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Chaos and Spatial pattern formation in phytoplankton dynamics

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ABSTRACT

In this paper, the dynamics of spatially extended infected phytoplankton with the Holling type II functional response and logistically growing susceptible phytoplankton system and also the phytoplankton dynamics with susceptible and infected class of populations with diffusion is studied. The reaction diffusion system exhibits spatiotemporal chaos and pattern formation in phytoplankton dynamics, which is particularly important role for the spatially extended systems. We obtained the condition of diffusion-driven stability for the spatial dynamical system. Also we obtained the numerical simulation for the chaos and pattern formation for understanding the dynamical behavior of the phytoplankton system.

Introduction

Marine viruses infect not only plankton but cultivated stocks of Crabs, Oysters, Mussels, Clams shrimp, Salmon and Catfish, etc. are all susceptible to various kinds of viruses. We observed that the viruses are nonliving organisms, in the sense, they have no metabolism when out side the host and they can reproduce only by infecting the living organisms. Viral infection of the phytoplankton cell is of two types, namely, Lysogenic and Lytic. In lytic viral infection, when a virus injects its DNA into a cell, it hijacks the cell's replication machinery and produces large a number of viruses. As a result, they rupture the host and are released into the environment. On the other hand, in lysogenic viral infection, the DNA of the viruses do not use the machinery of the host themselves, but their genes are duplicated each time as the host cell divides. Many papers have already been developed which have used this kind of lysogenic viral infection [1, 3, 12, 13].

Plankton pattern formation is dependent on the interplay of various physical (temperature, light) and biological (nutrient supply, fish predation) factors. The pattern formation focuses on environment, social and technological sciences where the nonlinearities conspire to from spatial patterns observe. The pattern formation in living systems is probably one of the most exciting subjects in modern biology and ecology.

We observed that the pattern formations in the population dynamics of both aquatic system and natural environment[7, 9]. Our mainly study the pattern formation in marine ecosystem taking the phytoplankton dynamics. Plankton system is study an important area for research in marine ecology. Sometimes are stationary, spiral, traveling or disordered in space and time often referred as spatiotemporal chaos. The diffusion of population is capturing the spatial distribution (i.e. pattern) of both susceptible and infected class of population. The reaction-diffusion equations modeling predator-prey interactions show a wide spectrum of ecologically relevant behavior resulting from intrinsic factors alone[6, 10].

This study is partially motivated by few works, namely, (i) a SIAM review paper [8] that considers the reaction - diffusion system as a model for marine plankton dynamics, (ii) a study on diffusion induced chaos [11] and (iii) a phytoplankton dynamics with susceptible and infective classes [4]. Our mathematical model is an extension of temporal model presented by [4], in spatiotemporal domain.

In this paper, we have introduced the spatially extended of the system and analyze four different cases of the system. In the first case, Proposed and analyzed the model , in second case, determined the diffusion driven stability condition, in third case, numerical simulation of chaos and pattern formation in 1-D and 2-D. Fourth case, conclusion and discuss are given.

Mathematical Model

A mathematical model of phytoplankton dynamics is proposed by considering the population densities of susceptible and infected phytoplankton as P_s and P_i respectively, at any instant of time T. The population of susceptible phytoplankton is assumed to be growing logistically with intrinsic growth rate r and carrying capacity K. Let a_1 be the disease contact rate and d_1 be the removal rate of the diseased phytoplankton population, out of which c_1 fraction of infected phytoplankton rejoin the susceptible phytoplankton population, because, dead infected phytoplankton become nutrients for the growth of susceptible phytoplankton after bacterial decomposition and partially through natural recovery process in the ecosystem. The proposed mathematical model can state by the following reaction diffusion equations:

$$\frac{\partial P_s}{\partial T} = rP_s\left(1-\frac{P_s}{K}\right)-\frac{a_1P_sP_i}{\eta+P_s}+cP_i+\delta_1\nabla^2P_s, \qquad (1)$$

$$\frac{\partial P_i}{\partial T} = \frac{a_1P_sP_i}{\eta+P_s}-dP_i+\delta_2\nabla^2P_i, \qquad (2)$$

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$$P_s(0) > 0,\; P_i(0) > 0,$$

where a_1, c_1, d_1, η are all positive constant and ∇^2 is the usual laplacian operator for two dimensional, δ_i, ($i = 1,2$) are diffusion coefficients and a, b, c, d are positive constants. The Holling type-II the functional response $\frac{a_1 P_s P_i}{\eta + P_s}$ is used [5] and many other researchers.

It is much easier to work with equation that have been scaled to non-dimensional form, in above system, we take $u = \frac{P_s}{K}$, $v = \frac{\alpha P_i}{rK}$, $t = rT$, $x_i = X_i\left(\frac{r}{\delta_1}\right)^{\frac{1}{2}}$ and re-scaling the parameters via, $\xi = K$,

$$\alpha = \frac{\eta}{K},\; \gamma = \frac{c}{a},\; \delta = \frac{\delta_1}{\delta_2},\; \omega = \frac{cK}{r},\; \beta = \frac{d}{r}.$$

Hence our spatial model reduces to

$$\frac{\partial u}{dt} = u(1-u) - \frac{\xi uv}{(u+\alpha)} + \gamma v + \Delta u, \quad (3)$$

$$\frac{\partial v}{dt} = \frac{\omega uv}{(u+\alpha)} - \beta v + \delta \Delta v, \quad (4)$$

$$u(x,y,0) > 0,\; v(x,y,0) > 0,\; (x,y)\;\varepsilon\;\Omega \quad (5)$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0,\; (x,y)\;\varepsilon\;\partial\Omega,\; t > 0, \quad (6)$$

where, n is the outward normal to $\partial\Omega$. and the parameters α, β, γ, ξ, ω are positive constant. We assume that the system is defined on two dimensional bounded domain, denoted by Ω and consider the zero-flux boundary conditions.

Diffusion-driven Instability

Now, in this subsection, we will explore the possibility of diffusion-driven instability with respect to the equilibrium solution, i.e., the spatially homogenous solution (u^*, v^*) of the reaction diffusion system. Obviously, the interior equilibrium point E^* for the non-spatial system is a spatially homogeneous steady-state for the reaction-diffusion system (3)-(4). We assume that E^* is stable in the non-spatial system, which means that the spatially homogeneous equilibrium is stable with respect to spatially homogeneous perturbations.

The conditions for the diffusion instability to occur in system (3) and (4), we take small heterogeneous perturbation following form:

$$u(x,y,t) = u^* + \varepsilon\, exp((kx+ky)i + \lambda_k t), \quad (7)$$

$$v(x,y,t) = v^* + \eta\, exp((kx+ky)i + \lambda_k t), \quad (8)$$

where ε and η are chosen to be small and $k = (k_x, k_y)$ is the wave number. Substituting (7)-(8) into (3) and (4), linearizing the system around the interior equilibrium E^*, we get the characteristic equation as follows:

$$|J_k - \lambda_k I_2| = 0, \quad (9)$$

with

$$J_k = \begin{bmatrix} a_{11} - k^2 & a_{12} \\ a_{21} & a_{22} - \delta k^2 \end{bmatrix}.$$

where, I_2 and k are second order identity matrix and wave number respectively and

$$a_{11} = 1 - 2u^* - \frac{\alpha \xi v^*}{(u^*+\alpha)^2},\; a_{12} = -\frac{\xi\beta}{\omega} + \gamma,$$

$$a_{21} = \frac{\omega\alpha v^*}{(u^*+\alpha)^2},\quad a_{22} = 0.$$

The diffusion instability conditions when at least one of (9) the eigenvalues of the systems matrix crosses the imaginary axis. The characteristic equation following form:

$$\lambda_k^2 - (a_{11} + a_{22} - k^2(1+\delta))\lambda_k + (a_{11} - k^2)(a_{22} - \delta k^2) - a_{12}a_{21} = 0. \quad (10)$$

We obtain that a change in stability will occur when at least one of the following two inequalities does not hold:

$$a_{11} + a_{22} - (1+\delta)k^2 < 0, \quad (11)$$

$$h(k^2) \equiv (a_{11} - k^2)(a_{22} - \delta k^2) - a_{12}a_{21} > 0, \quad (12)$$

where a_{ij} are the elements of the matrix J^*. Since δ and k^2 are positive, both the inequalities always holds as $a_{11} = tr(J^*) < 0$ by the stability condition of the non-spatial steady state. Hence in this system, the diffusion-driven instability never occurs.

Numerical simulation is carried out for the linear stability of the system (3)-(4) taking same parameter values as in the previous subsection. Moreover, it is observed that the increment of ratio diffusivity coefficient stabilizes the system (see Fig. 1).

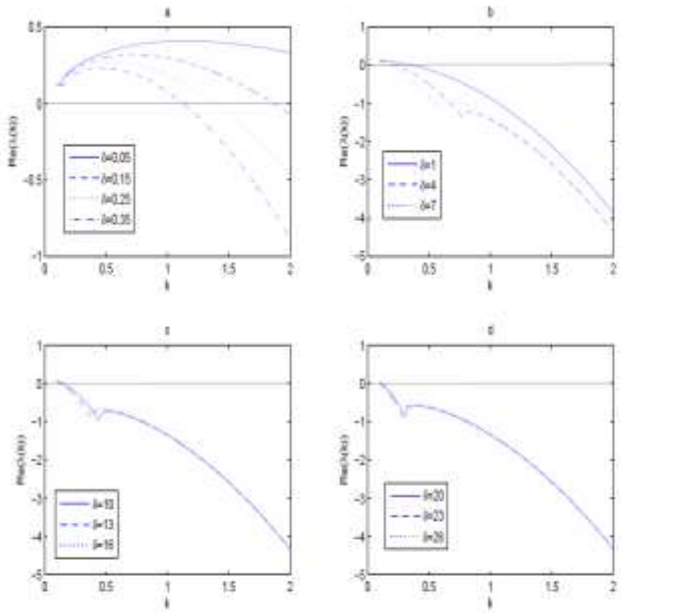

Figure 1: Plot of max Re $(\lambda(k))$ against k, parameter values are given in text.

Now, in the next two subsection, we perform numerical simulation of the spatiotemporal system (3)-(4) evolution of pattern with respect to the time T .

One Dimensional Case

In this subsection, we will study of the following system for one dimensional case.

$$\frac{\partial u}{dt} = u(1-u) - \frac{\xi uv}{(u+\alpha)} + \gamma v + \frac{\partial^2 u}{\partial x^2}, \qquad (13)$$

$$\frac{\partial v}{dt} = \frac{\omega uv}{(u+\alpha)} - \beta v + \delta \frac{\partial^2 u}{\partial x^2}, \qquad (14)$$

$$u(x,0) > 0, \;\; v(x,0) > 0, \;\; for \; x \, \varepsilon \, [0, R] \qquad (15)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0, \qquad (16)$$

The numerical solutions of the phytoplankton dynamics (i.e., u, v) are plotted with one space coordinate and time. Computer experiments are done in one dimension with domain size 6000 and we checked the sensitivity of the results to the choice of the time and space steps and their values are chosen sufficiently small. The parameter values and initial data: $\omega = 2$, $\beta = 0.8$, $\xi = 1$, $\gamma = 0.0001$, $\delta = 1$, $h = 4$, $\Delta t = 10^{-2}$, $\alpha = 0.3$. Varying the time to the four basic one-dimensional dynamics, namely stationary, intermittent chaos and chaos covering (almost all) of the domain (see Fig. 2).

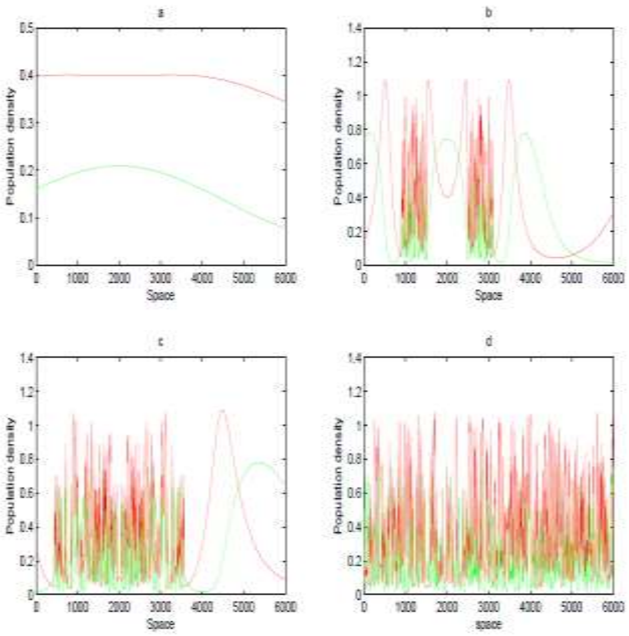

Figure 2: The red lines for susceptible phytoplankton, and green lines for infected phytoplankton population density. Simulations are obtained for different time scales: In figure (a)T=5, in (b) T=800, in (c) T=2000, in (d) T=10000.

Two-Dimensional Case

The numerical solutions of the phytoplankton dynamics (3)-(4) are plotted for two dimensional (i.e.,) space x and y coordinate with time t with square domain (600×600) is used for figure 3. The reaction diffusion equation is solved using finite difference technique semi implicit in time along with zero flux boundary condition and non-zero asymmetrical initial condition. The parameter values are $\xi = 1$, $\alpha = 0.4$, $\beta = 0.6$, $\omega = 2.0$, $\gamma = 0.0001$, $\delta = 0.5$, $h = 4$, $\Delta t = 1/6$ and initial condition (5)-(6). The time evolution of the system led to the formation of spiral patterns, followed by irregular patches covering the whole domain (see Fig. 3). The size of these patches has been related to the characteristic length of observed plankton patterns in the ocean.

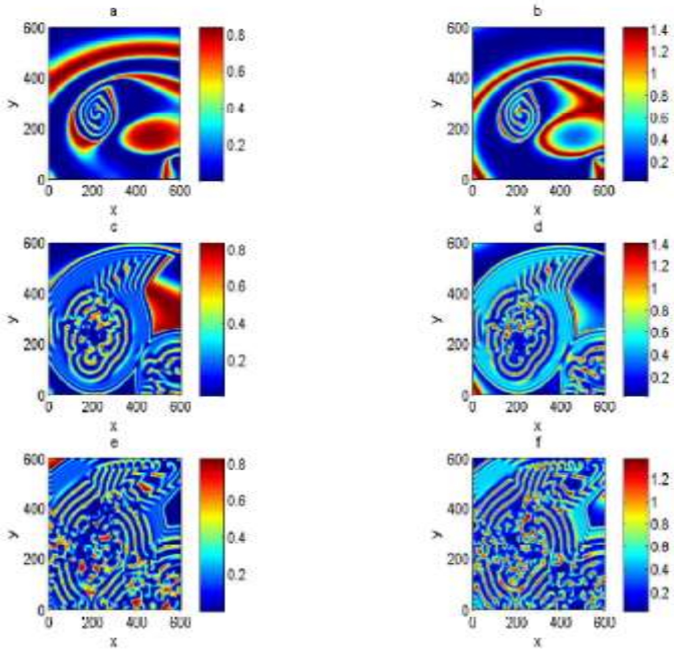

Figure 3: Susceptible phytoplankton densities [first column] and infected phytoplankton densities [second column] population density of the system. Spatial patterns are obtained different time scale. Plots show population density of (a)-(b) $T = 300$, (c)-(d) $T = 600$, (e)-(f) $T = 900$.

Conclusion

In this paper, a phytoplankton dynamics namely, susceptible and infected population with spatial movement have been studied. Also we have studied the reaction diffusion model in both one and two dimension space coordinates. For one dimensional case, we shown that how modest changes in a single parameter of the system time T, can lead to dramatic changes in the qualitative dynamics of solutions(see Fig.2). Furthermore, the dynamics of the spatially extended system are complicated and will depend on the system parameters, the initial data, and also the specifics of habitat geometry. There are situations where the local dynamics of solutions gives us important clues to the behavior in the spatially extended situation. For two dimensional case, numerical experiments for different values time T and different types of initial conditions for obtained different types of pattern (see Fig. 3). Hence, the the rate of growth of susceptible phytoplankton due to the dead infected phytoplankton (which become nutrients of susceptible phytoplankton after bacterial decomposition) is a major factor for the stability of the system.

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