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Existence results for some partial neutral functional integrodifferential equations with state-dependent delay via fractional operators

Nadaf Nabisab Yamanurasab[1] and M. Mallika Arjunan[2]

[1]Department of Mathematics, Anjuman Engineering College, Bhatkal-581320, Karnataka, India.

[2]Department of Mathematics, Karunya University, Karunya Nagar, Coimbatore-641 114, Tamil Nadu, India.

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ABSTRACT

This paper is mainly concerned with the existence of mild solutions for some partial neutral functional integrodifferential equations with state-dependent delay in Banach spaces. The results are obtained by using resolvent operators and Krasnoselski-Schaefer's type fixed point theorem.

Introduction

In this paper, we prove the existence of mild solutions for the following first order partial neutral functional integrodifferential equations with state–dependent delay:

$$\frac{d}{dt}[x(t)-g(t,x_t)] = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x(t-\rho(x(t)))),$$

$$t \in J = [0,b], \qquad (1.1)$$

$$x(t) = \phi(t), t \in [-r,0], \qquad (1.2)$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in a Banach space X, $B(t)$, $t \in J$ is a bounded linear operators, $g: J \times C \to X$ and $f: J \times X \to X$ are given functions. Here $C = C([-r,0],X)$ is a Banach space of all continuous functions $\phi: [-r,0] \to X$ endowed with the norm $\|\phi\| = \sup\{\phi(\theta): -r \le \theta \le 0\}$. Also, for $x \in C([-r,b], X)$, we have $x_t \in C$ for $t \in [0,b]$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$. Here $x_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time, ρ is a positive bounded continuous function on X and r is the maximal delay defined by

$$r = \sup_{x \in X} \rho(x).$$

The nonlinear integrodifferential equation with resolvent operators served as an abstract formulation of partial integrodifferential equations which arises in many physical phenomena [12, 13, 30]. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. However, the resolvent operator does not satisfy the semigroup properties, see for instance [8, 26].

Functional differential equations with state–dependent delay appear frequently in applications as models of equations and for this reason the study of this type of equation has received great attention in last few years, see for instance [2, 3, 7, 10, 17, 18, 19, 20, 23] and the references therein. For more details on differential equations with state–dependent delay, we refer the reader to the handbook by Canada et al.[6].

On the other hand, neutral differential equations arise in many areas of applied mathematics and for this reason these equations have been investigated extensively in the last few decades. There are many contributions relative to this topic and we refer the reader to [21, 22, 24], the handbook [31]. The literature related to ordinary neutral differential equations is very extensive, thus, we refer the reader to [16] only, which contains a comprehensive description of such equations. Similarly, for more on partial neutral functional differential equations and related issues, we refer to Adimy and Ezzinbi [1], Hale [15], Wu et al.[31] for finite delay equations, and Hernandez et al. [21] for unbounded delay.

Recently, much attention has been paid to the existence results for the partial functional differential equations with state–dependent delay such as [22, 23, 24, 32] and the references therein. Motivated by the works [26, 28], the main aim of this paper is to establish some existence results for the problem $(1.1)-(1.2)$ by using resolvent operators and Krasnoselski – Schaefer's type fixed point theorem with semigroup theory. Our main results can be seen as a generalization of the works in [8, 26, 28] and the above mentioned partial functional

Tele:

E-mail addresses: arjunphd07@yahoo.co.in

differential equations with state–dependent delay.

This paper is organized as follows. In section 2, we recall some notations, definitions and some Lemmas which are used throughout this paper. In section 3, we use the Krasnoselski-Schaefer's type fixed point theorem to prove the existence of mild solutions for the problem $(1.1)-(1.2)$.

Preliminaries

In this section, we give some notations, definitions and some results on resolvent operator that will be used to develop the main results.

Let $(X,\|\cdot\|)$ be the Banach space, the notation $L(X,Y)$ stands for the Banach space of all linear bounded operators from X into Y, and we abbreviate this notation to $L(X)$ when $X=Y$. R(t), $t>0$ is compact, analytic resolvent operator generated by A.

Assume that

$(\mathbf{A_1})$ A is a densely defined, closed linear operator in a Banach space $(X,\|\cdot\|)$ and generates a C_0 −semigroup $T(t)$. Hence $D(A)$ endowed with graph norm $|x|=\|x\|+\|Ax\|$ is a Banach space which will be denoted by $(Y,\|\cdot\|)$.

$(\mathbf{A_2})$ $\{B(t): t\in J\}$ is a family of continuous linear operators from $(Y,\|\cdot\|)$ into $(X,\|\cdot\|)$. Moreover, there is an integrable function $c:[0,b]\to R^+$ such that for each $y\in Y$, the map $t\to B(t)y$ belongs to $W^{1,1}(J,X)$ and

$$\left\|\frac{d}{dt}B(t)y\right\|\le c(t)|y|,\quad y\in Y,\qquad t\in J.$$

Definition 2.1 A family $\{R(t): t\ge 0\}$ of continuous linear operators on X is called a resolvent operator for

$$\frac{dx}{dt}=Ax(t)+\int_0^t B(t-s)x(s)ds,$$

if:

(R_1) $R(0)=I$(the identity operator on X),

(R_2) For all $x\in X$, the map $t\to R(t)x$ is continuous from J to X,

(R_3) For all $t\in J$, $R(t)$ is continuous linear operator on Y, and for all $y\in Y$, the map $t\to R(t)y$ belongs to $C(J,Y)\cap C'(J,X)$ and satisfies

$$\frac{d}{dt}R(t)y=AR(t)y+\int_0^t B(t-s)R(s)yds$$
$$=R(t)Ay+\int_0^t R(t-s)B(s)yds.$$

To prove the main results, we need the following theorem.

Theorem 2.1([26]) Let the assumptions (A_1) and (A_2) be satisfied. Then there exists a constant $H=H(b)$ such that

$\|R(t+h)-R(h)R(t)\|_{L(X)}\le Hh$, for all $0\le h\le t\le b$,

where $L(X)$ denotes the Banach space of continuous linear operators on X.

Next, if the C_0 −semigroup $T(\cdot)$ generated by A is compact (that is, T (t) is a compact operator for all t >0), then the corresponding resolvent operator $R(\cdot)$ is also compact (that is, R(t) is a compact operator for all t >0) and is operator norm continuous (or continuous in the uniform operator topology) for t >0.

Next, we give the basic definitions and lemmas which are used throughout this paper.

Let $A: D(A)\to X$ be the infinitesimal generator of a compact, analytic resolvent operator $R(t)$, $t\ge 0$. Let $0\in\rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0<\alpha\le 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|x\|_\alpha=\|(-A)^\alpha x\|,\ x\in D(-A)^\alpha$$

defines a norm on $D(-A)^\alpha$.

Furthermore, we have the following properties appeared in [29].

Lemma 2.1 The following properties hold.

(i) If $0<\beta<\alpha\le 1$, then $X_\alpha\subset X_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.

(ii) For every $0<\alpha\le 1$ there exists $C_\alpha>0$ such that

$$\|(-A)^\alpha R(t)\|\le\frac{C_\alpha}{t^\alpha},\qquad 0<t\le b.$$

Theorem 2.2 ([5]) Let F_1, F_2 be two operators satisfying:

(a) F_1 is contraction, and

(b) F_2 is completely continuous.

Then either

(i) the operator equation $F_1x+F_2x=x$ has a solution, or

(ii) the set $\mathcal{E}=\left\{x\in X: \lambda F_1\left(\frac{x}{\lambda}\right)+\lambda F_2x=x\right\}$ is unbounded for $\lambda\in(0,1)$.

Lemma 2.2 [21] Let $v(\cdot)$, $w(\cdot):[0,b]\to[0,\infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta>0$, $0<\alpha<1$ such that

$$v(t)\le w(t)+\theta\int_0^t\frac{v(s)}{(t-s)^{1-\alpha}}ds,$$

$t\in J$,

then

$$v(t)\le e^{\theta^n\Gamma(\alpha)^n t^{n\alpha}}/\Gamma(n\alpha)\sum_{j=0}^{n-1}\left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t\in[0,b]$ and every $n\in N$ such that $n\alpha>1$, and $\Gamma(\cdot)$ is the Gamma function.

Existence Results

In this section, we shall present and prove our main results. First, we define the mild solution for the problem $(1.1)-(1.2)$.

Definition 3.1 A function $x:[-r,\ b]\to X$ is called a mild solution of the problem $(1.1)-(1.2)$ if $x(t)=\phi(t)$ on $t\in[-r,0]$, the restriction of $x(\cdot)$ to the interval J is continuous, and for each $s \in [0,t)$, the function $AR(t-s)g(s,x_s)$, is integrable and the integral equation

$$x(t) = R(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AR(t-s)g(s,x_s)\,ds + \int_0^t R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)\,d\tau ds + \int_0^t R(t-s)\; f(s,x(s-\rho(x(s))))ds,$$

$t \in J$

is satisfied.

In order to prove the main results, we list the following hypotheses.

(H_1) ([see lemma 2.1]) A is the infinitesimal generator of a compact analytic resolvent operator $R(t), t > 0$ and $0 \in \rho(A)$ such that

$\|R(t)\| \le M_1$, $\|B(t)\| \le M_2$ for all $t \ge 0$ and $\|(-A)^{1-\beta}R(t-s)\| \le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}$, $0 < t \le b$.

(H_2) There exist constants $0 < \beta < 1, C_0, c_1, c_2, L_g$ such that g is X_β-valued, $(-A)^\beta g$ is continuous, and

(i) $\|(-A)^\beta g(t,x)\| \le c_1\|x\|_C + c_2$, $t \in J$, $x \in C$,

(ii) $\|(-A)^\beta g(t,x_1) - (-A)^\beta g(t,x_2)\| \le L_g\|x_1 - x_2\|_C$, $t \in J$, $x_i \in C$, $i =1, 2$ with $\|(-A)^{-\beta}\| = M_0$ and

$$C_0 = L_g\left\{M_0 + \frac{C_{1-\beta}b^\beta}{\beta}\right\} < 1.$$

(H_3) The function $f: J \times X \to X$ is Caratheodory that means that f is measurable with respect to the first argument and continuous with respect to the second argument.

(H_4) There exists a function $p \in L^1(J, R^+)$ and a continuous nondecreasing function $\psi: [0,\infty) \to [0,\infty)$ such that $|f(t,u)| \le p(t)\psi(\|u\|)$, for every $t \in J$, and for each $u \in X$ with

$$B_0K_3\int_0^b p(s) < \int_{B_0K_1}^\infty \frac{ds}{\psi(s)},$$

where

$$K_1 = \frac{1}{M}\left[M_1(\|\phi(0)\| + \|g(0,\phi)\|) + M_0c_2(1 + M_1b^2M_2) + \frac{c_2C_{1-\beta}b^\beta}{\beta}\right],$$

$$K_2 = \frac{c_1C_{1-\beta}}{M},\quad K_3 = \frac{M_1}{M},$$

$$B_0 = e^{K_2^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)}\sum_{j=0}^{n-1}\left(\frac{K_2b^\beta}{\beta}\right)^j K_1,$$

and $M = 1 - c_1M_0 - c_1M_1M_2b^2M_0$.

Theorem 3.1 If the assumptions $(H_1) - (H_4)$ are satisfied. Then the problem$(1.1) - (1.2)$ has at least one mild solution on $[-r,b]$.

Proof: Transform the problem $(1.1) - (1.2)$ into a fixed point problem. Consider the operator $F: C([-r,b],X) \to C([-r,b],X)$ defined by

$$F(x)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ R(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AR(t-s)g(s,x_s)\,ds \\ \quad + \int_0^t R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)\,d\tau ds \\ \quad + \int_0^t R(t-s)\; f(s,x(s-\rho(x(s))))ds, & t \in J. \end{cases}$$

From hypothesis (H_1), (H_2) and Lemma 2.1, the following inequality holds:

$$\|AR(t-s)g(s,x_s)\| \le \|(-A)^{1-\beta}R(t-s)\|\|(-A)^\beta g(s,x_s)\|$$

$$\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}[c_1\|x_s\|_C + c_2].$$

Then from Bochner theorem [33], it follows that $AR(t-s)g(s,x_s)$ is integrable on $[0,t)$.

Now we decompose F as $F = F_1 + F_2$ where

$$F_1x(t) = \begin{cases} 0, & t \in [-r,0], \\ -R(t)g(0,\phi) + g(t,x_t) + \int_0^t AR(t-s)g(s,x_s)\,ds, & t \in J, \end{cases}$$

and

$$F_2x(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ R(t)\phi(0) + \int_0^t R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)\,d\tau ds \\ \quad + \int_0^t R(t-s)\; f(s,x(s-\rho(x(s))))ds, & t \in J. \end{cases}$$

We claim that the operators F_1 and F_2 satisfy all the conditions of the Theorem 2.2 on $[-r,b]$.

First we show that F_1 is contraction on $C([-r,b],X)$.

Let $x, y \in X$. From hypothesis (H_1) and (H_2), we have

$$\|F_1x(t) - F_1y(t)\| \le \|g(t,x_t) - g(t,y_t)\| + \left\|\int_0^t AR(t-s)[g(s,x_s) - g(s,y_s)]\,ds\right\|$$

$$\le \|g(t,x_t) - g(t,y_t)\| + \left\|\int_0^t AR(t-s)[g(s,x_s) - g(s,y_s)]\right\| ds$$

$$\le \|(-A)^{-\beta}\|L_g\|x_t - y_t\|_C + L_g\|x_t - y_t\|_C\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}ds$$

$$\le M_0L_g\|x_t - y_t\|_C + L_g\|x_t - y_t\|_C\frac{C_{1-\beta}b^\beta}{\beta}$$

$$\le L_g\left\{M_0 + \frac{C_{1-\beta}b^\beta}{\beta}\right\}\|x_t - y_t\|_C$$

Taking supremum over t,

$$\|F_1x - F_2y\| \le C_0\|x - y\|,$$

where $C_0 = L_g\left\{M_0 + \frac{C_{1-\beta} b^\beta}{\beta}\right\} < 1.$

This shows that F_1 is contraction on $C([-r,b],X)$.

Next, we show that the operator F_2 is completely continuous on $C([-r,b],X)$. First we prove that F_2 maps bounded sets into bounded sets in $C([-r,b],X)$.

It is enough to show that for any $q > 0$ there exists a positive constant δ such that for each $x \in B_q = \{x \in C([-r,b],X): \|x\|_C \le q\}$, we have $F_2(x) \in B_\delta$. For each $t \in J$, we have

$$F_2x(t) = R(t)\phi(0) + \int_0^t R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)\,d\tau ds + \int_0^t R(t-s)\ f(s,x(s-\rho(x(s))))ds$$

Then

$$\|F_2x(t)\| \le \|R(t)\|\|\phi(0)\| + \int_0^t \|R(t-s)\|\int_0^s \|B(s-\tau)\|\|g(\tau,x_\tau)\|d\tau ds + \int_0^t \|R(t-s)\|\|f(s,x(s-\rho(x(s))))\|\,ds.$$

From hypothesis $(H_1) - (H_4)$, we have

$$\|F_2x(t)\| \le M_1\|\phi(0)\| + M_0M_1M_2b\int_0^t [c_1q + c_2]ds + M_1\int_0^t p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds$$

$$\le M_1\|\phi(0)\| + M_0M_1M_2b^2[c_1q + c_2] + M_1\psi(q)\int_0^t p(s)\,ds = \delta.$$

Thus for each $x \in B_q$, we have $\|F_2x(t)\|_C \le \delta$.

Next, we show that F_2 maps bounded sets into equicontinuous sets in $C([-r,b],X)$.

Let $0 < s_1 < s_2 \le b$, for each $x \in B_q = \{x \in C([-r,b],X): \|x\|_C \le q\}$ and let $\epsilon > 0$ be given. Now, let $s_1, s_2 \in J$ with $s_2 > s_1$.

Then, we have

$$\|F_2x(s_1) - F_2x(s_2)\| \le \|R(s_1) - R(s_2)\|\|\phi(0)\|$$

$$+\left\|\int_0^{s_1-\epsilon}[R(s_1-s) - R(s_2-s)]\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds\right\|$$

$$+\left\|\int_{s_1-\epsilon}^{s_1}[R(s_1-s) - R(s_2-s)]\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds\right\|$$

$$+\left\|\int_{s_1}^{s_2} R(s_2-s)\int_0^s B(s-\tau)g(\tau,x_\tau)\,d\tau ds\right\|$$

$$+\left\|\int_0^{s_1-\epsilon}[R(s_1-s) - R(s_2-s)]f(s,x(s-\rho(x(s))))ds\right\|$$

$$+\left\|\int_{s_1-\epsilon}^{s_1}[R(s_1-s) - R(s_2-s)]f(s,x(s-\rho(x(s))))ds\right\|$$

$$+\left\|\int_{s_1}^{s_2} R(s_2-s)\,f(s,x(s-\rho(x(s))))ds\right\|$$

$$\le \|R(s_1) - R(s_2)\|\|\phi(0)\|$$

$$+ M_0M_2b\int_0^{s_1-\epsilon}\|R(s_1-s) - R(s_2-s)\|_{L(X)}[c_1\|x_\tau\|_C + c_2]ds$$

$$+ M_0M_2b\int_{s_1-\epsilon}^{s_1}\|R(s_1-s) - R(s_2-s)\|_{L(X)}[c_1\|x_\tau\|_C + c_2]ds + M_0M_1M_2b\int_{s_1}^{s_2}[c_1\|x_\tau\|_C + c_2]ds$$

$$+ \int_0^{s_1-\epsilon}\|R(s_1-s) - R(s_2-s)\|_{L(X)}\, p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds$$

$$+ \int_{s_1-\epsilon}^{s_1}\|R(s_1-s) - R(s_2-s)\|_{L(X)}p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds$$

$$+ M_1\int_{s_1}^{s_2} p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds$$

$$\le$$

$$+M_0M_2b\int_0^{s_1-\epsilon}\|R(s_1-s) - R(s_2-s)\|_{L(X)}[c_1q + c_2]ds$$

$$+M_0M_2b\int_{s_1-\epsilon}^{s_1}\|R(s_1-s) - R(s_2-s)\|_{L(X)}[c_1q + c_2]ds$$

$$+M_0M_1M_2b\int_{s_1}^{s_2}[c_1q + c_2]ds$$

$$+ \psi(q)\int_0^{s_1-\epsilon}\|R(s_1-s) - R(s_2-s)\|_{L(X)}\, p(s)ds$$

$$+ \psi(q)\int_{s_1-\epsilon}^{s_1}\|R(s_1-s) - R(s_2-s)\|_{L(X)}p(s)ds + M_1\psi(q)\int_{s_1}^{s_2} p(s)ds.$$

From the Theorem 2.1 (see also [25, 26, Theorem 3.1; Theorem 2.5]), we deduce that the right hand side of the above inequality tends to zero as $s_2 - s_1$ goes to 0 and ϵ sufficiently small. Thus the set $\{F_2x : x \in B_q\}$ is equicontinuous. Here we consider only the case $0 < s_1 < s_2 \le b$, since the other cases $s_1 \le s_2 \le 0$ or $s_1 \le 0 \le s_2 \le b$ are very simple.

Next, we show that F_2 is continuous.

Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. Then

$$\|F_2x_n(t) - F_2x(t)\|$$

$$\le \left\|\int_0^t R(t-s)\int_0^s B(s-\tau)[g(\tau,x_{n\tau}) - g(\tau,x_\tau)]\,d\tau ds\right\|$$

$$+\left\|\int_0^t R(t-s)\left[f(s,x_n(s-\rho(x_n(s)))) - f(s,x(s-\rho(x(s))))\right]ds\right\|$$

$$\le \int_0^t \|R(t-s)\|\int_0^s \|B(s-\tau)\|\|g(\tau,x_{n\tau}) - g(\tau,x_\tau)\|\,d\tau ds$$

$$+ \int_0^t \|R(t-s)\|\left\|f(s,x_n(s-\rho(x_n(s)))) - f(s,x(s-\rho(x(s))))\right\|ds.$$

From hypothesis (H_3), the continuity of ρ and by Lebesgue dominated convergence theorem, the right hand side of the above inequality tends to zero as $n \to \infty$. Thus
$\|F_2 x_n(t) - F_2 x(t)\| \to 0$ as $n \to \infty$.
Next we show that F_2 maps B_q into precompact set in X.

Let $0 \le t \le b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_q$, we define the operators

$$(F_2^{\epsilon}x)(t) = R(t)\phi(0) + \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds$$

$$+ \int_0^{t-\epsilon} R(t-s)f(s,x(s-\rho(x(s))))ds$$

$$= R(t)\phi(0) + R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds$$

$$+R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)f(s,x(s-\rho(x(s))))ds$$

and

$$(\tilde{F}_2^{\epsilon}x)(t) = R(t)\phi(0) + \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds$$

$$+ \int_0^{t-\epsilon} R(t-s)f(s,x(s-\rho(x(s))))ds.$$

From the Theorem 2.1 and the compactness of the operator $R(\epsilon)$, the set $V_\epsilon(t) = \{(F_2^{\epsilon}x)(t) : x \in B_q\}$ is precompact in X, for every ϵ, $0 < \epsilon < t$. Moreover for each $x \in B_q$ and by Theorem 2.1, we have

$$\|(F_2^{\epsilon}x)(t) - (\tilde{F}_2^{\epsilon}x)(t)\|$$

$$\le \Big\|R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds - \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds\Big\|$$

$$+ \Big\|R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)f(s,x(s-\rho(x(s))))ds - \int_0^{t-\epsilon} R(t-s)f(s,x(s-\rho(x(s))))ds\Big\|$$

$$\le \Big\|\int_0^{t-\epsilon} R(\epsilon)R(t-s-\epsilon)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds - \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds\Big\|$$

$$+ \Big\|\int_0^{t-\epsilon} R(\epsilon)R(t-s-\epsilon)f(s,x(s-\rho(x(s))))ds - \int_0^{t-\epsilon} R(t-s)f(s,x(s-\rho(x(s))))ds\Big\|$$

$$\le \int_0^{t-\epsilon}\|R(\epsilon)R(t-s-\epsilon) - R(t-s)\|_{L(X)}\int_0^s\|B(s-\tau)\|\|g(\tau,x_\tau)\|d\tau ds$$

$$+\int_0^{t-\epsilon}\|R(\epsilon)R(t-s-\epsilon) - R(t-s)\|_{L(X)}\|f(s,x(s-\rho(x(s))))\|ds$$

$$\le \epsilon H\int_0^s\|B(s-\tau)\|\|g(\tau,x_\tau)\|d\tau ds + \epsilon H\int_0^{t-\epsilon}\|f(s,x(s-\rho(x(s))))\|ds.$$

So the set $V_\epsilon(t) = \{(F_2^{\epsilon}x)(t) : x \in B_q\}$ is precompact in X by using total boundedness. Applying this idea again and observing that

$$\|(F_2x)(t) - (F_2^{\epsilon}x)(t)\|$$

$$\le \Big\|\int_0^t R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds - \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds\Big\|$$

$$+ \Big\|\int_0^t R(t-s)f(s,x(s-\rho(x(s))))ds - \int_0^{t-\epsilon} R(t-s)f(s,x(s-\rho(x(s))))ds\Big\|$$

$$\le \int_{t-\epsilon}^t\|R(t-s)\|\int_0^s\|B(s-\tau)\|\|g(\tau,x_\tau)\|d\tau ds + \int_{t-\epsilon}^t\|R(t-s)\|\|f(s,x(s-\rho(x(s))))\|ds$$

$$\le M_0M_1M_2b\int_{t-\epsilon}^t[c_1\|x_t\|_C + c_2]ds + M_1\int_{t-\epsilon}^t p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right)ds$$

$$\le M_0M_1M_2b\int_{t-\epsilon}^t[c_1q + c_2]ds + M_1\psi(q)\int_{t-\epsilon}^t p(s)ds.$$

Therefore,
$\|(F_2x)(t) - (F_2^{\epsilon}x)(t)\| \to 0$ as $\epsilon \to 0^+$,
and there are precompact sets arbitrarily close to the set $\{(F_2x)(t) : x \in B_q\}$. Thus the set $\{(F_2x)(t) : x \in B_q\}$ is precompact in X. Therefore from Arzela-Ascoli Theorem, we can conclude that the operator F_2 is completely continuous.

In order to study the existence results for the problem $(1.1)-(1.2)$, we introduce a parameter $\lambda \in (0,1)$ and consider the following auxiliary system

$$\frac{d}{dt}[x(t) - \lambda g(t, x_t)] = Ax(t) + \lambda\int_0^t B(t-s)x(s)ds + \lambda f(t, x(t-\rho(x(t)))),$$
$$t \in J = [0,b], \qquad (3.1)$$
$$x(t) = \phi(t), \quad t \in [-r,0].$$

Then by Definition 3.1, the mild solution of the above system can be written as

$$F(x)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ R(t)[\phi(0) + \lambda g(0,\phi)] + \lambda g(t,x_t) + \lambda\int_0^t AR(t-s)g(s,x_s)ds \\ +\lambda\int_0^t R(t-s)\int_0^s B(s-\tau)g(\tau,x_\tau)d\tau ds \\ +\lambda\int_0^t R(t-s)f(s,x(s-\rho(x(s))))ds, & t \in J. \end{cases}$$

The following Lemma proves a priori bounds for the above defined system.

Lemma 3.1 If hypotheses $(H_1)-(H_4)$ are satisfied, and let $x(t)$ be a mild solution of the above system (3.1), then there exists a priori bounds $K > 0$ such that $\|x_t\|_C \le K$, $t \in J$, K depends only on b and on the functions $p(\cdot)$ and $\psi(\cdot)$.

Proof. From the above discussion we have

$$\|x(t)\| \le \|R(t)\|[\|\phi(0)\| + \|g(0,\phi)\|] + \|g(t,x_t)\| + \int_0^t \|AR(t-s)g(s,x_s)\|ds$$

$$+ \int_0^t \|R(t-s)\| \int_0^s \|B(s-\tau)\| \|g(\tau,x_\tau)\| d\tau ds$$
$$+ \int_0^t \|R(t-s)\| \left\|f\left(s, x(s-\rho(x(s)))\right)\right\| ds$$

$$\le M_1[\|\phi(0)\| + \|g(0,\phi)\|] + \|(-A)^{-\beta}\| \|c_1\|x_t\|_C + c_2\|$$

$$+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} [c_1\|x_s\|_C + c_2] ds + M_1 M_2 b^2 \|(-A)^{-\beta}\| \|c_1\|x_t\|_C + c_2\|$$
$$+ M_1 \int_0^t p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right) ds$$

$$\le M_1[\|\phi(0)\| + \|g(0,\phi)\|] + M_0 c_1\|x_t\|_C + M_0 c_2 + c_1 C_{1-\beta} \int_0^t \frac{\|x_s\|_C}{(t-s)^{1-\beta}} ds$$

$$+ \frac{c_2 C_{1-\beta} b^\beta}{\beta} + M_0 M_1 M_2 b^2 c_1 \|x_t\|_C + M_0 M_1 M_2 b^2 c_2$$
$$+ M_1 \int_0^t p(s)\psi\left(\left|x\left(s-\rho(x(s))\right)\right|\right) ds.$$

Since $-r \le s - \rho(x(s)) \le s$ for each $s \in J$ and consider the function μ defined by

$$\mu(t) = \sup\{|x(s)| : -r \le s \le t\},$$
$$0 \le t \le b.$$

For $t \in [0, b]$, we have

$$\mu(t) \le M_1[\|\phi(0)\| + \|g(0,\phi)\|] + M_0 c_1 \mu(t) + M_0 c_2 + c_1 C_{1-\beta} \int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} ds$$

$$+ \frac{c_2 C_{1-\beta} b^\beta}{\beta} + M_0 M_1 M_2 b^2 c_1 \mu(t) + M_0 M_1 M_2 b^2 c_2$$
$$+ M_1 \int_0^t p(s)\psi(\mu(s)) ds$$

$$\mu(t)[1 - M_0 c_1 - M_0 M_1 M_2 b^2 c_1] \le M_1[\|\phi(0)\| + \|g(0,\phi)\|] + M_0 c_2[1 + M_1 M_2 b^2]$$

$$+ \frac{c_2 C_{1-\beta} b^\beta}{\beta} + c_1 C_{1-\beta} \int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} ds + M_1 \int_0^t p(s)\psi(\mu(s)) ds.$$

Let us take $M = 1 - M_0 c_1 - M_0 M_1 M_2 b^2 c_1$, then

$$\mu(t) \le \frac{1}{M}\left\{M_1[\|\phi(0)\| + \|g(0,\phi)\|] + M_0 c_2[1 + M_1 M_2 b^2] + \frac{c_2 C_{1-\beta} b^\beta}{\beta}\right\}$$

$$+ \frac{c_1 C_{1-\beta}}{M} \int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} ds + \frac{M_1}{M} \int_0^t p(s)\psi(\mu(s)) ds$$

$$\le K_1 + K_2 \int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} ds + K_3 \int_0^t p(s)\psi(\mu(s)) ds$$

where

$$K_1 = \frac{1}{M}\left\{M_1[\|\phi(0)\| + \|g(0,\phi)\|] + M_0 c_2[1 + M_1 M_2 b^2] + \frac{c_2 C_{1-\beta} b^\beta}{\beta}\right\},$$

$$K_2 = \frac{c_1 C_{1-\beta}}{M}, \quad K_3 = \frac{M_1}{M}.$$

By using Lemma 2.2, we obtain

$$\mu(t) \le B_0\left(K_1 + K_3 \int_0^t p(s)\psi(\mu(s)) ds\right),$$

where

$$B_0 = e^{K_2^n (\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \left(\frac{K_2 b^\beta}{\beta}\right)^j K_1$$

Let us take the right hand side of the above inequality as $v(t)$, then

$$v(0) = B_0 K_1, \ \mu(t) \le v(t), \ t \in J \text{ and}$$
$$v'(t) = B_0 K_3 p(t)\psi(\mu(t)).$$

Using the nondecreasing character of ψ, we get

$$v'(t) \le B_0 K_3 p(t)\psi(v(t)), \quad t \in J.$$

This implies that for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \le B_0 K_3 \int_0^b p(s) ds < \int_{B_0 K_1}^{\infty} \frac{ds}{\psi(s)}.$$

This implies that $v(t) < \infty$. So there is a constant K such that $v(t) \le K$, $t \in J$, and hence $\|x_t\|_C \le \mu(t) \le v(t) \le K$, $t \in J$, where K depends only on b and on the functions $p(\cdot)$ and $\psi(\cdot)$. Consequently, by Theorem 2.2, the operator F has a fixed point in $C([-r, b], X)$. Thus the IVP $(1.1) - (1.2)$ has a solution on $[-r, b]$. This completes the proof.

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