}(t_i))| \\ & \quad + |I_i(t_i)| \\ & \leq \sum_{j=1}^n M_{ij} |x_j(t_j)| + \sum_{j=1}^n N_{ij} |x_j(t_j)| + |I_i(t_i)| \\ & = \sum_{j=1}^n (M_{ij} + N_{ij}) |x_j(t_j)| + |I_i(t_i)|, \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} c_1^- - (M_{11} + N_{11}) & -(M_{12} + N_{12}) & \cdots & -(M_{1n} + N_{1n}) \\ -(M_{21} + N_{21}) & c_2^- - (M_{22} + N_{22}) & \cdots & -(M_{2n} + N_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ -(M_{n1} + N_{n1}) & -(M_{n2} + N_{n2}) & \cdots & c_n^- - (M_{nn} + N_{nn}) \end{pmatrix} \times \begin{pmatrix} |x_1(t_1)| \\ |x_2(t_2)| \\ \vdots \\ |x_n(t_n)| \end{pmatrix} \leq \begin{pmatrix} |I_1(t_1)| \\ |I_2(t_2)| \\ \vdots \\ |I_n(t_n)| \end{pmatrix}.$$

$$\text{i.e., } \Gamma(|x_1(t_1)|, \dots, |x_n(t_n)|)^T \leq (|I_1(t_1)|, \dots, |I_n(t_n)|)^T. \tag{3.3}$$

According to (A_3) , Γ is a nonsingular matrix. By [33], it follows from (3.3) that

$$(|x_1(t_1)|, \dots, |x_n(t_n)|)^T \leq \Gamma^{-1}(|I_1(t_1)|, \dots, |I_n(t_n)|)^T.$$

In view of the continuity of $I_i(t)$, then there exists constant ζ_i such that $|x_i(t_i)| \leq \zeta_i$ for $i = 1, 2, \dots, n$.

Let $\Omega = \{x = (x_1, \dots, x_n)^T \in X, \|x\| < H\}$ where $H = \sum_{i=1}^n \zeta_i + M, M > 0$ is large enough such that $H\kappa > \sum_{i=1}^n |\bar{I}_i|$. Then for

$$x \in \partial\Omega \cap \text{Dom}L, \lambda \in (0, 1), Lx \neq \lambda Nx,$$

the condition (a) of Lemma 2.2 holds. When $x \in \partial\Omega \cap \text{Ker}L, x$ is a constant vector in R^n with $\|x\| = \sum_{i=1}^n |x_i| = H$, By easily computation, we have

$$\begin{aligned} \|QNx\| & = \frac{1}{\omega} \sum_{i=1}^n \left| \int_0^\omega (c_i(t)x_i - f_i(x_1, \dots, x_n) - I_i(t)) dt \right| \\ & \geq \sum_{i=1}^n c_i^- |x_i| - \sum_{i=1}^n \sum_{j=1}^n (M_{ij} + N_{ij}) |x_j| - |\bar{I}_i| \\ & = \sum_{i=1}^n \left(c_i^- - \sum_{j=1}^n (M_{ji} + N_{ji}) \right) |x_i| - \sum_{i=1}^n |\bar{I}_i| \\ & \geq \kappa \|x\| - \sum_{i=1}^n |\bar{I}_i| \\ & = \kappa H - \sum_{i=1}^n |\bar{I}_i| > 0. \end{aligned}$$

This implies that condition (b) of Lemma 2.2 holds as well.

Define $\psi : \text{Ker}L \times [0, 1] \rightarrow X$ by

$$\psi(z, \mu) = -\mu\phi z + (1 - \mu)QNz,$$

where $\phi(z) = (a_1^- x_1, \dots, a_n^- x_n)^T$.

When $x \in \partial\Omega \cap \text{Ker}L, x = (x_1, \dots, x_n)^T$ is a constant vector in R^n with $\sum_{i=1}^n |x_i| = H$, It is easy to verify that $\psi(z, \mu) \neq (0, \dots, 0)^T$. Therefore,

$$\text{deg}\{QNz, \Omega \cap \text{Ker}L, 0\} \neq 0,$$

which means that condition (c) of Lemma 2.2 holds. Employing Lemma 2.2, we conclude that system (1.2) admits at least one ω -periodic solution. This completes the proof.

Remark 3.1 In the process of discussing the priori bounds for equation $Lx = \lambda Nx, \lambda \in (0, 1)$, the matrix's theory is employed, which is new and much of interesting. It is in this sense that a new method to estimate the priori bounds is proposed. The conditions ensuring the existence of periodic solution of system (1.2) are new and interesting and much different from the known results in the literature [1, 19].

Stability of system (1.2)

Exponential periodic attractor

In this subsection, by using some analysis techniques and Lyapunov functional, we study the exponential periodic attractor of system (1.2).

Definition 4.1.1 [34] System (1.2) has an exponential periodic attractor if and only if there exists one ω -periodic solution $(x^T(t, \phi^*))$ with initial value ϕ^* and for any solution $x(t, \phi)$ with initial value ϕ , there exist positive constant α, β such that $\|(x^T(t, \phi^*))^T - (x^T(t, \phi))^T\| \leq \alpha e^{-\beta t} \|\phi^* - \phi\|$.

Theorem 4.1.1 Assume that $(A_1) - (A_4)$ hold. Further,

(A_5) $\tilde{\Gamma} = C - B$ is a nonsingular M- matrix,

(A_6) $\tau_{ij}(t)$ is continuous differential functions defined on $[0, \infty)$ such that $\tau'_{ij}(t) < 1$ for $i, j = 1, 2, \dots, n$, where

$$B = (M_{ji} + \tilde{\tau}N_{ji})_{n \times n}, \tilde{\tau} = \max_{1 \leq j, i \leq n} \left\{ \frac{1}{1 - \tau'_{ji}(t)}, t \in [0, \omega] \right\}, C$$

is defined in theorem 3.1. Then system (1.2) has an exponential periodic attractor.

Proof. Suppose that

$x^T(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))^T$ is an arbitrary solution of system (1.2) and $x^T(t, \phi^*)$ is a ω -periodic solution of (1.2). Then from system (1.2), we can obtain that:

$$\begin{aligned} \frac{d(x_i(t, \phi) - x_i(t, \phi^*))}{dt} & = -c_i(t)(x_i(t, \phi) - x_i(t, \phi^*)) + \sum_{j=1}^n (f_j(x_1(t, \phi), \\ & \dots, x_n(t, \phi), x_1(t - \tau_{i1}(t), \phi), \dots, x_n(t - \tau_{im}(t), \phi)) - f_j(x_1(t, \phi^*), \\ & \dots, x_n(t, \phi^*), x_1(t - \tau_{i1}(t), \phi^*), \dots, x_n(t - \tau_{im}(t), \phi^*))), \end{aligned} \tag{4.1}$$

for $t > 0, i = 1, 2, \dots, n$. For convenience, we denote $u_i(t) = x_i(t, \phi) - x_i(t, \phi^*)$. Then, from (4.1) and (A_2) , we have

$$D^+ |u_i(t)| \leq -c_i(t)|u_i(t)| + \sum_{j=1}^n M_{ij}|u_j(t)| + \sum_{j=1}^n N_{ij}|u_j(t - \tau_{ij}(t))|. \tag{4.2}$$

On the other hand, according to condition (A_5) and lemma 2.1, there exists a positive constant vector $(\rho_1, \rho_2, \dots, \rho_n)^T$ such that

$$\rho_i(c_i^- - M_{ii} - N_{ii}\tilde{\tau}) - \sum_{j=1, j \neq i}^n \rho_j(M_{ji} + N_{ji}\tilde{\tau}) > 0.$$

Let

$$\begin{aligned} \chi_i(\varepsilon) & = \rho_i(\varepsilon - c_i^- + M_{ii} + N_{ii}\tilde{\tau}e^{\tau_{ii}^+ \varepsilon}) \\ & \quad + \sum_{j=1, j \neq i}^n \rho_j(M_{ji} + N_{ji}\tilde{\tau}e^{\tau_{ij}^+ \varepsilon}). \end{aligned}$$

It is clear that $\chi_i(0) < 0$. Since $\chi_i(\varepsilon)$ is continuous on $[0, \infty)$ and $\chi_i(\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow +\infty$, and $\frac{d\chi_i(\varepsilon)}{d\varepsilon} > 0$, then there exist constant $\xi_i^* > 0$ such that

$$\chi_i(\xi_i^*) = \rho_i(\xi_i^* - c_i^- + M_{ii} + N_{ii}\tilde{\tau}e^{\lambda\tau_{ii}^+ \xi_i^*}) + \sum_{j=1, j \neq i}^n \rho_j(M_{ji} + N_{ji}\tilde{\tau}e^{\lambda\tau_{ij}^+ \xi_i^*}) = 0.$$

By choosing $\lambda = \min\{\xi_1^*, \xi_2^*, \dots, \xi_n^*\}$, we have,

$$\chi_i(\lambda) = \rho_i(\lambda - c_i^- + M_{ii} + N_{ii}\tilde{\tau}e^{\lambda\tau_{ii}^+}) + \sum_{j=1, j \neq i}^n \rho_j(M_{ji} + N_{ji}\tilde{\tau}e^{\lambda\tau_{ij}^+}) < 0 \tag{4.3}$$

for $i = 1, 2, \dots, n$.

Let $U_i(t) = e^{\lambda t}|u_i(t)|$, then it follows from (4.2) that

$$\begin{aligned} D^+U_i(t) &= \lambda e^{\lambda t}|u_i(t)| + e^{\lambda t} \operatorname{sgn}(u_i(t))u_i'(t) \\ &\leq \lambda e^{\lambda t}|u_i(t)| + e^{\lambda t} \left(-c_i(t)|u_i(t)| + \sum_{j=1}^n M_{ij}|u_j(t)| \right. \\ &\quad \left. + \sum_{j=1}^n N_{ij}|u_j(t - \tau_{ij}(t))| \right) \\ &\leq e^{\lambda t}(\lambda - c_i^-)|u_i(t)| + e^{\lambda t} \sum_{j=1}^n M_{ij}|u_j(t)| \\ &\quad + e^{\lambda t} \sum_{j=1}^n N_{ij}|u_j(t - \tau_{ij}(t))|. \end{aligned} \tag{4.4}$$

Define a Lyapunov functional as follows:

$$V(t) = \sum_{i=1}^n \rho_i U_i(t) + \sum_{i=1}^n \sum_{j=1}^n \rho_j \int_{t-\tau_{ij}(t)}^t \frac{N_{ij}e^{\lambda\tau_{ij}^+}}{1-\tau'_{ij}(t_{ij}(s))} e^{\lambda s}|u_j(s)| ds \tag{4.5}$$

where $t_{ij}(t)$ is the inverse function of $t - \tau_{ij}(t)$.

By calculating the derivative of $V(t)$ along the solution of (1.2) and from (4.3) and (4.4), we have

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^n \rho_i((\lambda - c_i^-)U_i(t) + \sum_{j=1}^n \rho_j e^{\lambda t} M_{ij}|u_j(t)| \\ &+ \sum_{i=1}^n \sum_{j=1}^n \rho_j e^{\lambda t} N_{ij}|u_j(t - \tau_{ij}(t))| + \sum_{i=1}^n \sum_{j=1}^n \rho_j e^{\lambda t} \frac{N_{ij}e^{\lambda\tau_{ij}^+}}{1-\tau'_{ij}(t_{ij}(t))}|u_j(t)| \\ &- \sum_{i=1}^n \sum_{j=1}^n \rho_j e^{\lambda(t-\tau_{ij}(t))} \frac{N_{ij}e^{\lambda\tau_{ij}^+}}{1-\tau'_{ij}(t_{ij}(t-\tau_{ij}(t)))}|u_j(t - \tau_{ij}(t))|(1 - \tau'_{ij}(t)) \\ &\leq \sum_{i=1}^n \rho_i(\lambda - c_i^-)U_i(t) + \sum_{i=1}^n \sum_{j=1}^n \rho_j M_{ij}U_i(t) + \sum_{i=1}^n \sum_{j=1}^n \rho_j N_{ji}e^{\lambda\tau_{ji}^+} \tilde{\tau}U_i(t) \\ &= \sum_{i=1}^n \left(\rho_i(\lambda - c_i^-) + \sum_{j=1}^n \rho_j M_{ji} + \sum_{i=1}^n \sum_{j=1}^n \rho_j N_{ji}e^{\lambda\tau_{ji}^+} \tilde{\tau} \right) U_i(t) \\ &< 0 \end{aligned} \tag{4.6}$$

for $t > 0$. Therefore, from (4.6) we obtain

$$V(t) \leq V(0) \quad \text{for all } t > 0.$$

(4.5) and (4.7) imply that

$$V(t) \leq \sum_{i=1}^n \left(\rho_i + \sum_{j=1}^n \rho_j N_{ij} \tilde{\tau} e^{\lambda\tau_{ij}^+} \right) \sup_{-\tau \leq s \leq 0} |x_i(s, \phi) - x_i(s, \phi^*)|$$

Let $\Delta = \min\{\rho_1, \rho_2, \dots, \rho_n\}$,

$$M = \frac{1}{\Delta} \left\{ \max_{1 \leq i \leq n} \rho_i + \sum_{j=1}^n \rho_j N_{ij} \tilde{\tau} e^{\lambda\tau_{ij}^+} \right\}, \text{ then}$$

$$\begin{aligned} \|(x^T(t, \phi) - x^T(t, \phi^*))\| &= \sum_{i=1}^n |x_i(t, \phi) - x_i(t, \phi^*)| \\ &\leq \frac{1}{\Delta} e^{-\lambda t} \sum_{i=1}^n \left(\rho_i + \sum_{j=1}^n \rho_j N_{ij} \tilde{\tau} e^{\lambda\tau_{ij}^+} \right) \\ &\quad \times \sup_{-\tau \leq s \leq 0} |x_i(s, \phi) - x_i(s, \phi^*)| \\ &\leq M e^{-\lambda t} \sup_{-\tau \leq s \leq 0} |x_i(s, \phi) - x_i(s, \phi^*)| \\ &\leq M e^{-\lambda t} \|\phi_x^T - (\phi_x^*)^T\|. \end{aligned}$$

Therefore, system (1.2) admits an exponential periodic attractor. This completes the proof.

Remark 4.1.1 Theorem 4.1.1 gives the sufficient conditions ensuring the existence of exponential periodic attractor of (1.2), which includes the stability of equilibrium point as its special case. Similarly, one can derive the conditions of the existence and stability of equilibrium point of (1.2).

Exponential convergence

In this subsection, we study the exponential convergence of system (1.2).

Definition 4.2.1 [35] system (1.2) is said to be exponentially convergent if there exist constants $\alpha_i > 0$ and $\delta > 0$ such that for any two solutions $(x(t, \phi))$ and $(x(t, \psi))$ with initial values ϕ and ψ , respectively, the following inequality holds:

$$|x_i(t, \phi) - x_i(t, \psi)| \leq \alpha_i e^{-\delta t}$$

for all $t > 0$ and $|\phi_{x_i}(s) - \psi_{x_i}(s)| \leq \alpha_i, i = 1, 2, \dots, n$. If $(x(t, \psi))$ is a periodic solution, then this periodic solution is said to be exponentially convergent.

Theorem 4.2.1 If $(A_1), (A_2)$ hold and there exist constants $\alpha_i > 0, \delta > 0$ such that for all $t > 0, i = 1, \dots, n$,

$$(A_7) \quad \alpha_i(\delta - c_i(t)) + \sum_{j=1}^n \alpha_j(M_{ij} + N_{ij}e^{\delta\tau_{ij}^+}) \leq 0.$$

Then system (1.2) is exponential convergent.

Proof. Let $x(t, \phi) = (x_1(t, \phi), \dots, x_n(t, \phi))^T, x(t, \psi) = (x_1(t, \psi), \dots, x_n(t, \psi))^T$ be two arbitrary solutions of (1.2) and $u_i(t) = x_i(t, \phi) - x_i(t, \psi)$, then we can obtain from system (1.2) and (A_2) that

$$\begin{aligned} D^+|u_i(t)| &= \operatorname{sgn}(u_i(t))[-c_i(t)u_i(t) + f_i(x_1(t, \phi), \dots, x_n(t, \phi), \\ &\quad x_1(t - \tau_{1j}(t), \phi), \dots, x_1(t - \tau_{1j}(t), \phi)) - f_i(x_1(t, \psi), \\ &\quad \dots, x_n(t, \psi), x_1(t - \tau_{1j}(t), \psi), \dots, x_1(t - \tau_{1j}(t), \psi))] \\ &\leq -c_i(t)|u_i(t)| + \sum_{j=1}^n M_{ij}|u_j(t)| + \sum_{j=1}^n N_{ij}|u_j(t - \tau_{ij}(t))|. \end{aligned} \tag{4.8}$$

Consider the following comparison system of (4.8), i.e.,

$$v_i(t) = -c_i(t)v_i(t) + \sum_{j=1}^n M_{ij}v_j(t) + \sum_{j=1}^n N_{ij}v_j(t - \tau_{ij}(t)),$$

By the comparison theorem, we have

$$|u_i(t)| \leq v_i(t) \quad \text{for } t > 0 \tag{4.9}$$

whenever $|u_i(s)| \leq v_i(s), s \leq 0, i = 1, 2, \dots, n$.

On the other hand, it follows from (A_7) that:

$$\begin{aligned} -\alpha_i \delta e^{-\delta t} &\geq -c_i(t)\alpha_i e^{-\delta t} + \sum_{j=1}^n \alpha_j M_{ij} e^{-\delta t} \\ &\quad + \sum_{j=1}^n \alpha_j N_{ij} e^{-\delta(t-\tau_{ij}(t))}. \end{aligned} \tag{4.10}$$

Let $\chi_i(t) = \alpha_i e^{-\delta t}$, then

$$\chi_i'(t) \geq -c_i(t)\chi_i(t) + \sum_{j=1}^n M_{ij}\chi_j(t) + \sum_{j=1}^n N_{ij}\chi_j(t - \tau_{ij}(t)).$$

Again employing comparison theorem, we have

$$v_i(t) \leq \chi_i(t) \quad \text{for } t > 0 \tag{4.11}$$

when $v_i(s) \leq \chi_i(s)$ for $s \leq 0, i = 1, 2, \dots, n$. (4.9) and (4.11) imply that $|u_i(t)| \leq \chi_i(t) = \alpha_i e^{-\delta t}$ for $t > 0$ and $|x_i(s, \phi) - x_i(s, \psi)| \leq \alpha_i, s \leq 0, i = 1, 2, \dots, n$.

That is, system (1.2) is exponential convergent, This completes the proof.

An application of Theorem 3.1 and Theorem 4.2.1 yields the following corollary.

Corollary 4.2.1 Suppose that $(A_1) - (A_4)$ and (A_7) hold true. Then the periodic solution of system (1.2) is exponentially convergent.

Corollary 4.2.2 Suppose that $(A_1) - (A_2)$ hold. Further, $(A_8) - \alpha_i c_i(t) + \sum_{j=1}^n \alpha_j (M_{ij} + N_{ij}) < 0$.

Then system (1.2) is exponentially convergent.

Proof. Actually, if (A_8) holds true, then (A_7) can be derived to be true as well. The result can be obtained immediately from **Theorem 4.2.1**.

Remark 4.2.1 Definition 4.2.1 gives a componentwise exponential convergence estimate. It is stronger than conventional Lyapunov stability, which implies globally exponential stability of system (1.2), see [35, 36]. In addition, (A_7) involves coefficient function $c_i(t)$ instead of its infimum or supremum value as in most previous studies (further see Proposition 5.2, Proposition 5.3, Proposition 5.5 and Proposition 5.6), which gives more flexibility in designing stable models.

Applications and an illustrative Example

In this section, we give applications and an example to show the effectiveness of the main results.

For system (1.1), we assume that

$(A_9) c_i(t) > 0, a_{ij}(t), b_{ij}(t), I_i(t)$ are continuous ω -periodic functions for $i, j = 1, \dots, n$ or $n + m$.

$(A_{10}) g_i : R \rightarrow R$ is continuous and there exists positive constant $G_i > 0$ such that $|g_i(u) - g_i(v)| \leq G_i |u - v|$ for any $u, v \in R, i = 1, \dots, n$ or $i = 1, \dots, n + m$.

We denote system (1.1) satisfying (A_9) and (A_{10}) by system (5.1), then by using Theorem 3.1 and Theorem 4.1.1, one can obtain the following conclusion.

Proposition 5.1 Assume that (A_9) and (A_{10}) hold. Further,

$(A_{11}) \tilde{\kappa} > 0,$

$(A_{12}) \Gamma^* = C - GD^*$ is a nonsingular M- matrix,

$(A_{13}) \tilde{\Gamma}^* = C - GB^*$ is a nonsingular M- matrix, where

$$\tilde{\kappa} = \min_{1 \leq i \leq n} \left\{ c_i^- - G_i \sum_{j=1}^n (a_{ij}^+ + b_{ij}^+) \right\},$$

$$G = \text{diag}(G_1, G_2, \dots, G_n), D^* = (a_{ij}^+ + b_{ij}^+)_{n \times n},$$

$$B^* = (a_{ji}^+ + \tilde{\tau} b_{ji}^+)_{n \times n}, C, \tilde{\tau} \text{ are defined as Theorem 3.1.}$$

Then system (5.1) has an exponential periodic attractor.

From Theorem 4.2.1 and Corollary 4.2.2, it is clear that the following two corollaries hold true.

Proposition 5.2 Assume that (A_9) and (A_{10}) hold. Further, there exist constants $\alpha_i > 0, \delta > 0$ such that for all $t > 0, i = 1, 2, \dots, n,$

$$(A_{14}) \alpha_i (\delta - c_i(t)) + \sum_{j=1}^n \alpha_j G_i (|a_{ij}(t)| + |b_{ij}(t)|) e^{\delta \tau_{ij}^+} \leq 0.$$

Then system (5.1) is exponential convergent.

Proposition 5.3 Assume that (A_9) and (A_{10}) hold. Further, $(A_{15}) - \alpha_i c_i(t) + \sum_{j=1}^n \alpha_j G_i (|a_{ij}(t)| + |b_{ij}(t)|) < 0$. Then system (5.1) is exponentially convergent.

Next consider the following BAM neural network with impulse and time-varying delays:

$$\begin{cases} x_i(t) = -c_i(t)x_i(t) + \sum_{j=m+1}^{m+n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=m+1}^{m+n} b_{ij}(t) \\ \quad \times g_j(x_j(t - \tau_{ij}(t))) + I_i(t), i = 1, 2, \dots, n, \\ x_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^m a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^m b_{ij}(t) \\ \quad \times g_j(x_j(t - \tau_{ij}(t))) + I_i(t), i = n + 1, \dots, n + m, \end{cases} \tag{5.2}$$

Similarly, one can derive the following conclusions.

Proposition 5.4 Assume that $(A_6), (A_9)$ and (A_{10}) hold. Further,

$(A_{16}) \hat{\kappa} > 0,$

$(A_{17}) \Gamma' = \begin{pmatrix} 0 & C - G\tilde{D} \\ \hat{C} - \hat{G}\hat{D} & 0 \end{pmatrix}$ is a nonsingular M- matrix,

$(A_{18}) \tilde{\Gamma}' = \begin{pmatrix} 0 & C - G\tilde{B} \\ \hat{C} - \hat{G}\hat{B} & 0 \end{pmatrix}$ is a nonsingular M- matrix, where

$$\hat{\kappa} = \min \left\{ \min_{1 \leq i \leq n} c_i^- - G_i \sum_{j=n+1}^{n+m} (a_{ij}^+ + b_{ij}^+), \min_{n+1 \leq i \leq n+m} c_i^- - G_i \sum_{j=1}^m (a_{ij}^+ + b_{ij}^+) \right\},$$

$$\tilde{D} = (a_{ij}^+ + b_{ij}^+)_{n \times n}, \tilde{B} = (a_{ji}^+ + \tilde{\tau} b_{ji}^+)_{n \times n} \text{ for } i = 1, 2, \dots, n, j = m + 1, m + 2, \dots, n + m,$$

$$\hat{C} = \text{diag}(-c_{n+1}^-, \dots, -c_{n+m}^-), \hat{G} = \text{diag}(G_{n+1}, \dots, G_{n+m}),$$

$$\hat{D} = (a_{ij}^+ + b_{ij}^+)_{m \times m}, \hat{B} = (a_{ji}^+ + \tilde{\tau} b_{ji}^+)_{m \times m} \text{ for } i = n + 1, n + 2, \dots, n + m, j = 1, \dots, m, C, G \text{ are}$$

defined in proposition 5.1. Then system (5.2) has an exponential periodic attractor.

Proposition 5.5 Assume that (A_9) and (A_{10}) hold. Further, there exist constants $\alpha_i > 0, \delta > 0$ such that for all $t > 0,$

$$(A_{19}) \alpha_i (\delta - c_i(t)) + \sum_{j=m+1}^{m+n} \alpha_j G_i (|a_{ij}(t)| + |b_{ij}(t)|) e^{\delta \tau_{ij}^+} \leq 0$$

for $i = 1, 2, \dots, n,$ and

$$\alpha_i (\delta - c_i(t)) + \sum_{j=1}^m \alpha_j G_i (|a_{ij}(t)| + |b_{ij}(t)|) e^{\delta \tau_{ij}^+} \leq 0 \text{ for } i = n + 1, \dots, n + m.$$

Then system (5.2) is exponential convergent.

Proposition 5.6 Assume that (A_9) and (A_{10}) hold. Further, for all $t > 0,$

$$(A_{20}) - \alpha_i c_i(t) + \sum_{j=m+1}^{m+n} \alpha_j G_i (|a_{ij}(t)| + |b_{ij}(t)|) < 0$$

for $i = 1, 2, \dots, n,$ and

$$- \alpha_i c_i(t) + \sum_{j=1}^m \alpha_j G_i (|a_{ij}(t)| + |b_{ij}(t)|) < 0$$

for $i = n + 1, \dots, n + m$. Then system (5.2) is exponentially convergent.

Obviously, for system (5.1) and (5.2), if $c_i(t)$, $a_{ij}(t)$ and $b_{ij}(t)$ are constants, or $b_{ij}(t) = 0$, one can similarly derive the sufficient conditions of the existence of the corresponding periodic attractor and exponential convergence in cases of time-varying delays and constant delays, they are omitted here. It shows that the results obtained in this paper contain many previously known results and are very general.

If delays are constant numbers, then (1.2) reads as

$$x'_i(t) = -c_i(t)x_i(t) + f_i(x_1(t), \dots, x_n(t), x_1(t - \tau_{i1}), \dots, x_n(t - \tau_{in}) + I_i(t), \tag{5.3}$$

which is studied in [28]. By using Banach fixed point theorem, the authors obtained the existence and global exponential stability of the equilibrium point. However, from Theorem 3.1 and Theorem 4.1.1, we can derive the existence of exponential periodic attractor of (5.3).

Proposition 5.7 Assume that $(A_1) - (A_4)$ and (A_6) hold. Further,

$$(A_{21}) \quad \hat{\Gamma} = C - \hat{B} \text{ is a nonsingular M- matrix,}$$

where $\hat{B} = (M_{ij} + N_{ij})_{n \times n}$, C is defined in theorem 3.1.

Then system (5.3) has an exponential periodic attractor.

Remark 5.1 The exponential convergence of system (5.3) and the exponential convergence of periodic solutions of system (5.1)-(5.3) can be derived similarly, they are omitted here.

Finally, an illustrative example is given to show the usefulness of the main results.

Example Consider the following neural networks:

$$\begin{cases} x'_1(t) = -c_1(t)x_1(t) + a_{11}(t)g_1(x_1(t)) + a_{12}(t)g_2(x_2(t)) \\ \quad + b_{11}(t)g_1(x_1(t - \tau_{11}(t))) + b_{12}(t)g_2(x_2(t - \tau_{12}(t))) + I_1(t), \\ x'_2(t) = -c_2(t)x_2(t) + a_{21}(t)g_1(x_1(t)) + a_{22}(t)g_2(x_2(t)) \\ \quad + b_{21}(t)g_1(x_1(t - \tau_{21}(t))) + b_{22}(t)g_2(x_2(t - \tau_{22}(t))) + I_2(t), \end{cases} \tag{5.4}$$

Take $g_1(u) = g_2(u) = \frac{1}{2}(|u + 2| - |u - 2|)$ for any $u \in R$. Then $G_1 = G_2 = 1$. Set

$$\begin{aligned} c_1(t) &= 2 + \sin 16\pi t, c_2(t) = 3 - \cos 16\pi t, a_{11}(t) = \frac{\sin 16\pi t}{8}, b_{11}(t) = \frac{\sin 16\pi t}{16}, a_{12}(t) = \frac{1 - \cos 16\pi t}{8}, b_{12}(t) = \frac{1 + \cos 16\pi t}{16}, a_{21}(t) = \frac{1 + 2\sin 16\pi t}{18}, b_{21}(t) = \frac{\cos 16\pi t}{12}, a_{22}(t) = \frac{1 - 2\sin 16\pi t}{18}, b_{22}(t) = \frac{1 + \sin 16\pi t}{8}, \\ \tau_{11}(t) &= \tau_{12}(t) = \tau_{21}(t) = \tau_{22}(t) = \frac{5 + 3\cos 16\pi t}{4}, I_1(t) = 1 + \sin 16\pi t, I_2(t) = 2 - \cos 16\pi t. \end{aligned}$$

$$\text{Then } c_1^- = 1, c_2^- = 2, a_{11}^+ = \frac{1}{8}, b_{11}^+ = \frac{1}{16}, a_{12}^+ = \frac{1}{4}, b_{12}^+ = \frac{1}{8},$$

$$a_{21}^+ = \frac{1}{6}, b_{21}^+ = \frac{1}{12}, a_{22}^+ = \frac{1}{6}, b_{22}^+ = \frac{1}{4}.$$

By simple calculation, $\tilde{\kappa} = \frac{7}{16} > 0, \tilde{\tau} = 4$,

$$\Gamma^* = \begin{pmatrix} \frac{13}{16} & -\frac{3}{8} \\ -\frac{1}{4} & \frac{19}{12} \end{pmatrix} \text{ and } \tilde{\Gamma}^* = \begin{pmatrix} \frac{5}{8} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{5}{6} \end{pmatrix}. \text{ One}$$

concludes from Lemma 2.1 that Γ and $\tilde{\Gamma}^*$ are all nonsingular M-matrix since there exist constant vectors

$$\xi^* = (1, 1)^T > 0, \tilde{\xi}^* = (\frac{3}{2}, 1)^T > 0 \text{ such that}$$

$\Gamma^* \xi^* > 0, \tilde{\Gamma}^* \tilde{\xi}^* > 0$, respectively. Hence $(A_{11}) - (A_{13})$ hold. It is clear that $(A_6), (A_9)$ and (A_{10}) hold as well.

Therefore, by proposition 5.1, system (5.4) admits an exponential periodic attractor.

Conclusions

In this paper, the existence of the exponential periodic attractor and exponential convergence of system (1.2) are investigated. The main method employed here is Mawhin's continuation theorem of coincidence degree theory, Lyapunov stability theory combining with comparison theorem. The main results are based on continuation theorem, matrix theory, some new estimation techniques for the priori bounds of the solutions of $Lx = \lambda Nx, \lambda \in (0, 1)$ and comparison theorem. Particularly, the conditions ensuring the existence of exponential periodic attractor and exponential convergence of system (1.2) are obtained, which are new and interesting and much different from the known results in the literature [1]. They complement or improve the previously known results [1, 17, 19, 28]. Finally, an illustrative example is given to show the effectiveness of the main results.

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