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Calculus of variations in physical problem

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ABSTRACT

Variation is an important mathematical physics method in mathematical physics problem. Many physical problems are always concerned with variation, and they can be solved by extremum of functional. We can obtain physical laws by way of variation calculus, such as the solution of problems of central field and electromagnetic field. By application of calculus of variation, the motion of matter can be unified.

Introduction

Variation is an important mathematical physics method in mathematical physics problems, its object is functional. We can apply variation methods in many physical regions. Many results in physics can be expressed as variational principles, and it is often when expressed in this form that their physical meaning is most clearly understood (K. F. Riley, 2002). All kinds of physical laws can be descripting as the extremum of functional each other. Thus, the motions of matter can be unified. The problem of mathematical physics can transform into a variational problem, and its mathematical type is unified. The calculus of variation is an approximate method to solve mathematical physics problems.

The extremum of functional and Euler-Lagrange Equation

Functional have many different definitions. We often make use of the integral definition (HU Si-zhu, 2002)

$$I[y(x)] = \int_a^b F(x, y, y')dx \ , \qquad (1)$$

where x_1=a, x_2=b, a, b is constant, $\delta y(a) = 0$ and $\delta y(b) = 0$; $y(x)$ has second-order derivative with respect to variable x; y' is the derivative of y with respect to x. The line integral has a stationary value relative to paths differing infinitesimally from the correct function $y(x)$.

In some cases, the definition of functional is (WU Cong-shi, 1999)

$$I[u(x, y)] = \iint_s F(x, y, u_x, u_y)dxdy \, . \qquad (2)$$

First variation of functional (1) is

$$\delta I[y(x)] = \int_a^b [\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}(\delta y)']dx \, . \qquad (3)$$

Its second variation is

$$\delta^2 I[y(x)] = \int_a^b [\frac{\partial^2 F}{\partial y^2}(\delta y)^2 + 2\frac{\partial^2 F}{\partial y \partial y'}\delta y \cdot (\delta y)' + \frac{\partial^2 F}{\partial y'^2}(\delta y')^2]dx \qquad (4)$$

When functional $I[y(x)]$ has extremum, the first variation of functional should equal to zero. The condition is necessary. So we have

$$\delta I[y(x)] = \int_a^b [\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}(\delta y)']dx = 0 \qquad (5)$$

Integrating by parts, the integral becomes

$$\delta I[y(x)] = \int_a^b [\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}(\delta y)']dx =$$

$$\int_a^b [\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}]\delta y dx = 0 \qquad (6)$$

In Eq.(6), δy is arbitrary, so Eq.(6) can be written

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \ . \qquad (7)$$

This is known as Euler-Lagrange(E-L) equation, and is a differential equation for $y(x)$, since the function F is known. In general, the extremum of functional is very complex. Because the extremum is related with numbers of independent variables, numbers of unknown functions, orders of derived functions and constraint conditions.

If we discuss the sufficient condition when the functional has extremum, positive and negative of second variation should be considered. In mathematical physics problems, functionals often have clearly physical meanings, the extremum certainly exists. So, we don't necessary consider second variation of functional.

By solving E-L equation, we can obtain the extremum of functional. When we calculate the extremum of functional, the condition usually is fixed boundary value or fixed end-points, i.e. $\delta y(a) = 0, \delta y(b) = 0$; when boundary is free, we have $\left.\frac{\partial F}{\partial y}\right|_{x=a} = 0, \left.\frac{\partial F}{\partial y}\right|_{x=b} = 0$.

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The extremum of functional in mathematical physics problem

When using methods of functional to solve mathematical physics problems, we can combine the mathematical physics problem with a variation, and make the problem to be an extremum of functional. By solving E-L equations, we can obtain the solution of the mathematical physics problem.

1 the extremum of functional in central field problem

In central field, a mechanical system moved upon a path $q=q(t)$ from point $q=q(t_1)$ to $q=q(t_2)$. Find the impossible motion path of the system.

We define $F(x,x',t)=L(q,q',t)$, where *q* is generalized coordinates and q' is generalized velocities. $L(q,q',t)$ is Lagrange function. The Lagrangian L is defined, in terms of kinetic energy T and potential energy V, by $L=T-V$. Here V is a function of the *q* only, not of q'. Applying Eq.(1) , we have functional $S=\int_{t_1}^{t_2}L(q,q',t)dt$, where *S* is Hamilton's action function.

From Eq.(3), we know, when

$$\delta S=\delta\int_{t_1}^{t_2}L(q,q',t)dt=0, \tag{8}$$

the motion of the mechanical system in a given time interval is to minimize the action integral. Eq.(8) referred to as "Hamilton's principle". It can be stated as

The motion of the system from time t_1 to time t_2 is such that the action has a stationary value; i.e. the integral along the given path has the same value to within first-order infinitesimals as that along all neighboring paths.

So, we have

$$\delta S=\int_{t_1}^{t_2}[\frac{\partial L}{\partial q}\delta q-\frac{\partial L}{\partial q'}(\delta q)']dt=0 \tag{9}$$

and

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial q'}=0. \tag{10}$$

Each is Hamilton's action function's integral equation and differential equation, when *S* has extremum.Thus, all of Newtonian mechanics can be summarized in a single statement.By inserting the Lagrangian, and one can then deduce the equation of motion for the mechanical system.

If L does not involve the time explicitly, i.e. $L=L(q,q')$, so that $\frac{\partial L}{\partial t}=0$. Then

$$\frac{d}{dt}[L-q'\frac{\partial L}{\partial q'}]=\frac{dL}{dt}-\frac{d}{dt}[q'\frac{\partial L}{\partial q'}]=q'[\frac{\partial L}{\partial q}-\frac{d}{dt}(\frac{\partial L}{\partial q'})] \tag{11}$$

Substituting in Eq.(11) with Eq.(10),we have $\frac{d}{dt}[L-q'\frac{\partial L}{\partial q'}]=0$, the solution of this equation is

$$q'\frac{\partial L}{\partial q'}-L=\text{constant} \tag{12}$$

i.e.

$$q'\frac{\partial L}{\partial q'}-L=q'\frac{\partial(T-V)}{\partial q'}-L=q'\frac{\partial T}{\partial q'}-L=\text{constant}. \tag{13}$$

Euler's theorem states that if *f* is a homogeneous function of degree *n* in the variables x_i, then (Herbet Goldstein, 2005)

$$\sum_i x_i\frac{\partial f}{\partial x_i}=nf\ . \tag{14}$$

Because T is a homogeneous function of the q''s of degree 2. Hence, applying Euler's theorem, so that

$$\sum_{\alpha=1}^{S}q'_\alpha\frac{\partial T}{\partial q'_\alpha}=2T, \tag{15}$$

where α is freedom degree of mechanical system.

Substituting in Eq.(13) with Eq.(15), we have

$$2T—L=T+V=E=constant. \tag{16}$$

Therefore, the mechanical energy *E* of the system is conservational in central field.

We discuss two-dimension motion of a particle in central field. In the polar coordinate system, we choose *r* and θ as general coordinates, then we have

$$F(q,q',t)=L(r,\theta,r',q';t)=\frac{1}{2}mr'^2+\frac{1}{2}mr^2\theta'^2+\frac{k^2m}{r}.$$

Appling Eq.(9), we have

$$\frac{\partial L}{\partial r}-\frac{d}{dt}\frac{\partial L}{\partial r'}=-mr''-mr\dot{\theta}^2-\frac{k^2m}{r^2}=0$$

$$\frac{\partial L}{\partial \theta}-\frac{d}{dt}\frac{\partial L}{\partial \theta'}=m(r\theta''+2r'\theta')=0$$

Hence,

$$m(r''+r\dot{\theta}^2)=-\frac{k^2m}{r^2} \tag{17}$$

$$m(r\theta''+2r'\theta')=0 \tag{18}$$

Eqs. (17) and (18) are the equations of motion in central force field.

Maxwell equations of electromagnetic field

There have an electromagnetic field, its electric field strength is $\vec{E}$ and magnetic field strength is $\vec{B}$. Both $\vec{E}$ and $\vec{B}$ are continuous functions of time and position. When charge density is ρ and circuit is $\vec{j}$, where ρ and $\vec{j}$ is constant. We define Lagrangian density function (Herbet Goldstein, 2005)

$$L=\frac{1}{2}(\frac{1}{\mu}B^2-\varepsilon E^2)+\rho\varphi-\vec{j}\cdot\vec{A}, \tag{19}$$

where, by $\vec{B}=\nabla\times\vec{A}$ and $\vec{E}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}$,we obtain a scalar potential φ and a vector potential $\vec{A}$.

From Eq.(19), we have $H=\frac{\partial L}{\partial B}=\frac{1}{\mu}B$ and $D=-\frac{\partial L}{\partial E}=\varepsilon E$.

Appling Eq.(2), we have

$$\delta\iint Ldxdt=\delta\iint\left[\frac{1}{2}(\frac{1}{\mu}\vec{B}^2-\varepsilon\vec{E}^2)+\rho\varphi-\vec{j}\cdot\vec{A}\right]dxdt$$

$$=\iint\left(\frac{1}{\mu}\vec{B}\cdot\delta\vec{B}-\varepsilon\vec{E}\cdot\delta\vec{E}-\vec{j}\cdot\delta\vec{A}+\rho\delta\varphi\right)dxdt$$

$$=\iint\left\{\vec{H}\cdot(\nabla\times\delta\vec{A})+\vec{D}\cdot\left[\nabla(\delta\varphi)+\frac{\partial(\delta\vec{A})}{\partial t}\right]-\vec{j}\cdot\delta\vec{A}+\rho\delta\varphi\right\}dxdt \qquad (20)$$

by

$$\vec{H}\bullet(\nabla\times\delta\vec{A})=\delta\vec{A}\cdot(\nabla\times\vec{H})=\nabla\cdot(\delta\vec{A}\times\vec{H})$$

$$\vec{D}\cdot(\nabla\delta\varphi)=\nabla\cdot[(\delta\varphi)\vec{D}]-\delta\varphi\cdot(\nabla\cdot\vec{D})$$

$$\vec{D}\cdot\frac{\partial}{\partial t}(\delta\vec{A})=\frac{\partial}{\partial t}[\vec{D}\cdot(\delta\vec{A})]-\delta\vec{A}\cdot\frac{\partial\vec{D}}{\partial t}$$

Substituting the three equations in Eq.(20), and considering the necessary condition of the extremum of functional: $\delta\iint Ldxdt=0$, we have

$$\delta\iint Ldxdt=\iint\left[\left(\nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}\right)-\vec{j}\right]\cdot\delta\vec{A}dxdt$$

$$+\iint\left[\rho-(\nabla\cdot\vec{D})\right]\delta\varphi dxdt$$

$$+\iint\frac{\partial}{\partial t}(\vec{D}\cdot\delta\vec{A})dxdt+\iint\nabla\cdot[(\delta\varphi)\vec{D}]dxdt=0. \qquad (21)$$

When the boundary of electromagnetic field is fixed, the variation $\delta\varphi\big|_{\Sigma}=0$ and $\delta\vec{A}\big|_{\Sigma}=0$. Hence, the third integral and the forth integral of Eq.(20) equals to zero. Considering $\delta\vec{A}$ and $\delta\varphi$ is independent and arbitrary, we have

$$\nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}=\vec{j} \qquad (22)$$

$$\nabla\cdot\vec{D}=\rho \qquad (23)$$

Appling vector identical equation $\nabla\times\nabla\varphi=0$ and $\nabla\cdot\nabla\times\vec{A}=0$, and taking into account the scalar potential φ and the vector potential $\vec{A}$, we have

$$-\nabla\times\nabla\varphi=\nabla\times\vec{E}+\frac{\partial(\nabla\times\vec{A})}{\partial t}=\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

And

$\nabla\bullet\vec{B}=\nabla\cdot\nabla\times\vec{A}=0$.

Namely,

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \qquad (24)$$

$$\nabla\bullet\vec{B}=0 \qquad (25)$$

Eqs. (22),(23) , (24) and (25) are the Maxwell equations of electromagnetic field.

Conclusion

The methods of variation played an important role in mathematical physics. Many mathematical physics problems can be transformed into a variational form. By solving E-L equation, we can obtain the solution of the mathematical physics problem. We can deduce physical laws by way of variation calculus, and its mathematical results are unified.

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