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P_{4k+1}-Factorization of complete bipartite graphs

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ABSTRACT

P_{2p}-factorization of a complete bipartite graph for p an integer was studied by Wang [1]. Further, Beiliang [2] extended the work of Wang [1], and studied the P_{2k}-factorization of complete bipartite multigraphs. For even value of k in P_k-factorization, the spectrum problem is completely solved [1, 2, 3]. However for odd value of k i.e. P_3, P_5, P_7 and P_9, the path factorization have been studied by a number of researchers [4, 5, 6, 7]. Again, P_3 —factorizations of complete bipartite multigraphs and symmetric complete bipartite multi-digraphs were studied by Wang and Beiliang[8]. Also, Beiliang and Wang have shown that Ushio conjecture is true for $4k-1$ factorization of complete bipartite graphs[9]. In the present paper we shall show that ushio conjecture is also true for $4k+1$ factorization of complete bipartite graphs. That is, we shall prove that a necessary and sufficient condition for the existence of a P_{4k+1}-factorization of $K_{m,n}$ is (1) $(2k+1)m \geq 2kn$, (2) $(2k+1)n \geq 2km$, (3) $m+n \equiv 0 \pmod{4k+1}$, (4) $(4k+1)mn/[4k(m+n)]$ is an integer.

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Introduction

Ushio conjecture [11] for path factorization of complete bipartite graphs is as follows:

If k is odd, and m and n be positive integers. Then $K_{m,n}$ has P_k-factorization if and only if:

(1) $(k-1)m \geq kn$,

(2) $(k-1)n \geq km$,

(3) $m+n \equiv 0 \pmod{k}$,

(4) $kmn/[(k-1)(m+n)]$ is an integer.

In this paper, we shall prove that Ushio conjecture is true for the path factorization of P_{4k+1}-factorization of complete bipartite graphs, that is we shall prove the theorem given below.

Theorem 1: Let k, m, n be positive integers, there exist a P_{4k+1}-factorization of $K_{m,n}$ if and only if:

(1) $(2k+1)m \geq 2kn$,

(2) $(2k+1)n \geq 2km$,

(3) $m+n \equiv 0 \pmod{4k+1}$,

(4) $(4k+1)mn/[4k(m+n)]$ is an integer.

Mathematical Analysis

We first give the proof of necessity of theorem 1, which is given in theorem 2. The sufficiency of theorem 1 is proved by theorem 3.

Theorem 2: Let k, m, n be positive integers. Then for P_{4k+1}- factorization it is necessary that:

(1) $(2k+1)m \geq 2kn$,

(2) $(2k+1)n \geq 2km$,

(3) $m+n \equiv 0 \pmod{4k+1}$,

(4) $(4k+1)mn/[4k(m+n)]$ is an integer.

Proof: Let r be the number of P_{4k+1} -factor in the factorization and e be the number of copies of P_{4k+1} in any factor.

Then $e = \frac{m+n}{4K+1}$, and $r = \frac{(4K+1)mn}{4K(m+n)}$

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hence conditions (3) and (4) are necessary.

Let a and b be the number of copies of P_{4k+1} with its end points in Y and X in a particular P_{4k+1}-factor respectively.

Then ,

$$(2k)a+(2k+1)b=m, \text{and } (2k+1)a+(2k)b=n$$

.

Hence,

$a=\frac{(2k+1)n-(2k)m}{4k+1}$, and $b=\frac{(2k+1)m-(2k)n}{4k+1}$.

Conditions (1) and (2) are therefore, necessary. This proves the necessecity of the theorem 1.

Now we will prove the sufficiency of theorem 1. Which is given by theorem 3.

Theorem 3: Let k, m, n be positive integers. Then for P_{4k+1}-factorization, it is sufficient that:

$(1)\ (2k+1)m \geq 2kn,$

$(2)(2k+1)n \geq 2km,$

$(3)\ m+n \equiv 0 \ (mod\ 4k+1),$

$(4)\ (4k+1)mn/[4k(m+n)]$ is an integer.

The proof of this theorem, consist of the following lemmas.

Lemma 1: Let a, b, p and q be positive integers. If gcd$(ap, bq)=1$, then gcd$(ab,\ ap+bq)=1$.

We prove the following result which is used later in the paper.

Lemma 2: If $K_{m,n}$ has a P_{4k+1}-factorization, then $K_{sm,sn}$ has a P_{4k+1}-factorization for every positive integer s.

Proof: Let $K_{s,s}$ is a 1- factorable [10] and { $F_1, F_2, \ldots, F_s$} be a 1- factorization of it. For each i with $1 \leq i \leq s$, replace every edge of F_i by a $K_{m,n}$ to get a spanning subgraph G_i of $K_{sm,sn}$ such that the graph $G_i's\ \{1 \leq i \leq s\}$ are pair wise edge disjoint and there union is $K_{sm,sn}$. Since $K_{m,n}$ has a P_{4k+1}-factorization, it is clear that the G_i is also$P_{4k+1}-$ factorable, and hence $K_{sm,sn}$ is also P_{4k+1}- factorization.

Lemma 2 implies that there are three cases to consider.

Case$(1)\ \boldsymbol{2km=(2k+1)n}$:

In this case, let

$$F_j=\{x_{i+2j}y_i, x_{i+2j+1}y_i : 1 \leq i \leq 4k\}, 1 \leq j \leq 2(2k+1).$$

It is easy to see that it is a $P_{4k+1}-$ factor of $K_{4k,4k+2}$. Then $\bigcup_{1\leq j\leq 2(2k+1)} F_j$ is a P_{4k+1}-factorization of $K_{4k,4k+2}$. $K_{m,n}$ has P_{4k+1}-factorization.

Case$(2)\ \boldsymbol{(2k+1)m=2kn}$:

Obviously, $K_{m,n}$ has aP_{4k+1}-factorization.

Case$(3)\ \boldsymbol{(2k+1)m>(2k)n}$ and $\boldsymbol{(2k+1)n>(2k)m}$:

In this case, let

$$a=\frac{(2k+1)n-2km}{4k+1},\quad b=\frac{(2k+1)m-2kn}{4k+1},$$

$$e=\frac{m+n}{4k+1},\quad \text{and } r=\frac{(4k+1)mn}{4k(m+n)}.$$

Then, from condition (1) to (4) in theorem (2), *a*, *b*, *e* and r are integers, and $0<a<m$ and $0<b<n$.

We have

$$(2k)a+(2k+1)b=m \text{ and } (2k)b+(2k+1)a=n.$$

Hence,

$$r=\frac{(k+1)(a+b)}{2}+\frac{ab}{4k(a+b)}.$$

Further, let $z=\frac{ab}{2k(a+b)}$ be a positive integer.

Let gcd$((2k)a, (2k+1)b)=d$.

Then, $2ka=dp$ and $(2k+1)b=dq$, where gcd$(p,q)=1$.

Therefore,

$$z=\frac{dpq}{2k((2k+1)p+2kq)}.$$

These equalities imply the following equalities:

$$d=\frac{2k[(2k+1)p+2kq]z}{pq},$$

$$r=\frac{(p+q)[(2k+1)^2p+4k^2q]z}{2pq},$$

$$m=\frac{2k(p+q)[(2k+1)p+2kq]z}{pq},$$

$$n=\frac{[(2k+1)^2p+4k^2q][(2k+1)p+2kq]z}{(2k+1)pq},$$

$$a=\frac{p[(2k+1)p+2kq]z}{pq} \text{ and}$$

$$b=\frac{2kq[(2k+1)p+2kq]z}{(2k+1)pq}.$$

Let $2k=p_1^{k_1}.p_2^{k_2}\ldots p_\gamma^{k_\gamma}$, where $p_1, p_2, \ldots, p_\gamma$ are distinct prime numbers with $k_1, k_2, \ldots, k_\gamma$ positive integers.

Also, let $2k+1=q_1^{h_1}.q_2^{h_2}\ldots q_\omega^{h_\omega}$, where $q_1, q_2, \ldots, q_\omega$ are distinct prime numbers with $h_1, h_2, \ldots, h_\omega$ positive integers.

Now we can establish the following lemma.
Lemma 3:
If
gcd
$(p,4k^2)=p_1^{i_1}.p_2^{i_2}...p_\alpha^{i_\alpha}.p_{\alpha+1}^{2k_{\alpha+1}-i_{\alpha+1}}.p_{\alpha+2}^{2k_{\alpha+2}-i_{\alpha+2}}...p_\beta^{2k_\beta-i_\beta}p_{\beta+1}^{2k_{\beta+1}}...p_\gamma^{2k_\gamma}$,

Where $1\le\alpha\le\beta\le\gamma, 0\le i_j\le k_j$(when $1\le j\le\alpha$) or $0<i_j<k_j$ (when $\alpha+1\le j\le\beta$);

gcd($q,(2k+1)^2$)
$=q_1^{j_1}.q_2^{j_2}...q_\mu^{j_\mu}.q_{\mu+1}^{2h_{\mu+1}-j_{\mu+1}}...q_\varepsilon^{2h_\varepsilon-j_\varepsilon}q_{\varepsilon+1}^{2h_{\varepsilon+1}}...q_\omega^{2h_\omega}$,

where $1\le\mu\le\varepsilon\le\omega$, $0\le j_i\le h_i$(when $1\le i\le\mu$) or $0<j_i<h_i$(when $\mu+1\le i\le\varepsilon$).

Let

$s=p_1^{i_1}.p_2^{i_2}...p_\alpha^{i_\alpha}$, $t=p_1^{k_1-i_1}p_2^{k_2-i_2}...p_\alpha^{k_\alpha-i_\alpha}$, $u=p_{\alpha+1}^{i_{\alpha+1}}.p_{\alpha+2}^{i_{\alpha+2}}...p_\beta^{i_\beta}$,

$v=p_{\alpha+1}^{k_{\alpha+1}-i_{\alpha+1}}p_{\alpha+2}^{k_{\alpha+2}-i_{\alpha+2}}...p_\beta^{k_\beta-i_\beta}$, $w=p_{\beta+1}^{k_{\beta+1}}p_{\beta+2}^{k_{\beta+2}}...p_\gamma^{k_\gamma}$,

$s'=q_1^{j_1}q_2^{j_2}...q_\mu^{j_\mu}$, $t'=q_1^{h_1-j_1}q_2^{h_2-j_2}...q_\mu^{h_\mu-j_\mu}$,
$u'=q_{\mu+1}^{j_{\mu+1}}q_{\mu+2}^{j_{\mu+2}}...q_\varepsilon^{j_\varepsilon}$,
$v'=q_{\mu+1}^{h_{\mu+1}-j_{\mu+1}}q_{\mu+2}^{h_{\mu+2}-j_{\mu+2}}...q_\varepsilon^{h_\varepsilon-j_\varepsilon}$,
$w'=q_{\varepsilon+1}^{h_{\varepsilon+1}}q_{\varepsilon+2}^{h_{\varepsilon+2}}...q_\omega^{h_\omega}$.

Also, let
$p=suv^2w^2p'$, $q=s'u'v'^2w'^2q'$.

Now three cases are possible:

Case (1): If $t'\equiv 1(mod\ 2)$ and $v'w'\equiv 1(mod\ 2)$, then for some positive integer z'
$m=4stut'(suv^2w^2p'+s'u'v'^2w'^2)(t'vwp'+tv'w'q')z'$,
$n=2suvwv'w'(s't'^2u'p'+st^2uq')(t'svwp'+ts'v'w'q')z'$,
$a=2suvwp't'(t'vwp'+tv'w'q')z'$,
$b=2stuv'w'q'(vwt'p'+tv'w'q')z'$,
$r=t'v'w'(suv^2w^2p'+s'u'v'^2w'^2q')(s't'^2u'p'+st^2uq')z'$,
$d=2stut'(vwt'p'+tw'v'q')z'$.

Case (2): If $t'\equiv 0(mod\ 2)$ and $v'w'\equiv 1(mod\ 2)$, then for some positive integer z'
$m=2stut'(suv^2w^2p'+s'u'v'^2w'^2)(t'vwp'+tv'w'q')z'$,
$n=suvwv'w'(s't^2u'p'+st^2uq')(t'svwp'+ts'v'w'q')z'$,
$a=suvwp't'(t'vwp'+tv'w'q')z'$,
$b=stuv'w'q'(vwt'p'+tv'w'q')z'$,
$r=t'w'v'(suv^2w^2p'+s'u'v'^2w'^2q')(s't'^2u'p'+st^2uq')z'/2$,
$d=stut'(vwt'p'+tw'v'q')z'$.

Case (3): If $t'\equiv 1\ (mod\ 2)$ and $v'w'\equiv 0(mod\ 2)$, then for some positive integer z'
$m=2stut'(suv^2w^2p'+s'u'v'^2w'^2)(t'vwp'+tv'w'q')z'$,
$n=suvwv'w'(s't^2u'p'+st^2uq')(t'svwp'+ts'v'w'q')z'$,
$a=suvwp't'(t'vwp'+tv'w'q')z'$,
$b=stuv'w'q'(vwt'p'+tv'w'q')z'$,
$r=t'w'v'(suv^2w^2p'+s'u'v'^2w'^2q')(s't'^2u'p'+st^2uq')z'/2$, and
$d=stut'(vwt'p'+tw'v'q')z'$.

Proof. We are now giving the proof of each case of lemma 3.

If $\gcd(p,q)$=1 $\gcd(p,4k^2)=suv^2w^2$, $\gcd(q,(2k+1)^2)=s'u'v'^2w'^2$ and $p=suv^2w^2p'$ and $q=s'u'v'^2w'^2q'$ hold, then

$\gcd(suv^2w^2p',s'u'v'^2w'^2q')=\gcd(s'u't'^2p',sut^2q')=1$

From lemma 1, it is clear that

$\gcd(suv^2w^2p'+s'u'v'^2w'^2q',p'q')=\gcd(s't'^2u'p'+st^2uq',p'q')=1.$

Since,

$$r=\frac{(suv^2w^2p'+s'u'v'^2w'^2q')(s't'^2u'p'+st^2uq')z}{2p'q'}$$

is an integer; hence, we see that $\frac{z}{2p'q'}$ must be an integer.

Let $z_1=\frac{z}{(2p'q')}$, then

$$d=\frac{2sut(t'vwp'+tv'w'q')z_1}{v'w'}.$$

Depending on the values of parameters t' and $v'w'$ the proof of three cases of lemma 3 are as follows.

Case (1): $t'\equiv 1(mod\ 2)$ and $v'w'\equiv 1(mod\ 2)$:

Since $\gcd(2,v'w')=\gcd(stu,v'w')=1$ and $\gcd(vwt'p'+tv'w'q',v'w')=1$, therefore, $\frac{z_1}{(v'w')}$ is an integer. Let $z_2=\frac{z_1}{(v'w')}$, then

$$n=\frac{2suvwv'w'(s't'^2u'p'+st^2uq')(t'svwp'+ts'v'w'q')z_2}{t'}.$$

Since $\gcd(2,t')=\gcd(suvwv'w',t')=1$ and $\gcd(t'svwp'+ts'v'w'q',t')=\gcd(s't'^2u'p'+st^2uq',t')=1$, therefore $\frac{z_2}{t'}$ is an integer.

Let $z'=\frac{z_2}{t'}$, then the equalities in Case (1) hold.

Case (2): $t'\equiv 0(mod\ 2)$ and $v'w'\equiv 1(mod\ 2)$:

Since $\gcd(2,v'w')=\gcd(stu,v'w')=1$ and $\gcd(vwt'p'+tv'w'q',v'w')=1$,

hence $\frac{z_1}{(v'w')}$ is an integer. Let $z_2=\frac{z_1}{(v'w')}$, then

$$n=\frac{2suvwv'w'(s't'^2u'p'+st^2uq')(t'svwp'+ts'v'w'q')z_2}{t'}.$$

Since,
$\gcd(2,t')=2$, $\gcd(suvwv'w',t')=\gcd(t'svwp'+ts'v'w'q',t')=1$
also
$\gcd(s't'^2u'p'+st^2uq',t')=1$, therefore $\frac{2z_2}{t'}$ is an integer.

Let $z'=\frac{2z_2}{t'}$, then the equalities in (2) hold.

Case (3): $t'\equiv 1\ (mod\ 2)$ and $v'w'\equiv 0(mod\ 2)$

Since, $\gcd(2,v'w')=2$, and
$\gcd(stu,v'w')=\gcd(vwt'p'+tv'w'q',v'w')=1$,

hence $\frac{2z_1}{(v'w')}$ is an integer.

Let $z_2=\frac{2z_1}{(v'w')}$, then

$$n=\frac{suvwv'w'(s't'^2u'p'+st^2uq')(t'svwp'+ts'v'w'q')z_2}{t'}$$

Since, $\gcd(2,t')=\gcd(suvwv'w',t')=1$ and
$\gcd(t'svwp'+ts'v'w'q',t')=\gcd(s't'^2u'p'+st^2uq',t')=1$
,
therefore $\frac{z_2}{t'}$ is an integer.

Let $z'=\frac{z_2}{t'}$ then the equalities in case (3) hold.

This proves the lemma 3.

We now give the direct construction of case (1) by taking $z'=1$ in lemma 3. We will call this as lemma 4.

Lemma 4. For any positive integers $s,t,u,v,w,s',t',u',v',w',p$ and q, let

$$m=4stut'(suv^2w^2p+s'u'v'^2w'^2q)(vwt'p+tv'w'q),$$
$$n=2suvwv'w'(s't'^2u'p+st^2uq)(vwp\,t's+qts'v'w').$$

Then $K_{m,n}$ has a P_{4k+1}- factorization.

Proof. Let
$a=2suvwpt'(t'vwp+tv'w'q)$, $b=2stuv'w'q(vwt'p+tv'w'q)$

and $r=t'v'w'(suv^2w^2p+s'u'v'^2w'^2q)(s't'^2u'p+st^2u\,q)$,

then $r_1=t'(suv^2w^2p+s'u'v'^2w'^2q)$ and $r_2=v'w'(s't'^2u'p+st^2u\,q)$.

Let X and Y be two partite set of $K_{m,n}$ such that

$$X=\{x_{i,j}:1\le i\le r_1;1\le j\le m_0\}$$

and

$$Y=\{y_{i,j}:1\le i\le r_2;1\le j\le n_0\},$$

where $m_0=\frac{m}{r_1}=4stu(vwpt'+tv'qw')$

and $n_0=\frac{n}{r_2}=2suvw(vwp\,t's+qts'v'w')$.

Now for each i,x,y,z and z',
$1\le i\le t'p$, $1\le x\le vw$, $1\le y\le suvw$,
$1\le z\le t$ and $0\le x'\le 1$,
let

$$f(i,x,y)=suv^2w^2(i-1)+suvw(x-1)+y,$$

$g(i,y,z,x')=s't'u'v'w'(i-1)+suvw(z-1)+y+x'$ and
$h(i,x,y,x')=2suvw(i-1)+2su(vwt'p+tv'w'q)(x-1)+2y+x'-1$,

here $stuvw+1=s't'u'v'w'$ and set

$$E_i=\{x_{f(i,x,y),j+4su(vwt'p+tv'w'q)(z-1)}y_{g(i,y,z,x'),j+h(i,x,y,x')}:$$
$$1\le j\le 4su(vwt'p+tv'w'q), 1\le x\le vw,$$
$$1\le y\le suvw, 1\le z\le t, 0\le x'\le 1\}.$$

For each i,x,y,z and x',
$1\le i\le v'w'q$, $1\le x\le stu$, $1\le y\le vw$, $1\le z\le t$ and $0\le x'\le 1$,
Let
$\varphi(i,x,y,x')=suv^2w^2t'p+s't'u'v'w'(i-1)+vw(x-1)+y+x'$,
$\psi(i,x,z)=s't'^2u'v'w'p+st^2u(i-1)+stu(z-1)+x$
and
$\emptyset(i,x,y,x')=2suvwt'p+2x+2stu(i-1)+2su(vwt'p+tv'w'q)(y-1)+x'-1$,
and let

$$E_{t'p+i}=\{x_{\emptyset(i,x,y,x'),j+4su(vwt'p+tv'w'q)(z-1)}y_{\psi(i,x,z),j+\emptyset(i,x,y,x')}:$$
$$1\le j\le 4su(vwt'p+tv'w'q), 1\le x\le stu, 1\le y\le vw, 1\le z\le t,$$
$$0\le x'\le 1\}.$$

Let $F=\cup_{1\le i\le t'p+v'w'q}E_i$, then it is easy to see that the graph F is a P_{4k+1}- factor of $K_{m,n}$. Define a bijection σ from $X\cup Y$ onto $X\cup Y$ in such a way that $\sigma(x_{i,j})=x_{i+1,j}$, $\sigma(y_{i,j})=y_{i+1,j}$.

For each $i\in\{1,2,\dots,r_1\}$ and each $j\in\{1,2,\dots,r_2\}$, let

$$F_{i,j}=\{\sigma^i(x)\sigma^j(y):x\in X,y\in Y,xy\in F\}.$$

It is easy to show that the graphs $F_{i,j}(1 \le i \le r_1, 1 \le j \le r_2)$ are P_{4k+1}-factor of $K_{m,n}$ and there union is $K_{m,n}$.

Thus $(F_{i,j}: 1 \le i \le r_1, 1 \le j \le r_2)$ is a P_{4k+1}-factorization of $K_{m,n}$.

This proves the lemma 4. Similarly we can prove the direct constructions of cases (2) and (3).

Proof (Theorem 3):

Applying lemmas 2-4, we see that for the parameters k, m and n satisfying conditions (1)-(4) in theorem 1, $K_{m,n}$ has a P_{4k+1}-factorization. This proves the sufficiency of the conditions given in theorem 3.

Proof (Theorem 1):

Combining theorem 2 and 3, we complete the proof of theorem 1. This proves that Ushio conjecture for P_{4k+1}-factorization of $K_{m,n}$ is true.

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