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# Numerical investigation of an industrial robot ARM control problem using RK method based on Centroidal mean

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**ABSTRACT**

In this paper, the parameters governing the arm model of a robot has been studied through a numerical technique “fourth order non-linear extended RK method based on Centroidal Mean (RKCeM)”. The exact solutions of the system of second order equations representing the arm model of a Robot have been compared with the corresponding discrete solutions (approximate solutions) at different time and also the absolute error between the exact and discrete solutions has been determined.



## Introduction

It is indeed true that a good number of researchers have contributed on a variety of aspects in the field of robust control, especially about the dynamics of a robotic motion and their governing equations, for the past three decades. But, a number of researchers are still contributing various principles and new techniques for the best use of robots in reality, especially in the field of industry, as this field of study is inexhaustible.

Said Oucheriah (1999) discussed ‘Robust Tracking and Model Following the Uncertain Dynamic Delay Systems by Memoryless Linear Controllers’. David Lim and Homayoun Seraji (1997) discussed ‘Configuration Control of a Mobile Dexterous Robot’. Polycarpou and Ioannoy (1996) discussed about a ‘Robust Adaptive Non-linear Control Design’.

Murugesan, Paul and Evans (1999) have analysed second order systems via Runge-kutta (RK) method and also studied second order multivariable linear system using Single Term Walsh Series (STWS) technique and RK method. The RK method have found wide applications in control engineering, communication and signal processing. Recently, Murugesan, Paul and Evans (2000) analysed a non-linear singular system from fluid dynamics using extended RK methods.

In this paper, the authors have observed that the robotic motion has been governed by second order linear and non-linear differential equations. Hence a meticulous attempt has been made to study the parameters concerning the control of a robot arm model by applying the flexible and suitable numerical method which is capable of solving a system of second order linear and non-linear differential equations representing the arm model of a robot .

## Robot arm model

*The dynamics of a robot arm is represented as*

$$T = A(Q)\ddot{Q} + B(Q,\dot{Q})\dot{Q} + C(Q) \qquad (2.1)$$

where $A(Q)$ = coupled inertia matrix

$B(Q,\dot{Q})$ = matrix of coriolis and centrifugal forces

$C(Q)$ = Gravity matrix

$T$ = Input torques applied at various joints

For a two degree of freedom robot under the assumption of lumped equivalent masses and mass less links, the dynamics are represented by

$$T_1 = D_{11}\ddot{q}_1 + D_{12}\ddot{q}_2 + D_{122}(\dot{q}_2)^2 + D_{112}(\dot{q}_1\dot{q}_2) + D_1$$

$$T_2 = D_{21}\ddot{q}_1 + D_{22}\ddot{q}_2 + D_{211}(\dot{q}_1)^2 + D_2 \qquad (2.2)$$

where

$D_{11} = (M_1 + M_2)\, d_1^2 + M_2 d_2^2 + 2M_2 d_1 d_2 \cos(q_2)$

$D_{12} = M_2 d_2^2 + M_2 d_1 d_2 \cos(q_2)$ $\quad D_{21} = D_{12}$

$D_{22} = M_2 d_2^2$ $\quad D_{112} = 2M_2 d_1 d_2 \sin(q_2)$

$D_{122} = -M_2 d_1 d_2 \sin(q_2)$ $\quad D_{211} = D_{122}$

$D_1 = [(M_1 + M_2) d_1 \sin(q_1) + M_2 d_2 \sin(q_1 + q_2)]g$

$D_2 = [M_2 d_2 \sin(q_1 + q_2)]g$

*For the set point regulation, the state vector is defined as*

$$X = (x_1, x_2, x_3, x_4)^T = (q_1 - q_{1d}, \dot{q}_1, q_2 - q_{2d}, \dot{q}_2)^T \qquad (2.3)$$

where $q_1$ and $q_2$ are the angles at the joint 1 and joint 2 respectively and $q_{1d}$, $q_{2d}$ are constants.

In state space representation, equation (2.2) can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{D_{22}}{d}(D_{122}x_2^2 + D_{112}x_2x_4 + D_1 + T_1) - \frac{D_{12}}{d}(D_{211}x_4^2 + D_2 + T_2) \qquad (2.4)$$

$$\dot{x}_3 = x_4$$

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$$\dot{x}_4 = \frac{-D_{12}}{d}(D_{122}x_2^2 + D_{112}x_2x_4 + D_1 + T_1) + \frac{D_{11}}{d}(D_{211}x_4^2 + D_2 + T_2)$$

The above system of equations is non-linear in nature and it can be seen that the synthesis of a control law is of the form

$$T_i = \psi_i x_1 + \psi_i^2 x_2 + \psi_i^3 x_3 + \psi_i^4 x_4 \quad (2.5)$$

It is observed that the synthesis of the control law would be very difficult due to the non-linear and interactive nature of the canonical equations (2.4). Hence it should be reduced to linear form.

Reduction of Robot dynamics to a second order linear system

Although the physical and mathematical structure of the complete dynamic robot model are analytically coupled and non-linear, it is observed that the transient responses of robot dynamics appear to resemble as transient responses of linear systems. Consequently each joint of the robot can be characterized as a single- input, single-output system (SISO). The input is the actuator torque (or) force and the output is the joint position. Hence the Mathematical model of a robot is taken as a 'Black Box'. The input into this 'Black Box' is the transient response of a linear model to a step input. The output are the motive forces or torques required by the robot to reproduce responses similar to the linear model. Samples of the input and output of the Black Box have been fed into an identification program which will match a low order decoupled linear time invariant model of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B_m s^m + B_{m-1}s^{m-1} + \ldots\ldots + B_1 s + B_0}{S^n + A_{n-1}s^{n-1} + \ldots\ldots + A_1 s + A_0} \quad (3.1)$$

The model order m and n are selected to give the lowest possible order that will characterize the structure of the mathematical model of the robot. This can be validated by comparing the response of the model based on the identified parameters $A_0$, $A_1$, …, $A_m$ and $B_0$, $B_1$,…,$B_m$ with the desired response from the linear time-invariant model, the input into the 'Black Box'.

It is determined that the non-linear model (2.4) of the two link robot arm model can be reduced to the following system of linear equations as (refer Warwick-1990)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= B_0T_1 - A_1x_2 - A_0x_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= B_0^2T_2 - A_1^2x_4 - A_0^2x_3 \end{aligned} \quad (3.2)$$

The above system has been reduced to a system of linear second order equations as

$$\ddot{x}_1 = -A_1\dot{x}_1 - A_0x_1 + B_0T_1$$

$$\ddot{x}_3 = -A_1^2\dot{x}_3 - A_0^2x_3 + B_0^2T_2 \text{ where } x_2 = \dot{x}_1 \text{ and } x_4 = \dot{x}_3 \quad (3.3)$$

i.e., $$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_3\end{bmatrix} = \begin{bmatrix}-A_1&0\\0&-A_1^2\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_3\end{bmatrix} + \begin{bmatrix}-A_0&0\\0&-A_0^2\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} + \begin{bmatrix}B_0&0\\0&B_0^2\end{bmatrix}\begin{bmatrix}T_1\\T_2\end{bmatrix} \quad (3.4)$$

Equation (3.4) is of the form $K\ddot{X} = A\dot{X} + BX + CU$, (3.5)

where $k = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $A = \begin{bmatrix}-A_1&0\\0&-A_1^2\end{bmatrix}$, $B = \begin{bmatrix}-A_0&0\\0&-A_0^2\end{bmatrix}$, $C = \begin{bmatrix}B_0&0\\0&B_0^2\end{bmatrix}$, $U = \begin{bmatrix}T_1\\T_2\end{bmatrix}$

**Example**

The values of the robot parameters used were $M_1=2$, $M_2=5$, $d_1=d_2=1$. The input into the "Black Box" is the response of $\ddot{y} + 2kn\dot{y} + n^2y = n^2y_s$, where k=1 is critical damping.

For n=5 and k=1, the initial conditions and the set points have been taken as

$$q_1(0)=0,\ q_2(0)=0,\ \dot{q}_1(0)=0,\ \dot{q}_2(0)=0,\ q_{1d}=1 \text{ and } q_{2d}=1 \quad (3.6)$$

The linear model parameters of joint 1 have been found as $A_0 = 0.1730$, $A_1 = -0.2140$, $B_o=0.0265$ and that of joint 2 have been determined as $A_0 = 0.0438$, $A_1 = 0.3610$, $B_0=0.0967$.Since the numerical solutions for the parameters governing the robot arm model have to be determined and to avoid the complexity, $A_0$, A1 and $B_0$ alone have been assigned non-zero values to maintain the linearity

Equation (3.4) becomes

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_3\end{bmatrix} = \begin{bmatrix}0.2140&0\\0&-0.130321\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_3\end{bmatrix} + \begin{bmatrix}-0.1730&0\\0&-0.00191844\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} + \begin{bmatrix}0.0265&0\\0&0.00935089\end{bmatrix}\begin{bmatrix}T_1\\T_2\end{bmatrix},$$

which is of the form $K\ddot{X} = A\dot{X} + BX + CU$.

In order to study the parameters that govern the dynamics of the Robot using numerical methods, the authors have chosen $T_1 = T_2 = 1$ unit. However, one can vary the values of $T_1$ and $T_2$ to visualize the effect of the parameters that control the arm model of the robot and the simulation can be done.

Hence the above equation becomes

$$\begin{aligned} \ddot{x}_1 &= 0.2140\,\dot{x}_1 - 0.1730\,x_1 + 0.0265 \\ \ddot{x}_3 &= -0.130321\,\dot{x}_3 - 0.00191844\,x_3 + 0.00935089 \end{aligned} \quad (3.7)$$

where $x_2 = \dot{x}_1$ and $x_4 = \dot{x}_3$

Since $(x_1, x_2, x_3, x_4) = \left(q_1 - q_{1d}, \dot{q}_1, q_2 - q_{2d}, \dot{q}_2\right)$ and using (3.6),the initial conditions are $x_1(0) = -1$, $x_3(0) = -1$,

$$\dot{x}_1(0) = 0, \dot{x}_3(0) = 0$$

*The exact solution of the system (3.7) is*

$x_1 = e^{0.107t}$ [-1.15317919 cos(0.401934074 t) + 0.306991074 sin(0.401934074 t)] + 0.15317919

$x_2 = e^{0.107t}$ [0.463502009 sin (0.401934074 t) + 0.123390173 cos (0.401934074 t)] + 0.107 $e^{0.107t}$[-1.15317919 cos (0.401934074 t) + 0.306991074 sin(0.401934074 t)] (3.8)

$x_3$ = 1.029908976 $e^{-0.11340416\,t}$ – 6.904124484 $e^{-0.016916839\,t}$ + 4.874215508

$x_4$ = - 0.116795962 $e^{-0.11340416\,t}$ + 0.116795962 $e^{-0.016916839\,t}$

Since $q_1$, $q_2$ (the angles at the joints 1 and 2), $\dot{q}_1$ and $\dot{q}_2$ are involving with $x_1$, $x_2$, $x_3$ and $x_4$

i.e., $(x_1, x_2, x_3, x_4) = \left(q_1 - q_{1d}, \dot{q}_1, q_2 - q_{2d}, \dot{q}_2\right)$, the exact solutions of $q_1$, $q_2$, $\dot{q}_1$ and $\dot{q}_2$ are given by

$q_1 = x_1 + q_{1d}$ ; $\dot{q}_1 = x_2$; $q_3 = x_3 + q_{2d}$ ;

$$\dot{q}_2 = x_4 \quad (3.9)$$

where $q_{1d} = q_{2d} = 1$ and $x_1, x_2, x_3, x_4$ are given in (3.8).

**Extended Runge-Kutta method based on Centroidal Mean (RKCeM)**

Recently, Murugesan, Paul and Evans [2000] have developed and extended the RK method based on the Harmonic mean to solve a system of second order differential equation which was earlier introduced by Sanugi and Evans [1994] based on the concept of harmonic mean averaging of functional values to solve a first order differential equation. In this paper RKCeM is being applied to study the dynamics of the robot problem. As a part of our contribution, we have extended the method introduced by Sanugi and Evans (1994) to suit the needs of this study.

Consider a system of second order differential equations

$$\ddot{x}_1(t) = f_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$$
$$\ddot{x}_2(t) = f_2(t, x_1, x_2, \dot{x}_1, \dot{x}_2) \quad (4.1)$$

with the initial conditions $x_1(0) = x_{10}$; $x_2(0) = x_{20}$

$\dot{x}_1(0) = x_{10}$; $\dot{x}_2(0) = x_{20}$ Put $u_1 = \dot{x}_1$ and $u_2 = \dot{x}_2$

Hence we have $\ddot{x}_1 = \dot{u}_1$ and $\ddot{x}_2 = \dot{u}_2$.

Therefore equation (4.1) becomes

$$\dot{u}_1(t) = f_1(t, x_1, x_2, u_1, u_2)$$
$$\dot{u}_2(t) = f_2(t, x_1, x_2, u_1, u_2) \quad (4.2)$$

with $x_1(0) = x_{10}$; $x_2(0) = x_{20}$
$u_1(0) = u_{10}$; $u_2(0) = u_{20}$

Let 'h' denote the interval between the equidistant values of t. If the initial values are t(0), $x_1(0)$, $x_2(0)$, $u_1(0)$ and $u_2(0)$ then the first increments in $x_1$, $x_2$, $u_1$ and $u_2$ are determined as follows

$x_1(1) = x_1(0) + \Delta x_1$
$x_2(1) = x_2(0) + \Delta x_2$
$u_1(1) = u_1(0) + \Delta u_1$
$u_2(1) = u_2(0) + \Delta u_2$ (4.3a)

In [1 – 3, 10-11], Evans and Yaakub have developed a new RK method of order 4 based on Centroidal mean to solve first order equation and it is to be noted that the Centroidal Mean of two points $x_1$ and $x_2$ is defined as

$$\frac{2}{3}\left(\frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2}\right)$$

Consider the first order equation (2.1) of the form

$$y' = f(x, y)$$

with $y(x_0) = y_0$.

Let h denote the interval between equidistant values of x. The fourth order RKAM formula (2.21) can be written as

$$y_{n+1} = y_n + \frac{h}{3}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right)$$

$$y_{n+1} = y_n + \frac{h}{3}\left(\sum_{i=1}^{3}\frac{k_i+k_{i+1}}{2}\right)$$

and substituting the arithmetic mean (AM) of $k_i$, $1 \le i \le 6$ with their Centroidal Means we obtain a new formula, similar to the above equation, as

$$y_{n+1} = y_n + \frac{h}{3}\left[\sum_{i=1}^{3}\frac{2\left(k_i^2 + k_ik_{i+1} + k_{i+1}^2\right)}{3(k_i + k_{i+1})}\right]$$

to obtain the fourth order formula in the form,

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + a_1h, y_n + ha_1k_1)$$
$$k_3 = f(x_n + (a_2 + a_3)h, y_n + ha_2k_1 + ha_3k_2)$$
$$k_4 = f(x_n + (a_4 + a_5 + a_6)h, y_n + ha_4k_1 + ha_5k_2 + ha_6k_3)$$

$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{2(k_1^2+k_1k_2+k_2^2)}{3(k_1+k_2)} + \frac{2(k_2^2+k_2k_3+k_3^2)}{3(k_2+k_3)} + \frac{2(k_3^2+k_3k_4+k_4^2)}{3(k_3+k_4)}\right]$$

that is $y_{n+1} = y_n + \dfrac{UPPER}{LOWER}$

where,

UPPER=

$$\frac{2h}{9}\left[(k_1^2+k_1k_2+k_2^2)(k_2+k_3)(k_3+k_4) + (k_2^2+k_2k_3+k_3^2)(k_1+k_2)(k_3+k_4) + (k_3^2+k_3k_4+k_4^2)(k_1+k_2)(k_2+k_3)\right]$$

and,

LOWER = $(k_1 + k_2)(k_2 + k_3)(k_3 + k_4)$,

while the Taylor series expansion of $y(x_n+1)$ may be given as,

TAYLOR =

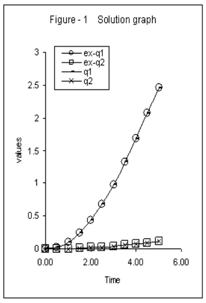
$$y_n + hf + \frac{h^2}{2}ff_y + \frac{h^3}{6}\left(ff_y^2 + f^2f_{yy}\right)\frac{1}{24}h^4\left(f^3f_{yyy} + ff_y^3 + 4f^2f_yf_{yy}\right) + \ldots$$

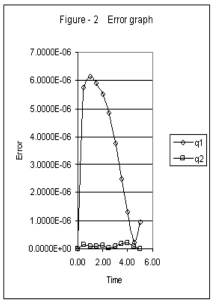
Hence the increments for the succeeding interval are computed exactly in the same way except that t(0), $x_1(0)$, $x_2(0)$, $u_1(0)$ and $u_2(0)$ are replaced by t(1), $x_1(1)$, $x_2(1)$, $u_1(1)$ and $u_2(1)$ in equation (4.5) and proceeded. Please refer Sangui and Evans (1994) and Murgesan et al., (1999) for the basic concepts involved in this method. The RKHM method has an accuracy of order 4. The local truncation error (LTE) of the RK method based on Centroidal Mean is

$$ERROR = TAYLOR - \frac{UPPER}{LOWER}$$

or, (TAYLOR x LOWER) - UPPER = (LOWER x ERROR).

The above discussed numerical method RKCeM with the step-size h = 0.01 has been applied to determine the discrete solutions of the example (refer eq. (3.7)),which represent the robot arm model involving the parameters $q_1, \dot{q}_1, q_2$ and $\dot{q}_2$.The exact and the discrete solutions are given in the Table-1 and the error between them is given in the Table-2. The graphs corresponding to the exact and discrete solutions and the absolute error between them have been given in the Figures 1 and 2 respectively.




## Conclusion

In this paper, the parameters governing the arm model of a robot and controller design have been studied by way of finding the discrete solutions at different time for the system of second order differential equations, which actually represents the dynamics of the arm model of the robot of two degree freedom.

*It has been shown that the complex and non-linear dynamics of a robot can be reduced to a set of linear models. In general, the obtained table of results and graphs related to the example considered, infer that the solutions obtained by the method RKCeM agree well with the exact solutions with an error ranging from $10^{-6}$ to $10^{-9}$. It helps to estimate the variation in the angles at the joints at different time and enables to identify the movement of the arm of the robot.*

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**Table – 1 Solution for system 3.7**

| | Exact solution | | Discrete solution (RKCeM) | |
|---|---|---|---|---|
| Time | ex-$q_1$ | ex-$q_2$ | $q_1$ | $q_2$ |
| 0.00 | 0 | 0 | 0 | 0 |
| 0.50 | 0.025757551 | 0.001378536 | 0.025751829 | 0.001378357 |
| 1.00 | 0.105704129 | 0.005396783 | 0.105697989 | 0.005396724 |
| 1.50 | 0.24232316 | 0.011886537 | 0.242317259 | 0.011886418 |
| 2.00 | 0.43580091 | 0.020688713 | 0.435795426 | 0.020688593 |
| 2.50 | 0.683758467 | 0.031653404 | 0.68375361 | 0.031653404 |
| 3.00 | 0.981070224 | 0.04463917 | 0.981066465 | 0.04463923 |
| 3.50 | 1.319786936 | 0.059512377 | 1.319784403 | 0.059512556 |
| 4.00 | 1.689178348 | 0.076147079 | 1.689177036 | 0.076147318 |
| 4.50 | 2.075907111 | 0.094424546 | 2.075906754 | 0.094424605 |
| 5.00 | 2.464341044 | 0.114232481 | 2.464341879 | 0.114232481 |

**Table – 2 Error in system 3.7**

| Time | $q_1$ | $q_2$ |
|---|---|---|
| 0.00 | 0.0000E+00 | 0.0000E+00 |
| 0.50 | 5.7240E-06 | 1.4900E-07 |
| 1.00 | 6.1240E-06 | 8.5000E-08 |
| 1.50 | 5.9010E-06 | 9.7000E-08 |
| 2.00 | 5.4840E-06 | 1.1400E-07 |
| 2.50 | 4.8280E-06 | 1.9000E-08 |
| 3.00 | 3.7550E-06 | 7.8000E-08 |
| 3.50 | 2.5030E-06 | 1.8600E-07 |
| 4.00 | 1.3110E-06 | 2.0900E-07 |
| 4.50 | 2.3800E-07 | 8.9000E-08 |
| 5.00 | 9.5400E-07 | 7.0000E-09 |