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Robust fuzzy solid transportation problems based on extension principle under uncertain demands

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ABSTRACT

The solid transportation problem (STP) arises when bounds are given on three item properties. The fuzzy solid transportation problem (FSTP) appears when the nature of the data problem is fuzzy. This paper deals with the robust fuzzy solid transportation problem based on extension principle under uncertain demands. The fuzzy solid transportation problem is transformed into a pair of mathematical programs that is employed to calculate the lower and upper bounds of the fuzzy total transportation cost at possibility level α. In this paper, we are interested in a robust version of location fuzzy transportation problem with an uncertain demand using a 2-stage formulation: one with inequality constraints and the other with equality constraints.

Introduction

The solid transportation problem (STP) was first presented by Haley [7] in 1962, in which three kinds of constraints are taken into consideration that is, source constraint, destination constraints and conveyance constraint. The STP degenerates into the classical transportation problem as the number of conveyance is only one. In a solid transportation problem more than one objective is normally considered. The solid transportation problem (STP) may be considered as a special case of LPP. In many industrial problems, a homogeneous product is delivered from an origin to a destination by means of different modes of transport called conveyances, such as trucks, cargo flights, goods trains, ships etc. The STP was proposed by Schell [14]. Bit et al. [3] used a fuzzy programming approach to solve a MOSTP, Ida et al. [8] presented a neural network method to solve a MOSTP. Furthermore there exists uncertainty in practical STP's. In fact, uncertainty exists everywhere in the practical world.

Solid transportation problem is a linear programming problem stemmed from a network structure consisting of a finite number of needs and arcs attached to them. For example, the unit shipping cost may vary in a time frame. The supplies, demands and conveyances may be uncertain due to some uncontrollable factors. Since the solid transportation problem a linear program, one straightforward idea is to apply the existing fuzzy linear programming techniques [10, 12, 13] to the fuzzy solid transportation problem. In this paper, we restrict attention on fuzzy total transportation cost measure and develop a solution procedure that is able to calculate the lower and upper bounds of the objective value of fuzzy solid transportation problem, where at least one of the parameters are fuzzy numbers. Based on extension principle [18, 19, 20], the fuzzy solid transportation problem is transformed into a pair of two-level mathematical programs to calculate the bounds of the objective value at possibility level α. Jimenenz and Verdegay [9] solved both interval and fuzzy solid transportation problems via an extension of auxiliary linear program proposed by Chanas et al. [5].

The method of Julien [10] and Parra et al. [12] is able to find the possibility distribution of the objective value provided all the inequality constraints are of $\geq$ type or $\leq$ type. In robust optimization, one divides approaches into two groups, depending on the decisional context. The first context can be identified as the single-stage context, where the decision-maker has to select a solution before knowing the real value of each uncertain parameter. Generally the single-stage approaches provide the worst case solutions (Soyster [15]) that are very conservative and far from optimality in real-world applications.
In this research, we propose a new approach for robust linear optimization that retains the advantages of the linear framework of Soyster (1973). More importantly, our approach offers full control on the degree of conservation for every constraint. In this paper a new method is proposed for solving fuzzy solid transportation problems by assuming that a decision maker is uncertain about the precise values of the transportation cost, availability, demand and conveyances of the product. To illustrate the proposed method a numerical example is solved and the obtained results are compared with the results of existing methods. So the proposed method is very easy to understand and to apply on real life transportation problem for the decision makers.

This paper is organized as follows. In section 2, solid fuzzy transportation and Robust solid transportation problems are presented. In section 3, fuzzy total transportation cost is formulated based on the extension principle is presented. In section 4, Robust location transportation problem and its solution is presented. In section 5, we use an example to illustrate the difference between inequality - constraint and equality - constraint problems. Finally, some conclusion is drawn from the discussions.

Solid Fuzzy Transportation Problem

Consider a transportation problem with *m* supply nodes and *n* demand nodes, in that $s_i > 0$ units are supplied by supply node *i* and $\delta_j > 0$ units are required by demand node *j*. Let

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e_k denote the units of this product that can be carries by k different modes of transportation called conveyance, such as the land transportation by car or train, flight and ocean shipping. Associated with each link *(i, j, k)* from supply node *i*, demand node *j* and *k*, there is a unit shipping cost c_{ijk} for transportation. The problem is to determine a feasible way of shipping the available amount to satisfy the demand that minimizes the total transportation cost.

Let x_{ijk} denote the number of units to be transported from supply *i*, demand *j* and conveyance capacity e_k. The mathematical description of the conventional transportation problem is:

$$\begin{cases} Z = \min \sum_{i=1}^{m}\sum_{j=i}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ \text{subject to} \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \text{ for } i = 1,2,\ldots,m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \text{ for } j = 1,2,\ldots,n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \text{ for } k = 1,2,\ldots,l \\ x_{ijk} \ge 0, \quad \forall\ i.j.k. \end{cases} \tag{2.1}$$

Intuitively, if any of the parameters c_{ijk}, s_i, or δ_j, or e_k is fuzzy, the total transportation cost *Z* becomes fuzzy as well. The conventional transportation problem defined in (2.1) then turns into the fuzzy transportation problem.

Suppose the unit shipping cost c_{ijk}, supply s_i, and demand δ_j, and conveyance capacity e_k are approximately known. They can be represented by the convex fuzzy numbers, $\tilde{C}_{ijk}$, $\tilde{S}_i$, $\tilde{\Delta}_j$ and $\tilde{E}_k$ respectively. Note that a fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min\{\ \mu_{\tilde{A}}(x_1),\ \mu_{\tilde{A}}(x_2)\},\ \ 0 \le \lambda \le 1.$

Let $\mu_{\tilde{C}_{ijk}}$, $\mu_{\tilde{S}_i}$, $\mu_{\tilde{\Delta}_j}$, and $\mu_{\tilde{E}_k}$ denote their membership functions. We have,

$$\tilde{C}_{ijk} = \{(c_{ijk}, \mu_{\tilde{C}_{ijk}}(c_{ijk})) \mid c_{ijk} \in S(\tilde{C}_{ijk})\}$$

$$\tilde{S}_i = \{(s_i, \mu_{\tilde{S}_i}(s_i)) \mid s_i \in S(\tilde{S}_i)\}$$

$$\tilde{\Delta}_j = \{(\delta_j, \mu_{\tilde{\Delta}_j}(\delta_j)) \mid \delta_j \in S(\tilde{\Delta}_j)\}$$

$$\tilde{E}_k = \{(e_k, \mu_{\tilde{E}_k}(e_k)) \mid e_k \in S(\tilde{E}_k)\},$$

where $S(\tilde{C}_{ijk})$, $S(\tilde{S}_i)$, $S(\tilde{\Delta}_j)$ and $S(\tilde{E}_k)$ which denote the universe sets of the unit shipping cost, the quantity supplied by i^{th} supplier and the quantity required by j^{th} customer through k^{th} conveyance respectively. The fuzzy solid transportation problem is of the following form:

$$\begin{cases} \tilde{Z} = \min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \tilde{C}_{ijk}x_{ijk} \\ \text{subject to} \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le \tilde{S}_i, \text{ for } i = 1,2,\ldots,m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \tilde{\Delta}_j, \text{ for } j = 1,2,\ldots,n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le \tilde{E}_k, \text{ for } k = 1,2,\ldots,l \\ x_{ijk} \ge 0, \quad \forall\ i.j.k. \end{cases} \tag{2.2}$$

Without loss of generality, the entire unit shipping costs, supply quantities and demand quantities, and conveyance capacities are assumed to be convex fuzzy numbers. In this model, as crisp values can be represented by degenerated membership functions which only have one value in their domains.

Robust Solid Transportation Problem

We consider the following transportation problem: a commodity has to be transported from each of *m* potential sources, to *n*-destinations through conveyance *l*. The sources capacities are C_i, *i = 1,2,…,m* and the demand at the destinations are Δ_j, *j = 1,2,…,n* . Let E_k denote the units of the product that can be carries by *k* different modes of transportation called conveyance. We assume that the total sum of the capacities at the sources is greater than or equal to the sum of the demands at the destinations through conveyance *k*. The fixed and variable costs of supplying from source *i = 1, 2,…,m* are δ_j and c_i respectively. The cost of the transporting one unit of the commodity from source *i* to destination *j* by means of the conveyance *k* is c_{ijk}. The goal is to determine which sources to open (r_i), the supply level s_i and the amounts x_{ijk} to be transported such that the total cost is minimized. The mathematical formulation of the nominal transportation problem is the following linear program (T):

$$(T) = \min \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m}\delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk}$$

subject to

$$\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \quad \text{for } i = 1,2,\ldots,m$$

$$\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \quad \text{for } j = 1,2,\ldots,n$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \quad \text{for } k = 1,2,\ldots,l$$

$$s_i \le C_i r_i,\ \ r_i \in \{0,1\}$$

$$x_{ijk} \ge 0, \quad \forall\ i,j,k. \tag{2.3}$$

Furthermore, it should be noted that the decision maker has to decide in two steps: first, the warehouses have to be located and filled and after, once the demands are known, the routing of commodities is decided. According to this context, the transportation part of the problem has a significant importance when deciding of the location part.

In practice, the customers' demands are often estimated, at the stage of construction of warehouses. To be realistic, it is common to assume some uncertainty on these demands. We define the uncertainty set as being interval numbers for each one of them. Formally, the j^{th} customer demand δ_j belongs to $[\tilde{\delta}_j - \hat{\delta}_j, \tilde{\delta}_j + \hat{\delta}_j]$ where $\tilde{\delta}_j \geq 0$ represents the nominal value of δ_j and $\hat{\delta}_j \geq 0$ its maximum deviation. Clearly, each demand δ_j can take on any value from the corresponding interval regardless of the values taken by other coefficients. We denote (T^{δ}) the location transportation problem for a given $\delta \in [\tilde{\delta} - \hat{\delta}, \tilde{\delta} + \hat{\delta}]$ with a nonempty feasible set. Finally, we denote $opt(T^{\delta})$ the optimal value (bounded value) of (T^{δ}) for a given δ.

Because of the context of our problem, we apply a 2-stage robust approach to the uncertain problem (T^{δ}). In fact, we recall that the decision maker has to size the capacities before knowing the demands, and once these demands are revealed, he has to satisfy them. Thus, according to this context, we define the r_i and s_i variables as first stage decisions, while the x_{ijk} variables represent the recourse variables or the second stage decisions. Total costs of all the decisions should be minimized. Furthermore, the decision maker wants to take decisions based on a realistic scenario and avoid the worst case demands. The model chosen to represent the uncertainty set is the one suggested by [1, 6, 17]. This model is a natural adaptation of the original Bertsimas and Sim approach (see [2, 4]), in which a parameter Γ, called the budget of uncertainty, is defined. The value of Γ represents the maximum range of the uncertain demands that can simultaneously deviate from their nominal values. As the uncertainty is on the right hand sides (demands), Γ belongs to [0, *n*]. For $\Gamma = 0$, every right hand side is equal to its nominal value, while $\Gamma = n$ leads to consider the problem with the greatest demands.

The aim of setting the parameter Γ in the robust formulation is to restrict the demands that are greater than the nominal ones. Hence, according to its predictions, the decision maker is free to choose any value of Γ in the interval [0, *n*] and solve the 2-stage robust problem. Then he can decide to open the sources and fill the warehouses, while the actual demands are not known yet. Nevertheless, when the demands are revealed, the decision maker must satisfy them, even if they are larger than those expected. Thus, we assume the total recourse hypothesis: given a solution s_i and r_i a solution for the transportation problem exists whatever are the demands. This problem may concern, for example, the installation of power plants, where the worst demands must be handled.

Solid Fuzzy Total Transportation Problem:

Define the α - cuts of $\tilde{C}_{ijk}, \tilde{S}_i, \tilde{\Delta}_j$ *and* $\tilde{E}_k$ *as*:

$$(C_{ijk})_\alpha = [(C_{ijk})^L_\alpha, (C_{ijk})^U_\alpha] = [\min_{c_{ijk}}\{c_{ijk} \in S(\tilde{C}_{ijk}) \mid \mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha\},$$

$$\max_{c_{ijk}}\{c_{ijk} \in S(\tilde{C}_{ijk}) \mid \mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha]$$

$$(S_i)_\alpha = [(S_i)^L_\alpha, (S_i)^U_\alpha] = [\min_{s_i}\{s_i \in S(\tilde{S}_i) \mid \mu_{\tilde{S}_i}(s_i) \geq \alpha\}, \max_{s_i}\{s_i \in S(\tilde{S}_i) \mid \mu_{\tilde{S}_i}(s_i) \geq \alpha]$$

$$(\Delta_j)_\alpha = [(\Delta_j)^L_\alpha, (\Delta_j)^U_\alpha] = [\min_{\delta_j}\{\delta_j \in S(\tilde{\Delta}_j) \mid \mu_{\delta_j}(\delta_j) \geq \alpha\}, \max_{\delta_j}\{\delta_j \in S(\tilde{\Delta}_j) \mid \mu_{\delta_j}(\delta_j) \geq \alpha]$$

$$(E_k)_\alpha = [(E_k)^L_\alpha, (E_k)^U_\alpha] = [\min_{e_k}\{e_k \in S(\tilde{E}_k) \mid \mu_{e_k}(e_k) \geq \alpha\}, \max_{e_k}\{e_k \in S(\tilde{E}_k) \mid \mu_{ek}(e_k) \geq \alpha].$$

These intervals indicate where the unit shipping cost, supply, demand, and conveyance lie at possibility level α. Suppose we are interested in deriving the membership function of the total transportation cost $\tilde{Z}$. The major difficulty lies on how to deal with the varying ranges of the unit shipping costs, the supply quantities, demand quantities and conveyance capacity. One idea is to apply Zadeh extension principle [18, 19, 20].

Based on the extension principle, the membership function $\mu_{\tilde{Z}}$ can be defined as:

$$\mu_{\tilde{Z}}(z) = \sup_{c,s,\delta,e} \min\{\mu_{\tilde{C}_{ijk}}(c_{ijk}), \mu_{\tilde{S}_i}(s_i), \mu_{\tilde{\Delta}_j}(\delta_j), \mu_{\tilde{E}_k}(e_k) \;\; \forall i,\ j,\ k \mid z = Z(c,\ s,\ \delta,\ e)\}, \tag{3.1}$$

where Z(c, s, δ,e) is defined in Model (2.1). If the α-cuts of $\tilde{Z}$ at all α values degenerate to the same point, and then the total transportation cost is a crisp number. Otherwise, it is a fuzzy number. In equation (3.1), several membership functions are involved. To derive $\mu_{\tilde{Z}}$ in closed form is hardly possible. According to (3.1), $\mu_{\tilde{Z}}$ is the minimum of $\mu_{\tilde{C}_{ijk}}, \mu_{\tilde{S}_i}, \mu_{\tilde{\Delta}_j}$ *and* $\mu_{\tilde{E}_k}$ $\forall\ i,\ j,\ k$. We need $\mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha,\ \mu_{\tilde{S}_i}(s_i) \geq \alpha, \mu_{\tilde{\Delta}_j}(\delta_j) \geq \alpha, \mu_{\tilde{E}_k}(e_k) \geq \alpha$, and atleast one $\mu_{\tilde{C}_{ijk}}(c_{ijk}),\ \mu_{\tilde{S}_i}(s_i), \mu_{\tilde{\Delta}_j}(\delta_j), \mu_{\tilde{E}_k}(e_k),\ \forall\, i, j, k$, equal to α such that z = Z(c,s,δ,e) to satisfy $\mu_{\tilde{Z}}(z) = \alpha.$ To find the membership function $\mu_{\tilde{Z}}$ it suffices to find the left shape function and right shape function of $\mu_{\tilde{Z}}$ which is equivalent to finding the lower bound Z^L_α and upper bound Z^U_α of the $\alpha-$cuts of $\tilde{Z}$. Since Z^L_α is the minimum of Z(c, s, δ, e) and Z^U_α is the maximum of Z(c, s, δ,e), they can be expressed as:

$$Z^L_\alpha = \min\{\, Z(c,s,\delta,e) \mid (C_{ijk})^L_\alpha \leq c_{ijk} \leq (C_{ijk})^U_\alpha, (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha,$$
$$(\Delta_j)^L_\alpha \leq \delta_j \leq (\Delta_j)^U_\alpha, (E_k)^L_\alpha \leq e_k \leq (E_k)^U_\alpha \;\;, \forall i, j, k\ \}$$

$$Z^U_\alpha = \max\{\;\; Z(c,s,\delta,e) \mid (C_{ijk})^L_\alpha \leq c_{ijk} \leq (C_{ijk})^U_\alpha, (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha,$$
$$(\Delta_j)^L_\alpha \leq \delta_j \leq (\Delta_j)^U_\alpha, (E_k)^L_\alpha \leq e_k \leq (E_k)^U_\alpha \;\;, \forall\ i, j, k\ \}.$$

Two Level Mathematical Programs

$$Z^L_\alpha = \min_{\substack{(C_{ijk})^L_\alpha \leq c_{ijk} \leq (C_{ijk})^U_\alpha \\ (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha \\ (\Delta_j)^L_\alpha \leq \delta_j \leq (\Delta_j)^U_\alpha \\ (E_k)^L_\alpha \leq e_k \leq (E_k)^U_\alpha, \forall i,\ j,\ k}} \begin{cases} \min\ \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m} \delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} C_{ijk} x_{ijk} \\ \textit{subject to} \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq s_i,\ \textit{for}\ i = 1,2,...,m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \geq \delta_j,\ \textit{for}\ j = 1,2,...,n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k,\ \textit{for}\ k = 1,2,...,l \\ x_{ijk} \geq 0 \end{cases} \tag{3.2}$$

$$Z_\alpha^U = \max_{\substack{(C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \forall i,j,k}} \left\{ \begin{array}{l} \min \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m} \delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} C_{ijk} x_{ijk} \\ subject \quad to \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \quad for \quad i = 1,2,\dots, m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \quad for \quad j = 1,2,\dots, n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \quad for \quad k = 1,2,\dots, l \\ x_{ijk} \ge 0 \end{array} \right. \qquad (3.3)$$

At least one c_{ijk}, s_i, δ_j or e_k must hit the boundary of their α-cuts to satisfy $\mu_{\tilde{Z}}(z) = \alpha.$ A necessary and sufficient condition for Model (3.2) and (3.3) to have feasible solutions is $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} \delta_j$ and $\sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} \delta_j$. In the first level of Model (3.2) and (3.3), s_i, δ_j and e_k are allowed to vary in the range of $\left[(S_i)_\alpha^L, (S_i)_\alpha^U\right], \left[(\Delta_j)_\alpha^L, (\Delta_j)_\alpha^U\right]$ and $\left[(E_k)_\alpha^L, (E_k)_\alpha^U\right]$ respectively. However, to ensure the transportation problem of the second level to be feasible, it is necessary that the constraint $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} \delta_j$ and $\sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} \delta_j$, be imposed in the first level. Hence, Model (3.2) and (3.3) becomes:

$$Z_\alpha^L = \min_{\substack{(C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \\ \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} \delta_j \\ \sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} \delta_j \\ \forall i,j,k}} \left\{ \begin{array}{l} \min \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m} \delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} C_{ijk} x_{ijk} \\ subject \ \ to \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \ for \ i = 1,2,\dots,m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \ for \ j = 1,2,\dots,n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \ for \ k = 1,2,\dots,l \\ x_{ijk} \ge 0 \end{array} \right. \qquad (3.4)$$

$$Z_\alpha^U = \max_{\substack{(C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U \\ \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} \delta_j \\ \sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} \delta_j \\ \forall i,j,k}} \left\{ \begin{array}{l} \min \sum_{i=1}^{m} c_i s_i + \sum_{i-1}^{m} \delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} C_{ijk} x_{ijk} \\ subject \quad to \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \ for \ i = 1,2,\dots,m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \ for \ j = 1,2,\dots,n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \ for \ k = 1,2,\dots,l \\ x_{ijk} \ge 0, \ \forall i,j,k. \end{array} \right. \qquad (3.5)$$

Model (3.4) and (3.5) will be infeasible for any α level if $\sum_{i=1}^{m} (S_i)_{\alpha=0}^U \le \sum_{j=1}^{n} (\Delta_j)_{\alpha=0}^L \le \sum_{k=1}^{l} (E_k)_{\alpha=0}^L$. In other words, a fuzzy transportation problem is feasible if the upper bound of the total fuzzy supply is greater than or equal to the lower bound of the total fuzzy demand. To derive the lower bound of the objective value in Model (3.4), we can directly set c_{ijk} to its lower bound $(C_{ijk})_\alpha^L \ \forall i,j,k$ to find the minimum objective value. Hence, Model (3.4) can be reformulated as:

$$Z_\alpha^L = \min_{\substack{(S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U \\ \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} \delta_j \\ \sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} \delta_j \\ \forall i,j,k}} \left\{ \begin{array}{l} \min \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m} \delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} C_{ijk} x_{ijk} \\ subject \quad to \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \quad for \quad i = 1,2,\dots, m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \quad for \quad j = 1,2,\dots, n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \quad for \quad k = 1,2,\dots, l \\ x_{ijk} \ge 0, \ \forall i,j,k. \end{array} \right. \qquad (3.6)$$

Since Model (3.6) is to find the minimum of all the minimum objective values, one can insert the constraints of level 1 into level 2 and simplify the two-level mathematical program to the conventional one-level program as follows:

$$Z_\alpha^L = \left\{ \begin{array}{l} \min \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m} \delta_i r_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} (C_{ijk})_\alpha^L x_{ijk} \\ subject \ to \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \ \ for \ i = 1,2,\dots,m \\ \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \delta_j, \ for \ j = 1,2,\dots,n \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \ for \ k = 1,2,\dots,l \\ \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} \delta_j \\ \sum_{j=1}^{n} \delta_j \ge \sum_{k=1}^{l} e_k \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, \ i = 1,2,\dots,m \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U, \ j = 1,2,\dots,n \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \ k = 1,2,\dots,l \\ x_{ijk} \ge 0, \ \ \forall i,j,k. \end{array} \right. \qquad (3.7)$$

This model is a linear program which can be solved easily. In this model, since all c_{ijk} have been set to the lower bounds of their α-cuts, that is, $\mu_{C_{ijk}}(C_{ijk}) = \alpha$ this assures $\mu_{\tilde{Z}}(Z) = \alpha$ as required by (3.1). To solve Model (3.5), the dual of the level 2 problem is formulated to become a maximization problem to be consistent with the maximization operation of level 1. It is well-known from the duality theorem of linear programming that the primal model and the dual model have the same objective value. Thus, Model (3.5) becomes:

$$Z_\alpha^{\ U} = \max_{\substack{(C_{jk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U \\ \sum_{i=1}^m s_i \ge \sum_{j=1}^n \delta_j \\ \sum_{k=1}^l e_k \ge \sum_{j=1}^n \delta_j \\ \forall\ i,j,k}} \begin{cases} \max\ -\sum_{i=1}^m s_i u_i + \sum_{j=1}^n \delta_j v_j + \sum_{k=1}^l e_k w_k \\ subject \quad to \\ -u_i + v_j - w_k \le c_{ijk} \\ \qquad for \quad i = 1,2,\ldots,\ m \\ \qquad for \quad j = 1,2,\ldots,\ n \\ \qquad for \quad k = 1,2,\ldots,\ l \\ u_i, v_j, w_k \ge 0 \\ \forall\ i,j,k, \end{cases} \tag{3.8}$$

since $(C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U\ ,\ \forall\, i,j,k.$ In Model (3.8), one can derive the upper bound of the objective value by setting c_{ijk} to its upper bound $(C_{ijk})_\alpha^U, \forall\, i,j,k$ because this gives the largest feasible region. Thus, we can reformulate model (3.8) as:

$$Z_\alpha^U = \max_{\substack{(S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U \\ \sum_{i=1}^m s_i \ge \sum_{j=1}^n \delta_j \\ \sum_{k=1}^l e_k \ge \sum_{j=1}^n \delta_j \\ \forall\ i,j,k}} \begin{cases} \max -\sum_{i=1}^m s_i u_i + \sum_{j=1}^n \delta_j v_j + \sum_{k=1}^l e_k w_k \\ subject \quad to \\ -u_i + v_j - w_k \le (C_{ijk})_\alpha^U \\ \qquad for\ i = 1,2,\ldots,\ m \\ \qquad for\ \ j = 1,2,\ldots,\ n \\ \qquad for\ k = 1,2,\ldots,\ l \\ u_i, v_j, w_k \ge 0 \\ \forall\, i,j,k. \end{cases} \tag{3.9}$$

Now, since both level 1 and level 2 perform the same maximization operation, their constraints can be combined to form the following one-level mathematical program:

$$\begin{cases} Z_\alpha^U \quad = \ \max -\sum_{i=1}^m s_i u_i + \sum_{j=1}^n \delta_j v_j + \sum_{k=1}^l e_k w_k \\ subject\ to \\ -u_i + v_j - w_k \le (C_{ijk})_\alpha^U \\ \qquad \sum_{i=1}^m s_i \ \ge \sum_{j=1}^n \delta_j \\ \qquad \sum_{k=1}^l e_k \ \ge \sum_{j=1}^n \delta_j \\ \qquad (S_i)_\alpha^L \ \le\ s_i \ \le\ (S_i)_\alpha^U,\ i = 1,\ldots,m \\ \qquad (\Delta_j)_\alpha^L \ \le\ \delta_j \ \le\ (\Delta_j)_\alpha^U,\ j = 1,\ldots,n \\ \qquad (E_k)_\alpha^L \ \le\ e_k \ \le\ (E_k)_\alpha^U,\ k = 1,\ldots,l \\ u_i, v_j, w_k \ge 0, \forall\, i,j,k. \end{cases} \tag{3.10}$$

This model is a linearly constrained nonlinear program. There are several effective and efficient methods for solving this problem. Similar to Model (3.7) since all c_{ijk} have been set to the upper bounds of their $\alpha-$cuts, that is, $\mu_{\tilde{c}_{ijk}}(c_{ijk}) = \alpha$ as required by (3.1). Problems (3.4) and (3.5) are assured to be feasible if the lower bound of the total fuzzy demand is smaller, than the upper bound of the total fuzzy supply, i.e.,

$$\sum_{j=1}^n (\Delta_j)_{\alpha=0}^L \ \le\ \sum_{i=1}^m (S_i)_{\alpha=0}^U \quad and \quad \sum_{j=1}^n (\Delta_j)_{\alpha=0}^L \ \le\ \sum_{k=1}^l (E_k)_{\alpha=0}^U .$$

If this condition is not satisfied, then the problem will be infeasible. In this case, a fictitious supply point *m + 1* with an amount of

$$s_{m+1} \ \ge\ \sum_{j=1}^n (\Delta_j)_{\alpha=0}^L \ -\ \sum_{i=1}^m (S_i)_{\alpha=0}^U \quad and$$

$$s_{m+1} \ \ge\ \sum_{j=1}^n (\Delta_j)_{\alpha=0}^L \ -\ \sum_{k=1}^l (E_k)_{\alpha=0}^U\ ,$$

just like the conventional transportation problem can be assumed to make the problem feasible. The amount to be shipped from the fictitious supply point is the shortage of that demand point.

For two possibility levels α_1 and α_2 such that $0 \ \le\ \alpha_2 \ \le\ \alpha_1 \le 1,$ the feasible regions defined by α_1 in Models (3.7) and (3.10) are smaller than those defined by α_2. Consequently $(Z)_{\alpha_1}^L \ \ge\ (Z)_{\alpha_2}^L$ and $(Z)_{\alpha_1}^U \ \ge\ (Z)_{\alpha_2}^U$: in other words, the left shape function is non decreasing and the right shape function is non increasing. This property, based on the definition of ‘‘convex fuzzy set’’ [20], assures the convexity of $\tilde{Z}$. If both Z_α^L and Z_α^U are invertible with respect to α, then a left shape function *L(z)* = $(Z_\alpha^L)^{(-1)}$ and a right shape function *R(z)* = $(Z_\alpha^U)^{(-1)}$ can be obtained. From *L(z)* and *R(z)*, the membership function $\mu_{\tilde{Z}}$ is constructed as:

$$\mu = \begin{cases} L(Z),\ (Z)_{\alpha=0}^L \le z \le (Z)_{\alpha=1}^L \\ 1, \qquad (Z)_{\alpha=1}^L \le z \le (Z)_{\alpha=1}^U \\ R(Z),\ (Z)_{\alpha=1}^U \le z \le (Z)_{\alpha=0}^U . \end{cases}$$

Robust location Transportation problem

The robust location transportation problem, denoted by $T_{rob}(\Gamma)$ is to choose the sources to open (with the r_i variables), and the amounts to store (with the s_i variables) such that the worst demand in the uncertainty set is satisfied with minimum cost. The robust problem is the following:

$$T_{rob}(\Gamma) \quad \begin{cases} \min\ \sum_{i=1}^m c_i s_i + \sum_{i=1}^m \delta_i r_i + opt\ (R(s,\Gamma)) \\ subject \quad to \\ \qquad s_i \ \le\ C_i r_i \\ \qquad \sum_{i=1}^m s_i \ \ge\ B \\ s_i \ \ge\ 0,\ r_i \in [0,1], \end{cases} \tag{4.1}$$

where B = $\sum_{j=1}^n \tilde{\delta}_j + \hat{\delta}_j$ and $opt\ (R(s,\Gamma))$ represents the optimal value of the recourse problem:

$$R(s,\Gamma) \begin{cases} \max_{\beta \in u(r)} \quad \min \quad \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l c_{ijk} x_{ijk} \\ \sum_{i=1}^m \sum_{k=1}^l x_{ijk} \le s_i,\ i = 1,\ldots,m \\ \sum_{j=1}^n \sum_{k=1}^l ijk \ge \delta_j,\ j = 1,\ldots,n \\ \sum_{i=1}^m \sum_{j=1}^n x_{ijk} \le e_k,\ k = 1,\ldots,l \\ i = 1,\ldots,m, \\ j = 1,\ldots,n, \\ k = 1,\ldots,l, \\ x_{ijk} \ge 0, \end{cases} \tag{4.2}$$

with the uncertainty set $u(\Gamma)$ is defined by

$$u(\Gamma) = \{\delta \in \Re^n : \delta_j = \tilde{\delta}_j + z_j \hat{\delta}_j,\ j = 1,...,n,\ \ z \in Z(\Gamma)\} \quad (4.3)$$

$$Z(\Gamma) = \{z \in \Re^n : \sum_{j=1}^{n} z_j \le \Gamma, 0 \le z_j \le 1,\ j = 1,...n\ \}. \quad (4.4)$$

The constraint $\sum_{i=1}^{m} s_i \ge B$ where B is given by (4.1) is due to the total recourse considered in our formulation, and the assumption of satisfaction of the greatest demands. At optimality $opt(R(s,\Gamma))$ represents the transportation cost for a fixed capacity level s and Γ worst deviations. Considering (4.2) and (4.3) rewrite recourse problem $R(s,\Gamma)$ as

$$R(s,\Gamma) = \begin{cases} \max \quad \begin{array}{l} \sum_{j=1}^{n} z_j \le \Gamma, \\ 0 \le z_j \le 1, \\ j = 1,...,n, \end{array} & \min \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk} x_{ijk} \\ & \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i,\ i = 1,...,m \\ & \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge \tilde{\delta}_j + \hat{\delta}_j z_j,\ j = 1,...,n \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k,\ k = 1,...,l \\ & i = 1,...,m, \\ & j = 1,...,n, \\ & k = 1,...,l \\ & x_{ijk} \ge 0. \end{cases}$$

By strong duality theorem, one can replace the minimization problem by its dual in the recourse problem as: (since the problem is feasible for all demands)

$$Q(s,\Gamma) \begin{cases} \max \ -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} \tilde{\delta}_j v_j + \sum_{j=1}^{n} \hat{\delta}_j v_j z_j - \sum_{k=1}^{l} e_k w_k \\ subject \quad to \\ \quad -u_i + v_j - w_k \le c_{ijk} \\ \quad \sum_{j=1}^{n} z_j \le \Gamma \\ \quad 0 \le z_j \le 1 \\ \quad u_i, v_j, w_k \ge 0,\ i = 1,...,m,\ j = 1,...,n,\ k = 1,...,l, \end{cases}$$

where u_i, v_j, w_k are dual variables. The recourse problem $Q(s,\Gamma)$ is of quadratic form. Now robust location transportation problem can be written as

$$T'_{rob}(\Gamma) \begin{cases} \min \sum_{i=1}^{m} c_i s_i + \sum_{i=1}^{m} \delta_i r_i + opt\ (Q(s,\Gamma)) \\ subject\ to \\ \quad s_i \ \le\ C_i r_i \\ \quad \sum_{i=1}^{m} s_i \ \ge\ B \\ s_i \ \ge\ 0,\ \ r_i \in [0,1], \end{cases}$$

which is the minimization of a convex function under linear constraints.

Numerical Examples

Example 1. To illustrate the proposed approach, consider a transportation problem with one fuzzy shipping cost, two fuzzy supplies, three fuzzy demands and two fuzzy conveyance capacities. Supply 1 and Demand 3 are trapezoidal fuzzy numbers and the remainders are triangular fuzzy numbers. The problem has the following form:

$$\begin{aligned} \tilde{Z} = \min\ & 20\ x_{111} + 40\ x_{112} + 60\ x_{121} + 60\ x_{122} + 50\ x_{131} + 30\ x_{132} + (70,80,90)\ x_{211} \\ & + 10\ x_{212} + 50\ x_{221} + 50\ x_{222} + 40\ x_{231} + 50\ x_{232} \end{aligned}$$

subject to

$$\begin{aligned} x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} &\le (40, 60, 70, 80) \\ x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} &\le (70, 90, 100) \\ x_{111} + x_{112} + x_{211} + x_{212} &\ge (40, 50, 70) \\ x_{121} + x_{122} + x_{221} + x_{222} &\ge (20, 30, 40) \\ x_{131} + x_{132} + x_{231} + x_{232} &\ge (10, 20, 30, 60) \\ x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} &\le (40, 50, 60) \\ x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} &\le (70, 90, 100) \end{aligned}$$

$$x_{111}, x_{112}, x_{121}, x_{122}, x_{131}, x_{132}, x_{211}, x_{212}, x_{221}, x_{222}, x_{231}, x_{232} \ge 0.$$

The total supply is $\tilde{S}_1 + \tilde{S}_2$ = (110,150,160,180) and the total demand is $\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3$ = (70,100,110,170), and the total conveyance capacity $E = \tilde{E}_1 + \tilde{E}_2$ = (110,150,170). Since $S \cap \Delta \cap E \ne \phi$. According to Models (3.7) and (3.10), the lower and upper bounds of $\tilde{Z}$ at possibility level α can be solved as:

$$\begin{aligned} Z_\alpha^L = \min\ & (s_1 c_1 + s_2 c_2) + (\delta_1 r_1 + \delta_2 r_2 + \delta_3 r_3) + 20\ x_{111} + 40\ x_{112} + 60\ x_{121} \\ & + 60\ x_{122} + 50\ x_{131} + 30\ x_{132} + (70, 80, 90)\ x_{211} + 10\ x_{212} \\ & + 50\ x_{221} + 50\ x_{222} + 40\ x_{231} + 50\ x_{232} \end{aligned}$$

subject to

$$\begin{aligned} x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} &\le s_1 \\ x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} &\le s_2 \\ x_{111} + x_{112} + x_{211} + x_{212} &\ge \delta_1 \\ x_{121} + x_{122} + x_{221} + x_{222} &\ge \delta_2 \\ x_{131} + x_{132} + x_{231} + x_{232} &\ge \delta_3 \\ x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} &\le e_1 \\ x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} &\le e_2 \\ s_1 + s_2 &\ge \delta_1 + \delta_2 + \delta_3 \\ e_1 + e_2 &\ge \delta_1 + \delta_2 + \delta_3 \\ 40 + 20\alpha \le s_1 &\le 80 - 10\alpha \\ 70 + 20\alpha \le s_2 &\le 100 - 10\alpha \\ 40 + 10\alpha \le \delta_1 &\le 70 - 20\alpha \\ 20 + 10\alpha \le \delta_2 &\le 40 - 10\alpha \\ 10 + 10\alpha \le \delta_3 &\le 60 - 30\alpha \\ 40 + 20\alpha \le e_1 &\le 70 - 10\alpha \\ 70 + 20\alpha \le e_2 &\le 100 - 10\alpha \end{aligned}$$

$$x_{111}, x_{112}, x_{121}, x_{122}, x_{131}, x_{132}, x_{211}, x_{212}, x_{221}, x_{222}, x_{231}, x_{232} \ge 0.$$

$$Z_\alpha^U = \max \; -s_1u_1 - s_2u_2 + \delta_1v_1 + \delta_2v_2 + \delta_3v_3 - e_1w_1 - e_2w_2$$

$$\begin{aligned}
&\textit{subject to}\\
&-u_1 + v_1 - w_1 \le 20\\
&-u_1 + v_1 - w_2 \le 40\\
&-u_1 + v_2 - w_1 \le 60\\
&-u_1 + v_2 - w_2 \le 60\\
&-u_1 + v_3 - w_1 \le 50\\
&-u_1 + v_3 - w_2 \le 30\\
&-u_2 + v_1 - w_1 \le (90 - 10\alpha)\\
&-u_2 + v_1 - w_2 \le 10\\
&-u_2 + v_2 - w_1 \le 50\\
&-u_2 + v_2 - w_2 \le 50\\
&-u_2 + v_3 - w_1 \le 40\\
&-u_2 + v_3 - w_2 \le 50\\
&s_1 + s_2 \ge \delta_1 + \delta_2 + \delta_3\\
&e_1 + e_2 \ge \delta_1 + \delta_2 + \delta_3\\
&40 + 20\alpha \le s_1 \le 80 - 10\alpha\\
&70 + 20\alpha \le s_2 \le 100 - 10\alpha\\
&40 + 10\alpha \le \delta_1 \le 70 - 20\alpha\\
&20 + 10\alpha \le \delta_2 \le 40 - 10\alpha\\
&10 + 10\alpha \le \delta_3 \le 60 - 30\alpha\\
&40 + 20\alpha \le e_1 \le 70 - 10\alpha\\
&70 + 20\alpha \le e_2 \le 100 - 10\alpha\\
&u_1, u_2, v_1, v_2, v_3, w_1, w_2 \ge 0.
\end{aligned}$$

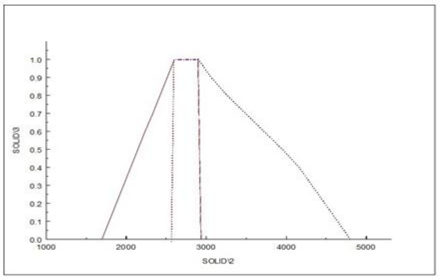

Fig. 1 Inequality Constraints

A mathematical programming solver Lingo [11] is used to solve the above mathematical programs. Table. 1 lists the α-cuts of the total transportation cost at eleven distinct α values: 0.1, 0.2,…, 1.0. The α-cut of $\tilde{Z}$ represents the possibility that the transportation cost will appear in the associated range. Specifically, the α = 1.0 cut shows the total transportation cost that is most likely to be and the α = 0 cut shows the range that the total transportation cost could appear. In this example, while the total transportation cost is fuzzy, it's most likely value falls between 2600 and 2900, and its value is impossible to fall outside the range of 1700 and 4800. The curve labeled ''Inequality-constraints'' in Fig. 1 is the membership function $\mu_{\tilde{Z}}$ of this example.

For the $\alpha = 0$ cut of $\tilde{Z}$, the lower bound of Z* = 1700 occurs at $x^*_{132} = 10$, $x^*_{212} = 40$, $x^*_{222} = 20$ with $\delta_1 = 40$, $\delta_2 = 20$, $\delta_3 = 10$, $s_1 = 80$, $s_2 = 100$, $e_1 = 70$ and the other decision variable are 0. The upper bound of Z*=4800 occurs at $u_1 = 0$, $u_2 = 0$ and $v_1 = 20$, $v_2 = 50$, $v_3 = 40$ with $\delta_1 = 70$, $\delta_2 = 40$, $\delta_3 = 60$, $e_1 = 70$, $e_2 = 100.$ At the other extreme end of $\alpha = 1$, the lower bound of Z* =2600 occurs at $x^*_{132} = 20, x^*_{212} = 50, x^*_{222} = 20$ with $\delta_1 = 50, \delta_2 = 30, \delta_3 = 20, e_1 = 60, e_2 = 90, s_1 = 70, s_2 = 90,$ and the other decision variables are 0. The upper bound of Z*=2900 occurs at $s_1 = 50$, $s_2 = 90$, $\delta_1 = 50$, $\delta_2 = 30$, $\delta_3 = 30$, $e_1 = 60$, $e_2 = 90$ and the other decision variables are 0. Notably, the values of the decision variables derived in this example are also fuzzy.

Equality Constraints

In the preceding section, the transportation model being considered has inequality constraints. The total supply must be greater than or equal to the total demand to assure feasibility. In this section, we discuss the transportation model with equality constraints:

$$\begin{cases}
Z = \min \sum_{i=1}^{m} \sum_{j=i}^{n} \sum_{k=1}^{l} c_{ijk} x_{ijk} \\
\textit{subject to} \\
\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = s_i, \quad \textit{for} \;\; i = 1, 2, \dots, m \\
\sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = \delta_j, \quad \textit{for} \;\; j = 1, 2, \dots, n \\
\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = e_k, \quad \textit{for} \;\; k = 1, 2, \dots, l \\
x_{ijk} \ge 0, \quad \forall \; i.j.k.
\end{cases} \tag{5.1}$$

This model is feasible if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} \delta_j, \; \sum_{k=1}^{l} e_k = \sum_{j=1}^{n} \delta_j.$$

When the shipping costs, supplies, and demands are not known exactly, we have the following fuzzy solid transportation problem:

$$\begin{cases}
\tilde{Z} = \min \sum_{i=1}^{m} \sum_{j=i}^{n} \sum_{k=1}^{l} \tilde{C}_{ijk} x_{ijk} \\
\textit{subject to} \\
\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = \tilde{S}_i, \; \textit{for} \; i = 1,2,\dots,m \\
\sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = \tilde{\Delta}_j, \; \textit{for} \; j = 1,2,\dots,n \\
\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = \tilde{E}_k, \; \textit{for} \; k = 1,2,\dots,l \\
x_{ijk} \ge 0, \quad \forall \; i.j.k.
\end{cases} \tag{5.2}$$

Similar to the discussion of the inequality-constraint case, the lower and upper bounds of $\tilde{Z}$ at possibility level α can be solved from the following pair of two-level mathematical programs:

$$Z_\alpha^L = \min_{\substack{(C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \\ \sum_{i=1}^m s_i = \sum_{j=1}^n \delta_j \ \forall\ i,j \\ \sum_{k=1}^l e_k = \sum_{j=1}^n \delta_j \ \forall\ j,k}} \left\{ \begin{array}{l} \min \sum_{i=1}^m c_i s_i + \sum_{i=1}^m \delta_i r_i + \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l c_{ijk} x_{ijk} \\ subject \quad to \\ \sum_{j=1}^n \sum_{k=1}^l x_{ijk} = s_i, \quad for \quad i = 1, 2, \ldots, m \\ \sum_{i=1}^m \sum_{k=1}^l x_{ijk} = \delta_j, \quad for \quad j = 1, 2, \ldots, n \\ \sum_{i=1}^m \sum_{j=1}^n x_{ijk} = e_k, \quad for \quad k = 1, 2, \ldots, l \\ \quad x_{ijk} \ge 0. \end{array} \right. \tag{5.3}$$

$$Z_\alpha^U = \max_{\substack{(C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \\ \sum_{i=1}^m s_i = \sum_{j=1}^n \delta_j \ \forall\ i,j \\ \sum_{k=1}^l e_k = \sum_{j=1}^n \delta_j \ \forall\ j,k}} \left\{ \begin{array}{l} \min \sum_{i=1}^m c_i s_i + \sum_{i=1}^m \delta_i r_i + \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l c_{ijk} x_{ijk} \\ subject \quad to \\ \sum_{j=1}^n \sum_{k=1}^l x_{ijk} = s_i, \quad for \quad i = 1, 2, \ldots, m \\ \sum_{i=1}^m \sum_{k=1}^l x_{ijk} = \delta_j, \quad for \quad j = 1, 2, \ldots, n \\ \sum_{i=1}^m \sum_{j=1}^n x_{ijk} = e_k, \quad for \quad k = 1, 2, \ldots, l \\ \quad x_{ijk} \ge 0. \end{array} \right. \tag{5.4}$$

The corresponding pair of one-level mathematical program is

$$Z_\alpha^L = \left\{ \begin{array}{l} \min \sum_{i=1}^m c_i s_i + \sum_{i=1}^m \delta_i r_i + \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l (C_{ijk})_\alpha^L x_{ijk} \\ subject\ to \\ \sum_{j=1}^n \sum_{k=1}^l x_{ijk} = s_i, \quad for\ i = 1,2,\ldots,m \\ \sum_{i=1}^m \sum_{k=1}^l x_{ijk} = \delta_j, \ for\ j = 1,2,\ldots,n \\ \sum_{i=1}^m \sum_{j=1}^n x_{ijk} = e_k, \ for\ k = 1,2,\ldots,l \\ \sum_{i=1}^m s_i = \sum_{j=1}^n \delta_j \\ \sum_{j=1}^n \delta_j = \sum_{k=1}^l e_k \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, \ i = 1,2,\ldots,m \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U, \ j = 1,2,\ldots,n \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \ k = 1,2,\ldots,l \\ \quad x_{ijk} \ge 0, \ \forall\ i, j, k. \end{array} \right. \tag{5.5}$$

$$\left\{ \begin{array}{l} Z_\alpha^U = \max \sum_{i=1}^m s_i u_i + \sum_{j=1}^n \delta_j v_j - \sum_{k=1}^l e_k w_k \\ subject\ to \\ u_i + v_j - w_k \le (C_{ijk})_\alpha^U \\ \sum_{i=1}^m s_i = \sum_{j=1}^n \delta_j \\ \sum_{k=1}^l e_k = \sum_{j=1}^n \delta_j \\ (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, \ i = 1,\ldots,m \\ (\Delta_j)_\alpha^L \le \delta_j \le (\Delta_j)_\alpha^U, \ j = 1,\ldots,n \\ (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \ k = 1,\ldots,l \\ u_i, v_j, w_k \ge 0, \ \forall\ i, j, k. \end{array} \right. \tag{5.6}$$

Example 2. Suppose the inequality constraints are replace by equality constraints:

$$\begin{aligned} \tilde{Z} = \min\ & 20x_{111} + 40x_{112} + 60x_{121} + 60x_{122} + 50x_{131} \\ & + 30x_{132} + (70, 80, 90)x_{211} + 10x_{212} + 50x_{221} \\ & + 50x_{222} + 40x_{231} + 50x_{232} \end{aligned}$$

$subject \quad to$

$$\begin{aligned} & x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} = (40,60,70,80) \\ & x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} = (70,90,100) \\ & x_{111} + x_{112} + x_{211} + x_{212} = (40, 50, 70) \\ & x_{121} + x_{122} + x_{221} + x_{222} = (20, 30, 40) \\ & x_{131} + x_{132} + x_{231} + x_{232} = (10, 20, 30, 60) \\ & x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} = (40, 50, 60) \\ & x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} = (70, 90, 100) \\ & x_{111}, x_{112}, x_{121}, x_{122}, x_{131}, x_{132}, x_{211}, x_{212}, x_{221}, x_{222}, x_{231}, x_{232} \ge 0. \end{aligned}$$

Based on Model (5.5) and (5.6) the lower and upper bounds of the α -cut of $\tilde{Z}$ can be derived by solving the following pair of mathematical programs:

$$\begin{aligned} Z_\alpha^L = \min(s_1c_1 + s_2c_2) + (\delta_1r_1 + \delta_2r_2 + \delta_3r_3) + 20x_{111} + 40x_{112} + 60x_{121} \\ + 60x_{122} + 50x_{131} + 30x_{132} + (70,80,90)x_{211} + 10x_{212} \\ + 50x_{221} + 50x_{222} + 40x_{231} + 50x_{232} \end{aligned}$$

$subject \quad to$

$$\begin{aligned} & x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} = s_1 \\ & x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} = s_2 \\ & x_{111} + x_{112} + x_{211} + x_{212} = \delta_1 \\ & x_{121} + x_{122} + x_{221} + x_{222} = \delta_2 \\ & x_{131} + x_{132} + x_{231} + x_{232} = \delta_3 \\ & x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} = e_1 \\ & x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} = e_2 \\ & s_1 + s_2 = \delta_1 + \delta_2 + \delta_3 \\ & e_1 + e_2 = \delta_1 + \delta_2 + \delta_3 \\ & 40 + 20\alpha \le s_1 \le 80 - 10\alpha \\ & 70 + 20\alpha \le s_2 \le 100 - 10\alpha \\ & 40 + 10\alpha \le \delta_1 \le 70 - 20\alpha \\ & 20 + 10\alpha \le \delta_2 \le 40 - 10\alpha \\ & 10 + 10\alpha \le \delta_3 \le 60 - 30\alpha \\ & 40 + 20\alpha \le e_1 \le 70 - 10\alpha \\ & 70 + 20\alpha \le e_2 \le 100 - 10\alpha \\ & x_{111}, x_{112}, x_{121}, x_{122}, x_{131}, x_{132}, x_{211}, x_{212}, x_{221}, x_{222}, x_{231}, x_{232} \ge 0. \end{aligned}$$

$$Z_\alpha^U = \max\ -s_1u_1 - s_2u_2 + \delta_1v_1 + \delta_2v_2 + \delta_3v_3 - e_1w_1 - e_2w_2$$

$subject \quad to$

$$\begin{aligned} & -u_1 + v_1 - w_1 \le 20 \\ & -u_1 + v_1 - w_2 \le 40 \\ & -u_1 + v_2 - w_1 \le 60 \\ & -u_1 + v_2 - w_2 \le 60 \\ & -u_1 + v_3 - w_1 \le 50 \\ & -u_1 + v_3 - w_2 \le 30 \\ & -u_2 + v_1 - w_1 \le (90 - 10\alpha) \\ & -u_2 + v_1 - w_2 \le 10 \\ & -u_2 + v_2 - w_1 \le 50 \\ & -u_2 + v_2 - w_2 \le 50 \\ & -u_2 + v_3 - w_1 \le 40 \\ & -u_2 + v_3 - w_2 \le 50 \\ & s_1 + s_2 = \delta_1 + \delta_2 + \delta_3 \\ & e_1 + e_2 = \delta_1 + \delta_2 + \delta_3 \\ & 40 + 20\alpha \le s_1 \le 80 - 10\alpha \\ & 70 + 20\alpha \le s_2 \le 100 - 10\alpha \\ & 40 + 10\alpha \le \delta_1 \le 70 - 20\alpha \\ & 20 + 10\alpha \le \delta_2 \le 40 - 10\alpha \\ & 10 + 10\alpha \le \delta_3 \le 60 - 30\alpha \\ & 40 + 20\alpha \le e_1 \le 70 - 10\alpha \\ & 70 + 20\alpha \le e_2 \le 100 - 10\alpha \\ & u_1, u_2, v_1, v_2, v_3, w_1, w_2 \ge 0. \end{aligned}$$

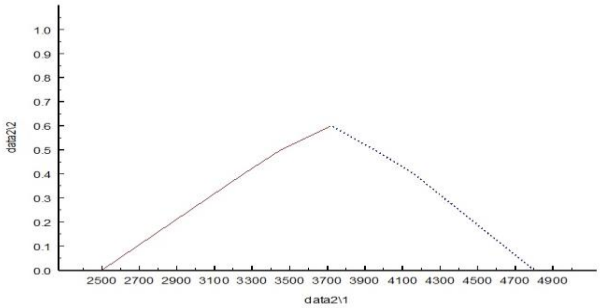

Fig.2 Equality Constraints

Table 2 lists the bounds of the total transportation cost at eleven α-cuts. The curve labeled Equality constraints in Fig. 2 is the membership function $\mu_{\tilde{Z}}$ of this example. For α greater than 0.6, the problem is infeasible. In other words, the maximum degree to which the constraints could be satisfied is equal to 0.6. It is also worthwhile to note that the membership function of the objective value of this example is contained in that of Example 1. The reason is simply because equality constraints are more restrictive than inequality constraints.

At $\alpha = 0$, the lower bound of the objective value is 2500, occurring at $x_{111} = 20$, $x_{132} = 20$, $x_{212} = 50$, $x_{221} = 20$ with $\delta_1 = 70$, $\delta_2 = 20$, $\delta_3 = 20$ and $s_1 = 80$, $s_2 = 90$, $E_1 = 40$, $E_2 = 70$. The upper bound is 4800, which occurs at $\delta_1 = 70$, $\delta_2 = 40$, $\delta_3 = 60$ and $E_1 = 70$, $E_2 = 100$. At $\alpha = 0.6$, the α-cut is a single point 3720.

Conclusion:

Transportation models have wide application in logistics and supply chain for reducing the cost. The solid transportation problem considers not only the supply and demand but also the conveyance capacity to satisfy the transportation requirements in a cost effective manner. This paper develops a procedure to derive the robust fuzzy transportation problem under certain demands to find the membership function of the fuzzy total transportation costs. The idea is based on the extension principle. In robust optimizations a part of the decisions must be taken before knowing the real values of the uncertain parameters and another part, called recourse decision is taken when the information is known. In this paper, we are interested in a robust version of the location fuzzy transportation problem with an uncertain demand using a 2-stage formulation. Two different types of the robust fuzzy transportation problem are discussed: one with inequality constraints and the other with equality constraint.

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Table. 1 The α -cuts of the transportation cost at 11 α values for Example 1

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z_α^L	1700	1790	1880	1970	2060	2150	2240	2330	2420	2510	2600
Z_α^U	4800	4640	4480	4320	4160	3950	3720	3490	3260	3060	2900

Table. 2 The α -cuts of the transportation cost at 11, α values for Example 2.

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z_α^L	2500	2690	2880	3070	3260	3450	3270	*inf*	*inf*	*inf*	*inf*
Z_α^U	4800	4640	4480	4320	4160	3950	3720	*inf*	*inf*	*inf*	*inf*