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0 – Modularity in the lattice of weak Congruences

A. Vethamanickam[1] and A. Veeramani[2]

[1]Department of Mathematics, Kamarajar Government Arts College, Surandai - 627 857.

[2]Department of Mathematics, H. H. The Rajah's College (Autonomous B+),Pudukkottai - 622 001.

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ABSTRACT

In this paper, we study 0 – modularity in the lattice of weak congruences. We are going to prove that $C_w(L)$ is 0 – modular if and only if L is a chain.

Introduction

The study on weak congruences in algebras have started in the seventies by H. Draskovicova[8], M. Kolibiar, F. Sik, T. D. Mai[15] and others, in the name of quasicongruences. Also, reflexive, symmetric relations called tolerances play an important role in algebra and applications. This concept occurs in automata theory[1], biology questions[21], linguistics[10] and it is used for some "inaccuracy" in abstract algebra.

Since then so many authors have made contributions to the weak congruence theory. For example, one can refer to [2], [4], [6], etc.

Several authors have attempted to characterize the structure of an algebra $\mathcal{A}$ in terms of its lattice of weak congruences $C_w(\mathcal{A})$. For example, Gradimir Vojvodic and Branimir Seselja proved that for an algebra , $C_w(\mathcal{A})$ is modular if and only if (i) $\mathcal{A}$ satisfied the CEP and CIP (ii) Con $\mathcal{A}$ and Sub $\mathcal{A}$ are modular[20]. In the case of a lattice L, Andreja Tepavcevic[17], etc., has proved that $C_w(L)$ is semimodular if and only if L is a two-element chain. These result motivated us to look for which $C_w(L)$ satisfy still weaker condition 0-modular.

Preliminary Notes

An algebra is a pair ($\mathcal{A}$, F) where $\mathcal{A}$ is a non-empty set and F is a set of finitary operations on $\mathcal{A}$. A weak congruences relation on $\mathcal{A}$ is a symmetric and transitive relation ρ on $\mathcal{A}$ which satisfies the substitution property namely, for each n-ary operation $f_i \in F$, whenever $a_j \rho b_j$, j = 1, 2,…n…$f_i(a_1, a_2,\ldots,a_n) \rho f_i(b_1, b_2,\ldots,b_n)$, $a_j, b_j \in \mathcal{A}$, j = 1, 2,…,n.

In other words, a weak congruence relation on $\mathcal{A}$ is a symmetric and transitive sub algebra of $\mathcal{A}^2$. The set of all weak congruence relations on $\mathcal{A}$ denoted by $C_w(\mathcal{A})$ becomes an algebraic lattice under the set inclusion[20].

The diagonal relation $\Delta = \{(x, x) \mid x \in L\}$ is always a co-distributive element in a weak congruence lattice, i.e., for all $\rho, \theta \in C_w(\mathcal{A})$, the following holds:

$$\Delta \wedge (\rho \vee \theta) = (\Delta \wedge \rho) \vee (\Delta \wedge \theta)$$

The filter $[\Delta) = \Delta\uparrow = \{\Theta \in C_w(\mathcal{A}) \mid \Theta \geq \Delta\}$ is isomorphic to the lattice of congruences Con $\mathcal{A}$, and the ideal $(\Delta] = \Delta\downarrow = \{\Theta \in C_w(\mathcal{A}) \mid \Theta \leq \Delta\}$, consisting of all the diagonal relations is isomorphic with Sub $\mathcal{A}$, under the mapping $\rho \rightarrow \{x \mid x \rho x\}$[20]. The atoms are always join-irreducible and the co-atoms are always meet-irreducible in any lattice. If L is a chain then Sub(L) is Boolean[13]. In $C_w(L)$, Sub(L) is atomic[13]. It is easily seen that Con(L) is Boolean in the case when L is a chain[13].

We use the following notations throughout the paper:

$C_w(L)$ is the set of all weak congruences on a lattice L.

Sub(L) is the set of all sublattices of L.

Con(L) is the set of all congruences on L.

Definition 2.1. A poset (P, $\leq$) is called a chain if any two elements in P are comparable . That is, for any two elements a, b $\in$ P either a $\leq$ b or b $\leq$ a, hold good. We denote an n-element chain by L_n.

Definition 2.2. An equivalence relation Θ (that is, a reflexive, symmetric, and transitive binary relation) on a lattice L is called a congruence relation on L iff $(a_0, b_0) \in \Theta$ and $(a_1, b_1) \in \Theta$ imply that $(a_0 \wedge a_1, b_0 \wedge b_1) \in \Theta$ and $(a_0 \vee a_1, b_0 \vee b_1) \in \Theta$ (Substitution Property).

Definition 2.3. [20] A binary relation Θ on a lattice L is called a weak congruence relation, if it is a symmetric and transitive binary relation satisfying the substitution property, that is $a_0, b_0, a_1, b_1 \in L$, $(a_0, b_0) \in \Theta$ and $\mathbf{(a_1, b_1)} \in \Theta$

Tele:

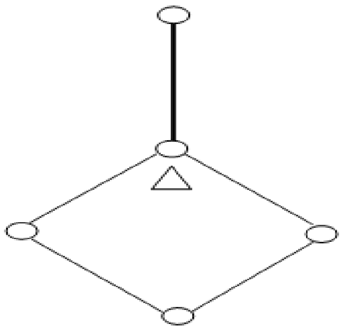
E-mail addresses: a.veeramani@yahoo.com

imply that $(a_0 \wedge a_1, b_0 \wedge b_1) \in \Theta$ and $(a_0 \vee a_1, b_0 \vee b_1) \in \Theta$.

Definition 2.4. [5] A lattice L is said to be 0 – modular if whenever $x \le y$ and $y \wedge z = 0$ then $x = (x \vee y) \wedge z$.

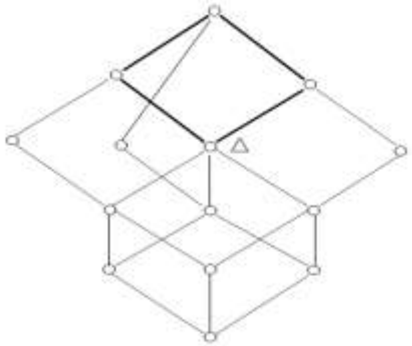
We produce below the lattice structures of weak congruences of chains up to four elements.

1.The lattice of weak congruences Cw(L2) of a two-element chain L2 is givenin Fig. 1[17].



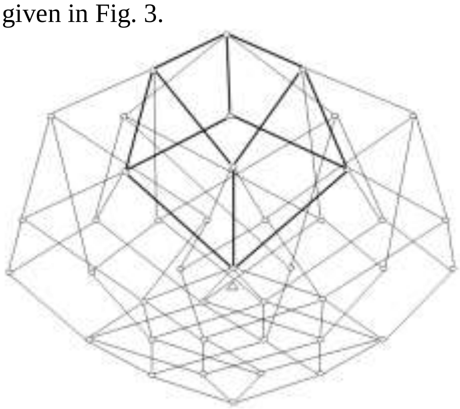
$C_w(L_2)$
Fig.1

2.The lattice of weak congruences $C_w(L_3)$ of a three-element chain L_3 is given in Fig. 2[17].



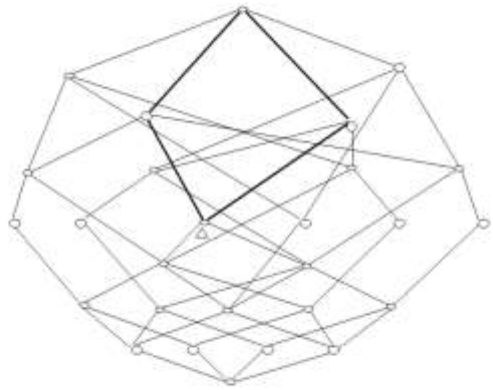
$C_w(L_3)$
Fig.2

3.The lattice of weak congruences $C_w(L_4)$ of a four-element chain L_4 is given in Fig. 3.



$C_w(L_4)$
Fig.3

4.The lattice of weak congruences $C_w(B_2)$ of a rank 2 Boolean lattice B_2 is given in Fig. 4.



$C_w(B_2)$
Fig. 4

Theorem 2.5. $C_w(L)$ is 0 – modular if and only if L is a chain.

Proof: Let $L \cong L_n$ be an n-element chain of the form $0 \prec x_1 \prec x_2 \prec \ldots \prec x_{n-2} \prec 1$.

Let $\Theta_1, \Theta_2 \in C_w(L_n)$ and $\Theta_1 \le \Theta_2$. Let $\Theta_3 \in C_w(L_n)$ such that $\Theta_2 \cap \Theta_3 = \emptyset$.

To prove that: $(\Theta_1 \vee \Theta_3) \wedge \Theta_2 = \Theta_1$.

Since a weak congruence relation on any algebra $\mathcal{A}$ is a union of squares of some subalgebras of $\mathcal{A}$ including one element subalgebras, we can assume that Θ_1 and Θ_2 are of the form $\Theta_1 = \{a_1, a_2,\ldots,a_k\}^2 \cup \{(a_{k+1}, a_{k+1})\ldots(a_{k+s}, a_{k+s})\}$ and $\Theta_2 = \{a_1, a_2,\ldots,a_k, a_l, a_{l+1},\ldots,a_{l+m}\}^2 \cup \{(a_{k+1}, a_{k+1})\ldots(a_{k+s}, a_{k+s}), \{(a_{k+s+1}, a_{k+s+1}),\ldots,(a_{k+n}, a_{k+n})\}$.

Without loss of geberality we can assume that $a_1 < a_2 < \ldots < a_k < a_{k+1} < a_{k+s} < a_l < \ldots < a_{l+m}$. Since $\Theta_2 \cap \Theta_3 = \emptyset$, Θ_3 contains no element of Θ_2.

So, Θ_3 is of the form $\Theta_3 = \{b_1, b_2,\ldots,b_r\}^2 \cup \{b_{r+1}, b_{r+1}),\ldots,(b_{r+t}, b_{r+t})\}$, where $b_1, b_2,\ldots,b_r, b_{r+1},\ldots b_{r+t} \notin \{a_1,\ldots a_k, a_l,\ldots,a_{l+m},a_{k+1},\ldots,a_{k+s},a_{k+s+1}, a_{k+n}\}$.

Now without loss of generality let us assume that $b_1 < b_2 < \ldots < b_{r+1} < \ldots < b_{r+t}$. We have three cases.

Case(i) $a_k < b_{r+1}$.

Case(ii) $b_{r+t} < a_1$.

Case(iii) $a_q < b_p < a_{q+1} < a_k$, for some p and q, such that $p \in \{r+1,\ldots,r+t\}$, $1 \leq q \prec k - 1$. From case(i) we get $a_i < b_j$, for every $i = 1, 2,\ldots,k$ and $j = 1, 2,\ldots,r+1,\ldots,r+t$.

Then $\Theta_1 \vee \Theta_3 = \Theta_1 \cup \Theta_3$. Clearly, $(\Theta_1 \vee \Theta_3) \cap \Theta_2 = (\Theta_1 \cup \Theta_3) \cap \Theta_2 = \Theta_1$.

Case(ii) is similar to case(i).

Case(iii) $a_1 < a_2 < \ldots < a_p < b_p < a_{q+1} < \ldots < a_k < a_{k+1} < \ldots < a_{k+s} < a_l \ldots < a_{l+m}$.

Then $\Theta_1 \vee \Theta_3 = \Theta_1 \cup \Theta_3 \cup \{(b_p, a_j)(a_j, b_p) / j = 1, 2,\ldots,k.\}$. So, $(\Theta_1 \vee \Theta_3) \wedge \Theta_2 = \Theta_1$. Therefore, in all the three cases we have established that the 0 – modularity condition is true in $C_w(L_n)$. So, $C_w(L_n)$ is 0 – modular.

Conversely, $C_w(L)$ be 0 – modular. Suppose that L is not a chain. Then there at least two elements x_1 and x_2 in L such that x_1 is not comparable with x_2.

Let L_1 = Sublattice of L generated by $\{x_1, x_2\}$. Let $\Theta_1 = \{(x_1, x_1)\}$, $\Theta_1 = \{(0, 0), (x_1, x_1)\}$ and $\Theta_3 = \{(x_2, x_2)\}$. Now $\Theta_1 \vee \Theta_3 = \Delta_{L_1}$, $[\emptyset, \Delta_{L_1}] \cong Sub(L_1)$, Δ_{L_1} is the top element in $SubL_1$.

So, $\Delta_{L_1} \wedge \Theta_2 = \Theta_2$. That means, $C_w(L)$ is not 0 – modular.

Our assumption is wrong. Therefore, $x_1 \leq x_2$. Therefore, L is a chain. ■

Remark 2.6

$C_w(B_n)$ is not 0 – modular. For, $C_w(B_2)$ is not 0 – modular and $C_w(B_2)$ is a sublattice of $C_w(B_n)$. Therefore, $C_w(B_n)$ is also not 0 – modular. Similarly, $C_w(M_3)$ and $C_w(N_5)$ are not 0 – modular.

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