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Total edge domination in graphs

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ABSTRACT

In this paper we discuss the concept of total edge domination in graphs. We prove that for any connected (p,q) – graph G with $\Delta' < q - 1$, $\gamma_t' \le q - \Delta'$ where Δ' denotes the maximum degree of an edge in G and characterize trees and unicyclic graphs which attain this bound. We also prove that $\gamma_t'(S(G)) \le 2(p-\beta_1)$ for any connected graph G. We also determine the value of total edge domatic number d_t' for some families of graphs.

Introduction

By a graph G = (V,E) we mean a finite undirected graph without loops or multiple edges Terms not defined have are used in the sense of Harary[1].

A set $S \subseteq E$ is said to be an edge dominating set if every edge in E – S is adjacent to some edge in S. The edge domination number of G is the cardinality of a smallest edge dominating set of G and is denoted by γ'. The degree of an edge e = uv of G is defined by deg e = deg u + deg v – 2. The minimum (maximum) degree of an edge in G is denoted by $\delta'(\Delta')$. For a real no x, $\lfloor x \rfloor$ denotes the largest integer $\le$ x and $\lceil x \rceil$ denotes the smallest integer $\ge$x. We need the following theorems.

Theorem 1.1[2] If G is a graph with $p \ge 3$ then $\gamma_t \le \left\lfloor \frac{2p}{3} \right\rfloor$.

Theorem 1.2[2] (i) If G is a graph without isolated vertices ,then

$$\gamma_t \le p - \Delta + 1$$

(ii)If G is connected and Δ = p – 1, then $\gamma_t \le$ p - Δ.

Theorem 1.3[2] For any (p,q) - graph G without isolated vertices, $d_t \le \min(\delta, p/\gamma_t)$

Theorem 1.3[2] If G is a connected graph, then $\gamma(S(G)) = \gamma_t(S(G))$

Main results

Definition 2.1 Let G = (V,E) be a graph without isolated edges. An edge dominating set X of G is called a total *edge dominating* set if the edge induced subgraph $\langle X \rangle$ has no isolated edges. The minimum cardinality of a total edge dominating set is called the *total edge domination number* of G and is denoted by $\gamma_t'(G)$ or γ_t. The *upper total edge domination number* of G is the maximum cardinality taken over all minimal total edge dominating sets of G and is denoted by Γ_t'

Example 2.2

(i) $\gamma_t'(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{ or } 2 \pmod 4 \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 3 \pmod 4 \end{cases}$

(ii) $\gamma_t'(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{ or } 3 \pmod 4 \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod 4 \end{cases}$

(iii) $\gamma_t'(K_{1,n}) = 2$

(iv) $\gamma_t'(K_{m,n}) = \min\{m,n\}$ if m,n >1.

Theorem 2.3 $\gamma_t'(K_p) = \left\lfloor \frac{2p}{3} \right\rfloor$ if $p \ge 3$.

Proof Let $V(K_p) = \{v_1, v_2, \ldots\ldots\ldots v_p\}$. Let $c_i = v_i v_{i+1}, 1 \le i \le p-1$. If p $\equiv 0$ or 1(mod 3), let $D = \{e_i \mid 1 \le i \le p-1 \text{ and } i \not\equiv 0 (\text{mod}\, 3)\} \bigcup \{e_{p-2}\}$.

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Clearly D is a total edge dominating set of K_p and $|D| = \left\lfloor \frac{2p}{3} \right\rfloor$. Hence $\gamma_t'(K_p) \le \left\lfloor \frac{2p}{3} \right\rfloor$.

Now let D be any total edge dominating set of K_p. If there exist two vertices, say v_i, v_j of K_p which are not incident with any edge of D, then the edge v_iv_j is not dominated by D. Hence D must cover at least p – 1 vertices of K_p. Further $\langle D \rangle$ has no isolated edges and hence $|D| \ge \left\lceil \frac{2(p-1)}{3} \right\rceil = \left\lfloor \frac{2p}{3} \right\rfloor$.

Thus $\gamma_t'(K_p) = \left\lfloor \frac{2p}{3} \right\rfloor$.

Theorem 2.4 $\gamma_t'(W_p) = \lfloor p/2 \rfloor$.

Proof Let $V(W_p) = \{v_1, v_2, \ldots\ldots\ldots, v_p\}$, $\deg v_1$ =p-1 and E(W_p) =

$\{v_1v_i \mid 2 \le i \le p\} \bigcup \{v_2v_3, v_3v_4, \ldots\ldots\ldots\ldots, v_{p-1}v_p, v_pv_2\}$.Then

$$S = \begin{cases} \{v_1v_2, v_1v_4, \ldots\ldots\ldots v_1v_p\} & \text{if p is even} \\ \{v_1v_2, v_1v_4, \ldots\ldots\ldots v_1v_{p-1}\} & \text{if p is odd} \end{cases}$$

is a total edge dominating set of W_p so that $\gamma_t'(W_p) \le \lfloor p/2 \rfloor$.

Now let S be a minimum total edge dominating set of W_p. Since any two adjacent edges e_1, e_2 of S dominate at most four edges of C_{p-1} = ($v_2,v_3,\ldots\ldots\ldots,v_p,v_2$) in W_p (including possibly e_1 and e_2), it follows that $|S| \ge \lfloor p/2 \rfloor$.Hence $\gamma_t'(W_p) = \lfloor p/2 \rfloor$.

Remark 2.5 If G is a graph without isolated edges, then $\gamma_t'(G) = \gamma_t(L(G))$ where L(G) denoted the line graph of G. Hence it follows from Theorems 1.1 and 1.2 that

(i) $\gamma_t' \le \left\lfloor \frac{2q}{3} \right\rfloor$

(ii) $\gamma_t' \le q - \Delta' + 1$

(iii) If G is connected and $\Delta' < q-1$ then $\gamma_t' \le q - \Delta'$

For the graphs given in Figure 4.1 $\gamma_t' = 4 = \left\lfloor \frac{2q}{3} \right\rfloor$.

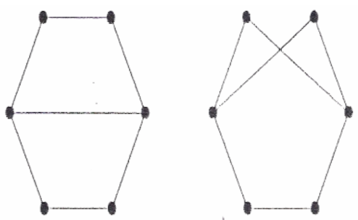

Figure 2.1

Theorem 2.6 For any tree T with $\Delta' < p-2, \gamma_t' \le q - \Delta'$ if and only if $4 \le diam(T) \le 6$ and T is isomorphic to one of the following.

(1) P_5,P_6 or P_7.

(2) Any tree with exactly one vertex u of degree ≥ 3 satisfying the following condition. If there exists a pendant vertex x with d(u,x) = 4, then there exists at most one pendant vertex y with d(u,y) = 2.

(3) Any tree obtained from a tree described in (2) by attaching any number of pendant vertices to exactly one vertex v of N(u) such that in the resulting tree v has degree $\ge$ 3.

Proof Let T be a tree with $\Delta' < p-2$. Suppose $4 \le diam(T) \le 6$ and T is isomorphic to one of the trees given in the hypothesis. Then $|S| = \Delta'$ where S is the set of all pendant edges of G and E(G) \ S is the unique minimum total edge dominating set of T so that $\gamma_t' \le q - \Delta'$

Conversely suppose that $\gamma_t' \le q - \Delta'$. Then diam(T) ≥ 4 and $|S| = \Delta'$. Since E(T)\S is the unique minimum total edge dominating set of T, it follows that diam(T) ≤ 6 and T has at most one vertex of degree ≥ 3 or two adjacent vertices of degree ≥ 3. We consider the following cases.

Case(i) T has no vertex of degree $\ge$ 3.

In this case, T=P_5,P_6 or P_7.

Case(ii) T has exactly one vertex, say u of degree ≥ 3.

Suppose there exists a pendant vertex x in T such that d(u,x) = 4. Let P = (u, u_1, u_2, u_3,u_4 = x) be the u – x path of length 4. If there exist two pendant vertices, say y_1,y_2 such that d(u,y_1) = d(u,y_2) = 2.then $E(T) \setminus (S \cup \{uu_1\}$ is a total edge dominating set of T so that $\gamma_t' \le q - \Delta' - 1 < q - \Delta'$ which is a contradiction. Hence there is at most one pendant vertex y with d(u,y) = 2.

Case(iii) T has exactly two adjacent vertices, say u, v of degree ≥ 3. First we claim that there exist at least deg(u) – 2 pendant vertices adjacent to v otherwise there exist vertices u_1,u_2,v_1,v_2, such that d(u,u_1) = d(u,u_2) = d(v,v_1) =d(v,v_2) = 2 and in this case $E(T) \setminus (S \cup \{uv\})$ is a total edge dominating set of T with cardinality$q - \Delta' - 1$, which is a contradiction. Hence we assume that there exist deg(u) – 2 pendant vertices adjecent to u in T. Let T_1 be the tree obtained by deleting these deg u – 2 pendant vertices adjacent to u. Clearly T_1 is a tree with exactly one vertex v of degree ≥ 3 and $\gamma_t' = q(T_1) - \Delta(T_1)$. Hence T_1 is of the form described in (2) and the result follows.

Notation2.7(i) Let G be a graph with vertex set V={v_1,v_2,........,v_n}). Let k_1,k_2,.........,k_m be non negative integers. Take k_i copies of P_{i+1}, i=1,2,........,n where P_i denotes the path on i vertices. Then the graph obtained from G by identifying one end vertex of each P_i with v1 is denoted by G*v_1(k_1P_2,k_2P_3,............k_mP_{m+1}).

If such paths are attached at more than one vertex of G, the above notation can be extended in an obvious way.

(ii)Let C_n be the cycle (v_1,v_2,..........v_n,v_1). The graph obtained by attaching a pendant edge, say wv_1 to C_n, is denoted by E_n. Thus $E_n = C_n * v_1(P_2)$

Example 2.8 Let $G = C_3 = (v_1, v_2, v_3, v_1)$. Then $G * v_1(2P_2, 2P_3, P_4) * v_2(2P_2, P_3) * v_3(3P_2)$ and $E_3 * v_1(2P_2, P_3, P_4) * w(3P_2)$ are given in Figure 2.1

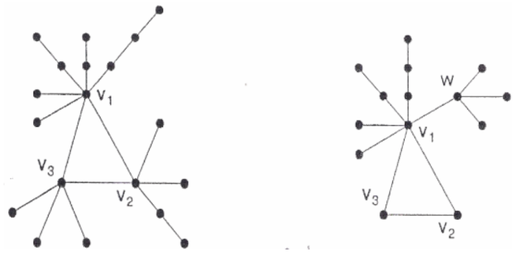

Figure 2.1

Theorem 2.9 Let G be a connected unicyclic graph with cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Then $\gamma_t' = q - \Delta'$ if and only if G is isomorphic to one of the following

1. $C_6 * v_1(a_{11}P_2) * v_6(a_{21}P_2)$ where $a_{11}, a_{21} \geq 0$.
2. $C_5 * v_1(b_{11}P_2) * v_5(b_{21}P_2, b_{22}P_3)$ where $b_{11}, b_{21} \geq 0$ and $b_{22} \leq 1$.
3. $E_5 * v_1(c_{11}P_2) * w(c_{21}P_2)$ where $c_{11}, c_{21} \geq 0$
4. $C_4 * v_1(d_{11}P_2) * v_4(d_{21}P_2)$ where $d_{11}, d_{21} \geq 0$
5. $C_4 * v_1(e_{11}P_2, P_3) * v_4(e_{21}P_2, e_{22}P_3)$ where $e_{11}, e_{21} \geq 0$ and $e_{22} \leq 1$
6. $C_4 * v_1(f_{11}P_2, f_{12}P_3, f_{13}P_4) * v_4(f_{21}P_2)$ where $f_{ij} \geq 0$ and $f_{12} \geq 2$ or $f_{13} \geq 1$.
7. $E_4 * w(g_{11}P_2, g_{12}P_3) * v_1(g_{21}P_2, g_{22}P_3, g_{23}P_4)$ where $g_{ij} \geq 0$ and $g_{12} \leq 1$.
8. $C_3 * v_1(h_{11}P_2, h_{12}P_5) * v_2(h_{21}P_2)$ where $h_{11}, h_{21} \geq 0$ and $h_{12} \leq 1$.
9. $C_3 * v_1(i_{11}P_2, i_{12}P_3) * v_2(i_{21}P_2, i_{22}P_3, i_{23}P_4)$ where $i_{11}, i_{21}, i_{22}i_{23} \geq 0$ and $i_{12} \leq 1$.
10. $C_3 * v_1(j_{11}P_2, j_{12}P_3, j_{13}P_4) * v_2(j_{21}P_2, j_{22}P_3)$ where $j_{11}, j_{12}, j_{13}, j_{21}, j_{22} \geq 0$ and if $j_{12} \geq 2$ or $j_{13} \geq 1$ then $j_{22} \leq 1$
$C_3 * v_1(k_{11}P_2, k_{12}P_3, k_{13}P_4) * v_2(k_{21}P_2) * v_3(P_3)$ where k_{11}, k_{12}, k_{13} and $k_{21} \geq 0$
11. $C_3 * v_1(l_{11}P_2, l_{12}P_3) * v_2(l_{21}P_2, P_3) * v_3(P_2)$ where $l_{11}, l_{21} \geq 0$ and $l_{12} \leq 1$
12. $C_3 * v_1(m_{11}P_2) * v_2(m_{21}P_2, m_{22}P_3, m_{23}P_4) * v_3(P_2)$ where $m_{ij} \geq 0$ and $m_{22} \geq 2$ or $m_{23} \geq 1$
13. $E_3 * w(n_{11}P_2, n_{12}P_3) * v_1(n_{21}P_2, n_{22}P_3, n_{23}P_4)$ where $n_{ij} \geq 0$ and $n_{12} \leq 1$
14. $E_3 * w(q_{11}P_2, q_{12}P_3, q_{13}P_4) * v_1(q_{21}P_2)$ where $q_{ij} \geq 0$ and $q_{13} \geq 1$

Proof Let G be a connected unicyclic graph with cycle C = C_n= $(v_1, v_2, \ldots, v_n, v_1)$, Let e = uv be an edge of maximum degree Δ' and let S denote the set of all pendant edges of G. Clearly $|S| \geq \Delta' - 2$

Now suppose $\gamma_t' \leq q - \Delta'$. Since $E(G) \setminus (S \cup \{e_1\})$ where e_1 is any edge of C is a total edge dominating set of G, it follows that $|S| = \Delta' - 1$ or Δ' - 2. Hence G has at most three vertices of degree ≥ 3

If there exist two non adjacent vertices w_1, w_2 with deg $w_1 \geq 3$ and deg $w_2 \geq 3$, then $|S| = \Delta' - 1$ and in this case there exist two adjacent edges e_1, e_2 of C such that $E(G) \setminus (S \cup \{e_1, e_2\})$ is a total edge dominating set of cardinality $q - \Delta' - 1$, which is a contradiction. Thus any two vertices of degree ≥ 3 are adjacent.

Hence if G has three vertices of degree ≥ 3, then these three vertices form a triangle and hence C = C_3 and in this case at least one vertex of C_3 has degree exactly 3.

We now claim that the edge e = uv of maximum degree lies on C or incident with a vertex of C. If both u and v are not on C, then $|S| = \Delta' - 1$ and there exist two adjacent edges e_1, e_2 of C such that $E(G) \setminus (S \cup \{e_1, e_2\})$ is a total edge dominating set of cardinality $q - \Delta' - 1$, which is a contradiction . Hence at least one of u,v lies on C.

Now if the length of C is at least 7 then there exist edges e_1, e_2, e_3 on C such that $E(G) \setminus (S \cup \{e_1, e_2, e_3\})$ is a total edge dominating set of cardinality $\leq q - \Delta' - 1$ which is a contradiction. Hence the length n of C is at most six.

Now, suppose $4 \leq n \leq 6$. Then G has at most two vertices of degree ≥ 3 and $|S| = \Delta' - 2$. If G has no vertex of degree ≥ 3, then G=C_4,C_5 or C_6 which is isomorphic to the graph given in (1),(2) or (4).

We now assume that G has at least one vertex of degree ≥ 3. Without loss of generally we assume that $\deg v_1 \geq 3$ and deg v_2 =2. Then D = $E(G) \setminus (S \cup \{v_2v_1, v_2v_3\})$ is a minimum total edge dominating set of cardinality $q - \Delta'$. We now consider the following cases.

Case(i) C = C_6.

If G contains an induced subgraph H isomorphic to the graph $C_6 * v_1(P_3)$, then D\{v_5v_6} is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph given in (1)

Case(ii) C = C_5.

If G contains an induced subgraph H isomorphic to the graph $C_5 * v_1(P_4)$, then D\{v_1v_5} is a total edge dominating set of G, which is a contradiction. Therefore distance of every vertex from C_5 is at most 2.

Subcase (a) e lies on C_5.

Let e = v_1v_5. If G contains an induced subgraph H isomorphic to $C_5*v_1(P_3)*v_5(P_3)$, then $D \cup \{v_1v_2\}) \setminus \{v_1v_5, v_3v_4\}$ is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph given in (2).

Subcase(b) e is incident with a vertex of C_5

Let e = v_1v. Since distance of any vertex from C_5 is at most 2, all the vertices adjacent to w are all pendant vertices. Since by Subcase(a), all the vertices adjacent to v_1 other than v are all pendant vertices, G is isomorphic to the graph given in (3).

Case(iii) C = C_4.

If G contains an induced subgraph H isomorphic to the graph $C_4*v_1(P_5)$. Then D\{v_1u_1} where u_1 is a vertex of H adjacent to v_1 in H is a total edge dominating set, which is a contradiction. Hence all the vertices not on C_4 ar at distance of at most 3 from C_4.

Subcase(a) e lies on C_4

Let e = v_1v_4. If G contains an induced subgraph H isomorphic to $C_4*v_1(P_3)*v_4(2P_3)$ or $C_4*v_1(P_3)*v_4(P_4)$, then $D\cup\{v_2v_1\})\setminus\{v_3v_4, v_1v_4\}$ is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph given in (4),(5) or (6).

Subcase(b) e is incident with a vertex of C_4.

Let e = v_1v. If G contains an induced subgraph H isomorphic to $E_4 * v_2(P_3)$.then D \ {e} is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph given in (7).

Case(iv) C = C_3.

Suppose G contains an induced subgraph H isomorphic to the graph $C_3 * v_1(P_6)$. Let $P = (v_1, u_1, u_2, u_3, u_4, w)$ be the v_1-w path of length 5 which is edge disjoint from C_3 in H. Then E(G)\(S ∪ {v_1v_3, v_1u_1, u_1u_2}) is a total edge dominating set of cardinality $\leq q-\Delta'-1$ which is a contradiction. Hence the distance of any vertex not on C_3 is at most 4 from C_3.

Suppose G contains an induced subgraph H isomorphic to $C_3*v(2P_5)$ or $C_3 * v_1(P_5) * v_2(P_5)$. If H = $C_3*v_1(2P_5)$ then E(G)\$(S\cup\{v_1v_3, v_1u_1, v_1u_2\})$ where u_1,u_2 are the vertices not on C_3 and adjacent to v_1 in H is total edge dominating set of cardinality $\leq q-\Delta'-1$ which is a contradiction. If H = $C_3*v_1(P_5)*v_2(P_5)$, then E(G)\$(S\,U\{v_1v_3, v_1u_1, v_2u_2\})$ where u_1,u_2 are the vertices not on C_3 and adjacent to v_1,v_2 respectively in H, is a total edge dominating set of cardinality $\leq q-\Delta'-1$ which is a contradiction. Hence there is at most one path of length four which is edge disjoint from C_3 in G.

If $|S| = \Delta'-1$, then D = $E(G)\setminus(S\,U\{v_1v_3\})$ is a minimum total edge dominating set of G with cardinality $q-\Delta'$. If $|S| = \Delta'-2$, then G contains a vertex of C_3, say v_3, of degree 2 and $D_1 = E(G)\setminus(S\,U\{v_3v_1, v_3v_2\})$ is a minimum total edge dominating set of G with cardinality $q-\Delta'$. .

Subcase(a) e lies on C_3 .

Suppose $|S| = \Delta'-2$. Then e=v_1v_2. If G contains an induced subgraph H isomorphic to $C_3*v_1(P_5)*v_2(P3)$ or $C_3*v_1(P_5,P_3)$, then D_1\{v_1u_1} where u_1 is a vertex not on C_3 and adjacent to v_1 in H, is a total edge dominating set of G, which is a contradiction. Hence all the vertices of G other than u_1, adjacent to both v_1 and v_2 are all pendant vertices so that G is isomorphic to the graph given in (8). If G contains an induced subgraph H isomorphic to $C_3*v_1(2P_3)*v_2(2P_3)$ or $C_3*v_1(P_4)*v_2(2P_3)$ or $C_3*v_1(P_4)*v_2(P_4)$, then D_1\{v_1v_2} is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph in (9) or (10).

Suppose $|S| = \Delta'-1$. Without loss of generality we assume that deg v_3 = 3 and e =v_1v_2. If G contains an induced subgraph H isomorphic to $C_3*v_1(2P_3)*v_2(p_3)$ or $C_3*v_1(P_4)*v_2(P_3)*v_3(P_2)$ then D\{v_1v_2} is a total edge dominating set, which is a contradiction. Hence G is isomorphic to one of the graphs given in either (11),(12) or (13).

Subcase(b) e is incident with a vertex of C_3.

Let $e = v_1v$. In this case G has at most two vertices of degree ≥ 3 and $|S| = \Delta'-2$. It G contains an induced subgraph H isomorphic to $E_3*v_1(P_3)*v(2P_3)$ or $E_3*v_1(P_3)*v(P_4)$, then D_1\{e} is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to one of the graphs given in (14) and (15).

Conversely suppose that G is isomorphic to one of the graphs given in the hypothesis. Then if $|S| = \Delta'-1$, then $E(G)\setminus S\cup\{e\}$ is a minimum total edge dominating set of cardinality $q-\Delta'$ and if t$|S| = \Delta'-1$, then $E(G)\setminus S\,U\{e_1, e_2\})$ where e_1, e_2 are the adjacent edges of C incident with a vertex of degree 2, is a minimum total edge dominating set of cardinality $q-\Delta'$. Hence $\gamma_t' = q-\Delta'$.

Theorem 2.10 For any graph G of order p, $\gamma'_t(S(G)) \leq 2(p-\beta_1)$.

Proof Let $X = \{u_iv_i \mid 1 \leq i \leq \beta_i\}$ be a maximum edge independent set of G. Then X is an edge dominating set of G. Let w_i be the vertex of S(G) which is adjacent to both u_i and v_i. Let S be the set of vertices of G which are not incident with any edge of X. If S = $\emptyset$ then $D = \{u_1w_1, w_1v_1, u_2w_2, w_2v_2, \ldots\ldots, u_{\beta_1}w_{\beta_1}, w_{\beta_1}v_{\beta_1}\}$ is a total edge dominating set of S(G) so that $\gamma'_t(S(G)) \leq 2\beta_1 = 2(p-\beta_1)$. Suppose $S \neq \emptyset$. Let S={$x_1,x_2,\ldots\ldots x_n$}. Since G is connected and $\langle S\rangle$ is independent, each x_i is adjacent to some z_i ($z_i = u_l$ or v_m) in G. Let y_i be the vertex of S(G) adjacent to both x_i and z_i in S(G). Then $D_1 = \{u_1w_1, w_1v_1, u_2w_2, w_2v_2, \ldots\ldots, u_{\beta_1}w_{\beta_1}, w_{\beta_1}v_{\beta_1}, x_1y_1, x_2y$ forms a total edge dominating set of G so that $\gamma'_t(S(G)) \leq |D| = 2\beta_1 + 2n = 2\beta_1 + 2(p-2\beta_1) = 2(p-\beta_1)$.

The inequality in Theorem 2.10 cannot be improved further. In fact equality holds for K_p and $K_{m,n}$ as shown in the following theorem.

Theorem 2.11 (1) $\gamma'_t(S(K_p)) = 2\lceil p/2\rceil$.

(2) $\gamma'_t(S(K_{m,n})) = 2n$ (m ≤ n).

Proof (1) Since $\beta_1(K_p) = \left\lfloor \frac{p}{2} \right\rfloor$ by Theorem 2.10, it follows that $\gamma'_t(S(Kp)) \leq 2p - 2\lfloor p/2\rfloor = 2\lceil p/2\rceil$. To prove the reverse inequality, let D be a total edge dominating set for $S(K_p)$. Suppose there exist two vertices, say u,v of K_p such that neither u nor v is incident with any edge of D. Let w be adjecent to u and v in $S(K_p)$. Now the edges uw and wv are not dominated by D so that D is not a total edge dominating set of $S(K_p)$. Hence D must cover at least p – 1 vertices of K_p. Further D has no isolated edges and hence it follows that $|D| \geq 2\lceil p/2\rceil$. Thus $\gamma'_t(S(K_p)) = 2\lceil p/2\rceil$.

(2) Let (X,Y) be the bipartition of $K_{m,n}$ with $|X| = m$ and $|Y|$ = n. By Theorem 2.10, we have $\gamma'_t(S(Km,n)) \leq 2(p-\beta_1) = 2(m+n) - 2m = 2n.$ Further any total edge dominating set for $S(K_{m,n})$ must contain at least 2n edges for dominating the edges incident with the vertices of X and m of the vertices of Y and 2(n-m) edges for dominating the edges incident with the remaining n – m vertices. Hence $|D| \geq 2m + 2(n-m) = 2n.$ Thus $\gamma'_t(S(K_{m,n})) = 2n.$

Theorem 2.12 Let T be any tree of order $p \geq 3$ and n be the number of pendant edges of T.

(1) $n \leq \gamma'_t(S(T)) \leq 2p-2-n.$

(2) $\gamma'_t(S(T)) = 2p-2-n$ if and only if T is isomorphic to $K_{1,n}$ or P_4.

(3) $\gamma'_t(S(T)) = n$ if and only if any internal vertex of T is adjacent to at least two pendant vertices.

Proof (1) Let $u_1v_1, u_2v_2, \ldots, u_nv_n$ be the pendant edges of T such that $deg_T v_i = 1$. Let w_i be the vertex of S(T) that subdivides the edge u_iv_i. Any total edge dominating set of S(T) contains the edges u_iw_i, i = 1,2,......,n and hence $\gamma'_t(S(T) \geq n.$ Further E(S(T)\S where S is the set of all pendant edges of S(T) forms a total edge dominating set of S(T) and hence $\gamma'_t(S(T)) \leq 2p-2-n.$

(2) Suppose $\gamma'_t(S(T)) \leq 2p-2-n.$ We claim that diam(T) ≤ 3. Suppose diam(T) ≥ 4. Let P = $(v_1, v_2, \ldots, v_k, v_{k+1})$ be path of length k in T. Let w_i be the vertex of S(T) subdividing the edge v_iv_{i+1}, $1 \leq i \leq k$. Now $E(S(T)) \setminus S \cup \{w_2v_3\})$ is a total edge dominating set of cardinality 2p – 2 – n - 1, which is a contradiction. Hence diam(T) ≤ 3. If diam(T)=2 then T is isomorphic to $K_{1,n}$.

Suppose diam(T) = 3. Let $P_4=(u_1,u_2,u_3,u_4)$ be a path of length 3 in T. We claim that T = P_4. Suppose deg $u_2 \geq 3$. Since diam(T) = 3 any vertex $w \neq u_3$ which is adjacent to u_2 is a pendant vertex of T. Let $N(u_2) = \{u_1,u_3,w_1,w_2,\ldots,w_r\}$. Let x be a vertex subdividing the edge u_2u_3. Now $E(S(T)) \setminus (S \cup \{u_2x\})$ is a total edge dominating set of cardinality 2p-2-n-1, which is a contradiction. Hence deg u_2 = 2. Similarly deg u_3 = 2 and hence T is isomorphic to P_4.

Conversely,

if T = P_4 then $\gamma'_t(S(T)) = 4 = 2p-2-n$ and

if T = $k_{1,n}$ then $\gamma'_t(S(T)) = p-1 = 2p-2-n.$

(3)Suppose $\gamma'_t(S(T)) = n.$ Let $u_1v_1, u_2v_2, \ldots, u_nv_n$ be the pendant edges of T such that $\deg_T v_i = 1$. Let w_i be the vertex of S(T) that subdivide the edge u_iv_i. Since any total edge dominating set of S(T) contains the edges u_iw_i for all i = 1,2,....,n and $\gamma'_t = n, S = \{u_iw_i \mid i = 1,2,\ldots,n\}$ is the unique minimum total edge dominating set of S(T). The edge induced subgraph $\langle S \rangle$ is isomorphic to a union of stars K_{1,n_i} with $n_i > 1$ and every non pendant edge of T joins the centers of two such stars. Hence any internal vertex of T is adjacent to at least two pendant vertices.

The converse is obvious.

Definition 2.13 The total edge domatic number of G, denoted by $d'_t(G)$ or d'_t is the maximum order of a partition of the edge set E into total edge dominating sets of G.

Example 2.14 (i) $d'_t(P_n) = 1$.

(ii) $d'_t(C_n) = \begin{cases} 2 \text{ if } n \equiv 0 (\text{mod } 4 \\ 1 \text{ otherwise.} \end{cases}$

(iii) $d'_t(K_{1,n}) = \lfloor n/2 \rfloor$.

(iv) $d'_t(K_{m,n}) = \max\{m,n\}$ if $m,n \geq 2$.

Remark 2.15 Since for any graph G, $d'_t(G) = d_t(L(G))$, by **Theorem 1.3** it follows that for any (p,q)-graph G without isolated edges, $d'_t(G) \leq \min((\delta', q/\gamma'_t)$.

Theorem 2.16

(1) For any (p,q)-graph without isolated edges, $\gamma'_t + d'_t \leq q+1$ and equality holds if and onlly if G = mP_3.

(2) If G is connected and $p \geq 4$, then $\gamma'_t + d'_t \leq q$ and equality holds if and only if G=C_4,$K_{1,4}$,$K_{1,3}$ or P_4.

Proof (1) By Remarks 2.5 and 2.15, we have $\gamma'_t \leq q - \Delta' + 1$ and $d'_t \leq \delta'$. Hence $\gamma'_t + d'_t \leq q - \Delta' + 1 + \delta' \leq q+1$ Further $\gamma'_t + d'_t = q+1$ if and only if $\gamma'_t = q - \Delta' + 1, d'_t = \delta'$ and $\Delta' = \delta'$ Since $d'_t \leq \frac{q}{\gamma'_t}$ and $\gamma'_t + d'_t = q+1$, it follows that $d'_t \leq \frac{q}{q-d'_t+1}$ which implies $(q-d'_t)(d'_t-1) \leq 0.$ Since $q-d'_t > 0,$ we have $d'_t = 1 = \delta'$ and hence G=P_3.

(2) By(1), P_3 is the only connected graph with $\gamma'_t + d'_t \leq q+1$ and hence $\gamma'_t + d'_t \leq q$ for any connected graph with $p \geq 4$. Suppose $\gamma'_t + d'_t = q$. We consider the following cases.

Case(i) $\Delta' < q-1$

Then $\gamma'_t = q - \Delta' + 1, d'_t = \delta'$ and $\Delta' = \delta'$. Since $d'_t \leq \frac{q}{\gamma'_t}$, we have $d'_t \leq \frac{q}{q-d'_t}$ so that $d'_t \geq q(d'_t-1)$. Further $q \geq 2d'_t$ and hence $d'^2_t \geq 2d'_t(d'_t-1)$ so that $d'_t \leq 2.$ If $d'_t = 1 = \delta'$ then G = P_3, which is a contradiction since $p \geq 4$. If $d'_t = 2 = \delta'$ then G = C_4.

Case (ii) $\Delta' = q-1$

Then $\gamma'_c = 2$ and d'_t = q - 2. Now Since $d'_t \leq \frac{q}{\gamma'_t}$, we have $q-2 \leq \frac{q}{\gamma'_t} = \frac{q}{2}$ so that $q \leq 4.$

If q=3, then G = P_4 or $K_{1,3}$. If q=4, then G = $K_{1,4}$. Thus G is isomorphic to C_4, $K_{1,4}$, $K_{1,3}$ or P_4.

The converse is obvious.

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