



Heuristic algorithm for multi-index fixed charge fuzzy transportation problem

W.Ritha* and J.Merlin Vinotha

Department of Mathematics, Holy Cross College (Autonomous), Tiruchy – 620002, Tamil Nadu, India.

ARTICLE INFO

Article history:

Received: 19 March 2012;

Received in revised form:
25 April 2012;

Accepted: 16 May 2012;

Keywords

Multi-index transportation problem,
Fixed charge,
Fuzzy transportation problem,
Trapezoidal fuzzy numbers.

ABSTRACT

This paper develops a heuristic algorithm for the multi-index fixed charge transportation problem. The efficiency of this algorithm for solving the multi-index fixed charge transportation problem has been proved by comparing the results obtained by using this algorithm with the existing exact method of solving multi-index fixed charge transportation problem. Further the proposed algorithm is extended to multi-index fixed charge fuzzy transportation problem in which all the parameters are considered as trapezoidal fuzzy numbers. A numerical example is presented to illustrate the proposed method.

© 2012 Elixir All rights reserved.

Introduction

As supply chain management becomes increasingly important in global economics, knowledge of the transportation system is fundamental to the efficient and economical operation of a company's responsibility. Transportation models play an important role in logistics and supply chain management for reducing cost and improving service.

Multi-index transportation problems are the extension of conventional transportation problems and are appropriate for solving transportation problems with multiple supply points, multiple demand points as well as problems using diverse modes of transportation demands or delivering different kinds of merchandises. Thus, the forward problem would be more complicated than conventional transportation problems. Hayley,1962; considered the multi-index transportation problem and presented an algorithm to solve multi-index transportation problem. Junginer ,1993; who proposed a set of logic problems to solve multi-index transportation problems, has also conducted a detailed investigation regarding the characteristics of multi-index transportation model. Rautman et al,1993; used multi-index transportation model to solve the shipping scheduling and suggested that the employment of such transportation problems model would not only enhance the entire transportation efficiency but also optimize the integral system. Anu Ahuja and S.R. Arora ,2001; presented an algorithm for identifying the efficient cost – time trade off pairs in a multi-index fixed charge bi-criterion transportation problem. This is an exact method of finding solution to this problem.

The fixed charge problem is a non-linear programming problem of practical interest to business and industry. The existence of fixed charges in its objective function has prevented the development of any extensive theory, for solution of the fixed charge problem. Since the problems with fixed charges are usually NP-hard (non deterministic polynomial time), the computational time to obtain exact solutions increases in a polynomial fashion and very quickly becomes extremely long as the dimensions of the problem increases.

The fixed charge problem was originally formulated by G.B. Dantzig and W. Hirsch in 1954. The literature provides only a few exact methods for solving the fixed charge problem. Steinberg ,1970; provided an exact algorithm based on branch and bound method. But, the exact branch and bound method is applicable to small problems only. Since the effort to solve an FCP grows substantially with the size of the problem as explained in Walker ,1976;A good deal of effort has been devoted to finding approximate solutions to fixed charge problems. The heuristic methods try to reach the optimum through simplex like iterations. Copper and Drebes,1967; Denzler,1969; Steinberg ,1970; and Walker,1976; have developed heuristic adjacent – extreme – point algorithms for the general FCP. Salkin ,1975;presented the capacities plant location problem as an FCP model and discussed the heuristic branch and bound algorithm proposed by Efroymson and Ray ,1966;. Adalakha and Kowalski ,2003; developed a heuristic algorithm for the fixed charge problem.

Some algorithms that purport to be FCP algorithms are in fact algorithms for fixed charge transportation problems (FCTPs), which are only a subset of FCPs. The total value of the charges / loads varies for an FCP, making it significantly harder to solve than an FCTP where the total value of the charges / loads is fixed. In a classical transportation problem the cost of transportation is directly proportional to the number of units transported. But the transportation cost may not be linear on account of price – break quantity discounts etc. Thus in real world situation when a commodity is transported, a fixed cost is incurred in the objective function. It may represent the cost of hiring a vehicle, landing fee in an airport, setup costs for machines in a manufacturing environment etc. Many distribution problems in practice can be modeled as fixed charge transportation problems. For example, rail, roads and trucks have invariably used freight rate which consists of a fixed cost and variable cost. Therefore, most of the recent efforts are concentrated on finding solution methods for FCTPs.

The most widely known methods for FCTP are ranking the extreme points method. (Murthy,1968; Sadagopan and Ravindran,1982;) and the branch and bound method. Hirsch and Dantzig,1968; established that the feasible region of an FCTP is a bounded convex set and that the objective function is concave. An optimal solution occurs at an extreme point of the constraint set, and for a non-degenerate problem with all positive fixed costs, every extreme point of the feasible region is a local minimum. Gray ,1971; attempted to find an exact solution to the FCTP by decomposing it into a master integer program and a series of transportation subprograms. In contrast, Palekar et al ,1990; and Steinberg 1970; provided exact algorithms based on branch and bound method applicable to small problems only.

Since the available exact algorithms generally require long computation times and large amounts of storage, many authors have turned to efficient heuristic algorithms for solving FCTP. The well known heuristic approaches are presented by Diaby ,1991; and Sun et al ,1998;Kuhn and Baumol 1962; suggested that an approximate solution maybe found by forcing a highly degenerated solution which is accomplished by making small adjustments to the demand and supply quantities.

Balinski ,1961; replaced the non-linear fixed – charge objective function with an approximate linear objective function and solved the resulting problem using the standard transportation algorithm. Sandrock ,1988; provided a heuristic simplex type algorithm for a source – induced FCTP where the fixed charge is associated with the supply points instead of the routes. Adlakha and Kowalski ,2004; presented a simple algorithm for the source – induced FCTP. Adlakha et al,2006;presented a heuristic algorithms for the fixed charge transportation problem. Veena Adlakha et al,1999; presented an approximation methods for regular FCTP. However, the above exact and approximate solution methods for solving FCTP are useful only for conventional (ie two dimensional) transportation problems.

In this paper we develop a heuristic algorithm for the multi-index fixed charge transportation problems.

There are cases that the parameters in the multi-index fixed charge transportation problem can't be presented in a precise manner. For example, the unit shipping cost and fixed cost may vary in a time frame. The supplies and demands may be uncertain due to uncontrollable factors. Fuzzy numbers introduced by Zadeh, 1965; may represent this data. So, fuzzy decision making method is needed here.

Zimmerman,1978; showed that solutions obtained by fuzzy linear programming are always efficient subsequently. Zimmerman's fuzzy linear programming has developed into several fuzzy optimization methods for solving the transportation problems. Oheigeartaigh ,1982; proposed an algorithm for solving transportation problems where the capacities and requirements are fuzzy sets with linear or triangular membership functions. Chanas et al, 1993; presented a fuzzy linear programming model for solving transportation problems with crisp cost coefficients and fuzzy supply and demand values. Chanas et al ,1984; formulated the fuzzy transportation problems in three different situations and proposed method for solving the formulated fuzzy transportation problems. Chanas ad Kuchta ,1996; proposed the concept of the optimal solution for the transportation problem with fuzzy coefficients expressed as fuzzy numbers, and developed an algorithm for obtaining the optimal solution. Chanas and Kuchta ,1998; proposed a new method for solving fuzzy integer

transportation problem by representing the availability and demand parameters as L-R type fuzzy numbers. Liu and Kao ,2004; described a method for solving fuzzy transportation problems based on extension principle. Chiang ,2005; proposed a method to find the optimal solution of transportation problems with fuzzy demand and fuzzy product. Lie et al ,2008; proposed a new method based on goal programming for solving fuzzy transportation problems with fuzzy costs. Chen et al ,2008; proposed the methods for solving transportation problems on a fuzzy network. Lin ,2009; used genetic algorithm for solving transportation problems with fuzzy coefficients. Stephen Dineger and Palanivel ,2009; investigated fuzzy transportation problem, with the aid of trapezoidal fuzzy numbers and proposed fuzzy modified distribution method to find the optimal solution interms of fuzzy numbers. Amitkumer et al ,2011; proposed fuzzy linear programming approach for solving fuzzy transportation problems with transshipment. Ojhe et al, 2010; investigated solid transportation problem for an item with fixed charge via genetic algorithm. Lixing Yang et al, 2007; discussed the solution algorithm for solving bi-criteria fixed charge solid transportation problem under stochastic environment. In the above transportation models the optimal solutions are crisp value.

In this paper we develop a heuristic algorithm for the multi-index fixed charge transportation problem. This algorithm is an extension work of Veena Adlakha et al, 2010; for fixed charge problems. The obtained results by using this algorithm are compared with the existing exact method of solving multi-index fixed charge transportation problem (Anu Ahuja and S.R. Arora ,2001;). Further the proposed algorithm is extended to multi-index fixed charge fuzzy transportation problem on which all the parameters are considered as trapezoidal fuzzy numbers.

This paper is organized as follows,. In section 2, the formulation of MIFCTP is given and the heuristic algorithm to solve MIFCTP is proposed. In section 3, the preliminaries of fuzzy set theory are reviewed. In section 4, the formulation of MIFCFTP is given and the heuristic algorithm to solve MIFCFTP is proposed. The numerical example is also given in this section to illustrate the proposed method. In section 6, conclusion is given.

Multi-Index Fixed Charge Transportation Problem (MIFCTP)

The general model of the problem considered is as follows :

$$P : \text{Minimize } \left\{ \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk} x_{ijk} + \sum_{i=1}^m \sum_{k=1}^p F_{ik} \right\}$$

subject to

$$\sum_{i=1}^m x_{ijk} = A_{jk}$$

$$\sum_{j=1}^n x_{ijk} = B_{ki}$$

$$\sum_{k=1}^p x_{ijk} = E_{ij}$$

$$\text{and } x_{ijk} \geq 0 ; i = 1, 2, \dots, m ; j = 1, 2, \dots, n ; k = 1, 2, \dots, p.$$

$$\text{where } \sum_{j=1}^n A_{jk} = \sum_{i=1}^m B_{ki}, \sum_{k=1}^p B_{ki} = \sum_{j=1}^n E_{ij},$$

$$\sum_{i=1}^m E_{ij} = \sum_{k=1}^p A_{jk}.$$

2.1

$$\sum_{j=1}^n \sum_{k=1}^p A_{jk} = \sum_{k=1}^m \sum_{i=1}^n B_{ki} = \sum_{i=1}^m \sum_{j=1}^n E_{ij}$$

Here $i = 1, 2, \dots, m$ are the origins.
 $j = 1, 2, \dots, n$ are the destinations.

$k = 1, 2, \dots, p$ are the various types of commodities.

x_{ijk} = the amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination
 c_{ijk} = the variable cost per unit amount of k^{th} type of commodity from the i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported, so long as $x_{ijk} > 0$.

F_{ik} = the fixed cost associated with origin i and commodity k .

$$F_{ik} = \sum_{j=1}^n F_{ijk} S_{ijk}, \quad i = 1, 2, \dots, m, \quad k = 1, 2, \dots, p.$$

where $S_{ijk} = 1$ if $x_{ijk} > 0$.
 $S_{ijk} = 0$ if $x_{ijk} = 0$.

A_{jk} = the total quantity of k^{th} type of commodity to be sent to the j^{th} destination.

B_{ki} = the total quantity of k^{th} type of commodity available at the i^{th} origin.

E_{ij} = the total quantity to be sent from i^{th} origin to the j^{th} destination.

Finding upper and lower bounds of the decision variables

Now extending the procedure of veena Adlakha,2010; to MIFCTP

For each $i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p$.

Consider the following Max LP_{ijk}.

Maximize $Z = x_{ijk}$

Show that

$$\begin{aligned} \sum_{i=1}^m x_{ijk} &= A_{jk} \\ \sum_{j=1}^n x_{ijk} &= B_{ki} \\ \sum_{k=1}^p x_{ijk} &= E_{ij} \end{aligned} \quad \dots (2.2)$$

and $x_{ijk} \geq 0, i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p$.

where $\sum_{j=1}^n A_{jk} = \sum_{i=1}^m B_{ki}, \sum_{k=1}^p B_{ki} = \sum_{j=1}^n E_{ij}, \sum_{i=1}^m E_{ij} = \sum_{k=1}^p A_{jk}$.

$$\sum_{j=1}^n \sum_{k=1}^p A_{jk} = \sum_{k=1}^m \sum_{i=1}^n B_{ki} = \sum_{i=1}^m \sum_{j=1}^n E_{ij}$$

Let x_{ijk}^{max} denote the optimal solution of problem Max LP_{ijk}.

Now consider the following Min LP_{ijk}.

Minimize $z = x_{ijk}$
 Subject to (2.1) ... (2.3)

Let x_{ijk}^{min} denote the optimal solution of problem Min LP_{ijk}.

Approximation method to find the solution of MIFCTP

Now consider the relaxed version of problem p.

$$\text{Minimize } z = \left\{ \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk} x_{ijk} + \sum_{i=1}^m \sum_{k=1}^p F_{ik} \delta_{ijk} \right\}$$

Show that

$$\begin{aligned} \sum_{i=1}^m x_{ijk} &= A_{jk} \\ \sum_{j=1}^n x_{ijk} &= B_{ki} \\ \sum_{k=1}^p x_{ijk} &= E_{ij} \end{aligned}$$

where $\sum_{j=1}^n A_{ijk} = \sum_{i=1}^m B_{ki}, \sum_{k=1}^p B_{ki} = \sum_{j=1}^n E_{ij}, \sum_{i=1}^m E_{ij} = \sum_{k=1}^p A_{jk}$.

$$\sum_{j=1}^n \sum_{k=1}^p A_{jk} = \sum_{k=1}^m \sum_{i=1}^n B_{ki} = \sum_{i=1}^m \sum_{j=1}^n E_{ij}$$

$$0 \leq \delta_{ijk} \leq 1, \delta_{ijk} \text{ integer.} \quad \dots (2.4)$$

$$0 \leq x_{ijk} \leq m_{ijk} \delta_{ijk} \quad i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p. \quad \dots (2.5)$$

Set $m_{ijk} = x_{ijk}^{\text{max}}$ for all $i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p$.

As demonstrated in Balinski,1961, $\{x_{ijk}, \delta_{ijk}\}$ is a solution of the above problem with the integer constraint ignored only if $x_{ijk} = m_{ijk} \delta_{ijk}$

Now consider the following problem P' where

$$C_{ijk} = C_{ijk} + f_{ijk} / m_{ijk}$$

P' : Minimize $z = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk} x_{ijk}$

Show that $\sum_{i=1}^m x_{ijk} = A_{jk}$

$$\sum_{j=1}^n x_{ijk} = B_{ki}$$

$$\sum_{k=1}^p x_{ijk} = E_{ij}$$

where $\sum_{j=1}^n A_{ijk} = \sum_{i=1}^m B_{ki}, \sum_{k=1}^p B_{ki} = \sum_{j=1}^n E_{ij}, \sum_{i=1}^m E_{ij} = \sum_{k=1}^p A_{jk}$.

$$\dots (2.6)$$

$$\sum_{j=1}^n \sum_{k=1}^p A_{jk} = \sum_{k=1}^m \sum_{i=1}^n B_{ki} = \sum_{i=1}^m \sum_{j=1}^n E_{ij}$$

$$0 \leq x_{ijk} \leq m_{ijk}, i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p.$$

Theorem 2.2.1:

The optimal value of P', Z(P') provides a lower bound to the optimal value Z*(P) of the corresponding MIFCTP, Problem P.

Remark 1 : The optimal solution $\{x'_{ijk}\}$ of problem P' can be used to create an upper bound to the optimal value Z*(P). Note that the solution $\{x'_{ijk}\}$ is easily modified into a feasible solution of $\{x'_{ijk}, \delta'_{ijk}\}$ of P as follows :

$$\begin{aligned} \delta'_{ijk} &= 0 \text{ if } x'_{ijk} = 0 \\ \delta'_{ijk} &= 1 \text{ if } x'_{ijk} > 0 \end{aligned}$$

Solution $\{x'_{ijk}, y'_{ijk}\}$ being a feasible solution provides an upperbound on Z*(P). Consequently the optimal solution $\{x'_{ijk}\}$ can be used to provide a lower and upper bound for the optimal value of problem P as follows :

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk} x'_{ijk} \leq Z^*(P) \leq \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk} x'_{ijk} + f_{ijk} \delta_{ijk} \dots (2.7)$$

Remark 2 : If the lower bound is equal to the upper bound in Equation 2.7, then $\{x_{ijk}, \delta_{ijk}\}$ is the optimal solution of problem P with optimal value of Z*(P).

Reformulation of MIFCTP by extracting fixed cost

“Extracting Constant Values” from any mathematical problem eases the optimization process and brings the solution closer to the optimal at a minimum expense. It is also done as

part of the Gradient Method. Some methods like Hungarian Method are totally based on ‘extraction’.

Now the problem P can be rewritten as follows :

$$P^* : \text{Minimize } Z = K\Delta + \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p (C_{ijk} x'_{ijk} + f_{ijk}^* \delta_{ijk})$$

$$\text{Subject to } \begin{aligned} \sum_{i=1}^m x_{ijk} &= A_{jk} \\ \sum_{j=1}^n x_{ijk} &= B_{ki} \\ \sum_{k=1}^p x_{ijk} &= E_{ij} \end{aligned}$$

$$\text{where } \sum_{j=1}^n A_{ijk} = \sum_{i=1}^m B_{ki}, \sum_{k=1}^p B_{ki} = \sum_{j=1}^n E_{ij}, \sum_{i=1}^m E_{ij} = \sum_{k=1}^p A_{jk}.$$

$$\sum_{j=1}^n \sum_{k=1}^p A_{ijk} = \sum_{k=1}^p \sum_{i=1}^m B_{ki} = \sum_{i=1}^m \sum_{j=1}^n E_{ij}.$$

where $\Delta = \min f_{ijk}$ (the smallest fixed charge coefficient)

$$f_{ijk}^* = f_{ijk} = \Delta. \dots (2.8)$$

The variable K is a consolidation coefficient equal to the number of non-zero basic variables. In the case of MITP, the number of basic variables should be equal to $mnp - (m - 1)(n - 1)(p - 1)$. This extraction simplifies the problem by eliminating atleast one δ_{ijk} variable from problem P. Since the FCTP is NP-hard, eliminating even one variable in a large size problem can bring significant savings in the computational time.

The MIFCTP Heuristic Method

In this section we outline steps for solving the MIFCTP Problem P with cost coefficients C_{ijk} and fixed costs f_{ijk} .

The Algorithm

- Step 1 : Formulate and solve the problems Max LP_{ijk} and Min LP_{ijk} to determine lower and upper bounds X_{ijk}^{\min} and X_{ijk}^{\max} , for all $i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p$.
- Step 2 : Identify $\Delta =$ the smallest fixed charge coefficient.
- Step 3 : Subtract Δ from all the fixed charges. Extract Fixed cost $K\Delta$. Corresponding to the number of non-zero basic variables.
- Step 4 : Formulate the problem P' with $C_{ijk} = C_{ijk} + f_{ijk} / m_{ijk}$.
- Step 5 : Solve the problem P' using LINDO Software and record solution $\{x'_{ijk}\}$.
- Step 6 : Determine δ'_{ijk} values as outlined in Remark 1.
- Step 7 : Stop. Record the solution for problem P as $\{x'_{ijk}, \delta'_{ijk}\}$.

Numerical Example

To explain the MIFCTP heuristic method the following example is presented.

Consider a 3 x 3 x 3 multi-index fixed charge transportation problem (Anu Ahuja, Arora. S.R., 2001)

The variable cost C_{ijk} is given in Table 1 at the top left corner.

The fixed costs are

$F_{111} = 10$	$F_{121} = 30$	$F_{131} = 20$
$F_{112} = 20$	$F_{122} = 20$	$F_{132} = 20$
$F_{113} = 30$	$F_{123} = 20$	$F_{133} = 10$
$F_{211} = 10$	$F_{221} = 20$	$F_{231} = 20$

$F_{212} = 10$	$F_{222} = 10$	$F_{232} = 30$
$F_{213} = 40$	$F_{223} = 10$	$F_{233} = 10$
$F_{311} = 10$	$F_{321} = 40$	$F_{331} = 20$
$F_{312} = 20$	$F_{322} = 10$	$F_{332} = 30$
$F_{313} = 20$	$F_{323} = 10$	$F_{333} = 10$

The above MIFCTP can be formulated as a linear programming problem and using the steps of the algorithm described in 3.3.1. we get the optimal solution to the problem P' using LINDO is as follows :

$x_{113} = 10$	$x_{122} = 6$	$x_{131} = 6$	$x_{132} = 3$
$x_{211} = 5$	$x_{212} = 14$	$x_{213} = 2$	$x_{221} = 8$
$x_{223} = 1$	$x_{233} = 14$	$x_{311} = 10$	$x_{312} = 3$
$x_{313} = 8$	$x_{322} = 5$	$x_{323} = 8$	$x_{331} = 5$
$x_{332} = 5$	$x_{333} = 2$		

with $Z(P') = 1126.02$

Preliminaries of Fuzzy Set Theory

In this section some basic definitions and arithmetic operations are reviewed. (Stephen Dinagar, S., Palanivel, K.,2009)

Basic Definitions

In this section, some basic definitions are reviewed.

Definition 3.1 : Trapezoidal Fuzzy Number

The fuzzy number $\mathcal{A} = [a_1, a_2, a_3, a_4]$ is a trapezoidal fuzzy number, denoted by $[a_1, a_2, a_3, a_4]$ its membership function $\mu_{\mathcal{A}}$ is given in the figure below :

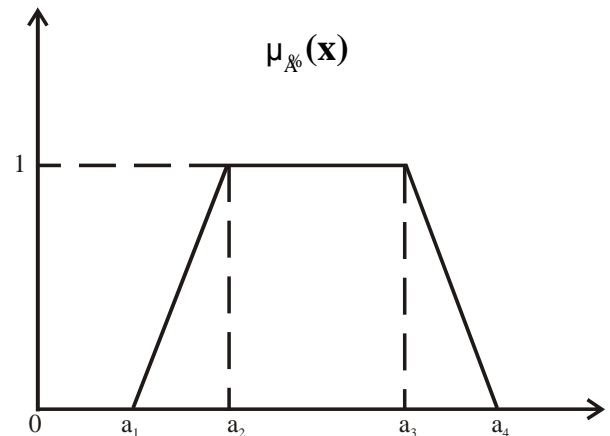


Fig.1 : Membership function of a trapezoidal fuzzy number

Definition : 3.2

Two trapezoidal fuzzy numbers $\mathcal{A}_1 = (a_1, b_1, c_1, d_1)$ and $\mathcal{A}_2 = (a_2, b_2, c_2, d_2)$ are said to be equal ie. $\mathcal{A}_1 = \mathcal{A}_2$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

Definition : 3.3

A Ranking function is a function $\mathbb{R} : F(\mathbb{R}) \rightarrow \mathbb{R}$ where $F(\mathbb{R})$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line. Let $\mathcal{A} = (a, b, c, d)$ be a trapezoidal fuzzy number, then $\mathbb{R} : (\mathcal{A}) = \frac{a + b + c + d}{4}$.

Arithmetic Operations on fuzzy numbers:

In this section addition, subtraction, multiplication and division operations of trapezoidal fuzzy numbers are reviewed.

Let $\mathcal{A}_1 = (a_1, b_1, c_1, d_1)$ and $\mathcal{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, then

(i) Addition : $\mathcal{A}_1 + \mathcal{A}_2 : (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

(ii) Subtraction : $X_1^0 - X_2^0 : (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$

(iii) Multiplication : $X_1^0 \cdot X_2^0 : \left[\frac{a_1}{4}(a_2 + b_2 + c_2 + d_2), \right.$

$$\left. \frac{b_1}{4}(a_2 + b_2 + c_2 + d_2), \right.$$

$$\left. \frac{c_1}{4}(a_2 + b_2 + c_2 + d_2), \right.$$

$$\left. \frac{d_1}{4}(a_2 + b_2 + c_2 + d_2) \right] \text{ if } \mathbb{R}(X_1^0) > 0. = \left[\frac{d_1}{4}(a_2 + b_2 + c_2 + d_2), \right.$$

$$\left. \frac{c_1}{4}(a_2 + b_2 + c_2 + d_2), \right.$$

$$\left. \frac{b_1}{4}(a_2 + b_2 + c_2 + d_2), \right.$$

$$\left. \frac{a_1}{4}(a_2 + b_2 + c_2 + d_2) \right] \text{ if } \mathbb{R}(X_1^0) < 0.$$

(iv) Division : $X_1^0 / X_2^0 : \left(\frac{a_1}{d_2}, \frac{b_1}{c_2}, \frac{c_1}{b_2}, \frac{d_1}{a_2} \right)$

Multi-index fixed charge fuzzy transportation problem

The general model of the problem is considered as follows :

$$\text{(FP) Minimize } \left\{ \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk} X_{ijk} + \sum_{i=1}^m \sum_{j=1}^n F_{ik} \right\}$$

$$\text{Subject to } \sum_{i=1}^m X_{ijk} = A_{jk}$$

$$\sum_{j=1}^n X_{ijk} = B_{ki}$$

$$\sum_{k=1}^p X_{ijk} = E_{ij}$$

and $\mathbb{R}(X_{ijk}^0) \geq 0, i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots,$

$p.$

$$\text{where } \sum_{j=1}^n X_{ijk}^0 = \sum_{i=1}^m B_{ki}^0, \sum_{k=1}^p B_{ki}^0 = \sum_{j=1}^n E_{ij}^0, \sum_{i=1}^m E_{ij}^0 = \sum_{k=1}^p A_{jk}^0.$$

$$\sum_{j=1}^n \sum_{k=1}^p X_{ijk}^0 = \sum_{k=1}^p \sum_{i=1}^m B_{ki}^0 = \sum_{i=1}^m \sum_{j=1}^n E_{ij}^0. \dots (4.1)$$

Here $i = 1, 2, \dots, m$ are the origins.

$j = 1, 2, \dots, n$ are the destinations.

$k = 1, 2, \dots, p$ are the various types of commodities.

X_{ijk}^0 - the fuzzy amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination.

C_{ijk}^0 - the fuzzy variable cost per unit amount of k^{th} type commodity from the i^{th} origin to the j^{th} destination.

$$F_{ijk}^0 = \sum_{j=1}^n F_{ijk}^0 \delta_{ijk}, i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots,$$

$p.$

$$\text{where } \delta_{ijk} = 1 \text{ if } \mathbb{R}(X_{ijk}^0) > 0$$

$$= 0 \text{ if } \mathbb{R}(X_{ijk}^0) = 0$$

A_{jk}^0 - the total fuzzy quantity of k^{th} type of commodity to be sent to the j^{th} destination.

B_{ki}^0 - the total fuzzy quantity of k^{th} type of commodity available at the i^{th} origin.

E_{ij}^0 - the total fuzzy quantity to be sent from i^{th} origin to the j^{th} destination.

Finding upper and lower bounds of the fuzzy decision variables

Solving the following two FLPP using the methodology proposed in Amit Kumar et.al, 2011, will yield the upper and lower bounds of the fuzzy decision variables.

For each $i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p.$

Consider the following Max FLP_{ijk}

$$\text{Maximize } Z = X_{ijk}^0$$

$$\text{Subject to } \sum_{i=1}^m X_{ijk} = A_{jk}$$

$$\sum_{j=1}^n X_{ijk} = B_{ki}$$

$$\sum_{k=1}^p X_{ijk} = E_{ij}$$

and $\mathbb{R}(X_{ijk}^0) \geq 0, i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots,$

$p.$

$$\text{where } \sum_{j=1}^n X_{ijk}^0 = \sum_{i=1}^m B_{ki}^0, \sum_{k=1}^p B_{ki}^0 = \sum_{j=1}^n E_{ij}^0, \sum_{i=1}^m E_{ij}^0 = \sum_{k=1}^p A_{jk}^0.$$

$$\sum_{j=1}^n \sum_{k=1}^p X_{ijk}^0 = \sum_{k=1}^p \sum_{i=1}^m B_{ki}^0 = \sum_{i=1}^m \sum_{j=1}^n E_{ij}^0. \dots (4.2)$$

Let X_{ijk}^{max} denote the fuzzy optimal solution of the problem Max FLP_{ijk}.

Now consider the following Min FLP_{ijk}

$$\text{Minimize } Z = X_{ijk}^0$$

$$\text{Subject to (4.1) } \dots (4.3)$$

Let X_{ijk}^{min} denote the fuzzy optimal solution of the problem Min FLP_{ijk}.

Approximation method to find the solution of (MIFCFIP)

Now consider the relaxed version of the problem FP

$$\text{Minimise } Z = \left\{ \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p C_{ijk}^0 X_{ijk}^0 + \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p F_{ijk}^0 \delta_{ijk} \right\}$$

$$\text{Subject to } \sum_{i=1}^m X_{ijk}^0 = A_{jk}$$

$$\sum_{j=1}^n X_{ijk}^0 = B_{ki}$$

$$\sum_{k=1}^p X_{ijk}^0 = E_{ij}$$

$$\text{where } \sum_{j=1}^n X_{ijk}^0 = \sum_{i=1}^m B_{ki}^0, \sum_{k=1}^p B_{ki}^0 = \sum_{j=1}^n E_{ij}^0, \sum_{i=1}^m E_{ij}^0 = \sum_{k=1}^p A_{jk}^0.$$

$$\sum_{j=1}^n \sum_{k=1}^p X_{ijk}^0 = \sum_{k=1}^p \sum_{i=1}^m B_{ki}^0 = \sum_{i=1}^m \sum_{j=1}^n E_{ij}^0. \dots (4.4)$$

$$0 \leq \delta_{ijk} \leq 1, \delta_{ijk} \text{ integer.}$$

$$0 \leq \mathbb{R}(\mathbb{X}_{ijk}^0) \leq \mathbb{R}(\mathbb{m}_{ijk} \delta_{ijk}), i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p. \dots (4.5)$$

Set $\mathbb{m}_{ijk}^0 = \mathbb{X}_{ijk}^{\max}$ for all $i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p$.

$\{\mathbb{X}_{ijk}^0, \mathbb{Y}_{ijk}^0\}$ is a solution of the above problem with the integer constraint ignored only if $\mathbb{X}_{ijk}^0 = \mathbb{m}_{ijk}^0 \delta_{ijk}$.

Now consider the problem FP' where

$$\mathbb{C}_{ijk}^0 = \mathbb{C}_{ijk} + \mathbb{F}_{ijk} / \mathbb{m}_{ijk}^0$$

$$\text{FP}' : \text{Minimise } \mathbb{Z}' = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p \mathbb{C}_{ijk}^0 \mathbb{X}_{ijk}^0$$

$$\text{Subject to } \sum_{i=1}^m \mathbb{X}_{ijk}^0 = \mathbb{A}_{jk}^0$$

$$\sum_{j=1}^n \mathbb{X}_{ijk}^0 = \mathbb{B}_{ki}^0$$

$$\sum_{k=1}^p \mathbb{X}_{ijk}^0 = \mathbb{E}_{ij}^0$$

where

$$\sum_{j=1}^n \mathbb{A}_{ijk}^0 = \sum_{i=1}^m \mathbb{B}_{ki}^0, \sum_{k=1}^p \mathbb{B}_{ki}^0 = \sum_{j=1}^n \mathbb{E}_{ij}^0, \sum_{i=1}^m \mathbb{E}_{ij}^0 = \sum_{k=1}^p \mathbb{A}_{jk}^0.$$

$$\sum_{j=1}^n \sum_{k=1}^p \mathbb{A}_{jk}^0 = \sum_{k=1}^p \sum_{i=1}^m \mathbb{B}_{ki}^0 = \sum_{i=1}^m \sum_{j=1}^n \mathbb{E}_{ij}^0 \dots (4.6)$$

$$0 \leq \mathbb{R}(\mathbb{X}_{ijk}^0) \leq \mathbb{R}(\mathbb{m}_{ijk}^0), i = 1, 2, \dots, m, j = 1, 2, \dots, n, k = 1, 2, \dots, p.$$

Theorem 4.2.1 :

The fuzzy optimal value of FP', $\mathbb{Z}'(\text{FP}')$ provides a lower bound to the optimal value $\mathbb{Z}^*(\text{FP}')$ of the corresponding MIFCFTP, Problem FP.

Remark 3 :

The fuzzy optimal solution $\{\mathbb{X}_{ijk}^0\}$ of problem FP can be used to create an upperbound to the optimal value $\mathbb{Z}^*(\text{FP})$.

Note that the solution $\{\mathbb{X}_{ijk}^0\}$ is easily modified into a fuzzy feasible solution of $\{\mathbb{X}_{ijk}^0, \delta'_{ijk}\}$ FP as follows :

$$\delta'_{ijk} = 0 \text{ if } \mathbb{R}(\mathbb{X}_{ijk}^0) = 0.$$

$$\delta'_{ijk} = 1 \text{ if } \mathbb{R}(\mathbb{X}_{ijk}^0) > 0.$$

Solution $\{\mathbb{X}_{ijk}^0, \delta'_{ijk}\}$ being a fuzzy feasible solution, provides an upperbound on $\mathbb{Z}^*(\text{FP})$.

Consequently the optimal solution $\{\mathbb{X}_{ijk}^0\}$ can be used to provide a lower and upperbound for the fuzzy optimal value of problem FP as follows :

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p \mathbb{C}_{ijk}^0 \mathbb{X}_{ijk}^0 \leq \mathbb{Z}^*(\text{FP}) \leq \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p \mathbb{C}_{ijk}^0 \mathbb{X}_{ijk}^0 + \mathbb{F}_{ijk}^0 \delta'_{ijk}. \dots (4.7)$$

Remark 4 : If the lower bound is equal to the upperbound in equation (4.7), then $\{\mathbb{X}_{ijk}^0, \delta'_{ijk}\}$ is the fuzzy optimal solution

of problem p, with fuzzy optimal value of $\mathbb{Z}^*(\text{FP})$.

Reformation of MIFCFTP by extracting fixed costs

Now the problem FP can be rewritten as follows :

$$\text{FP}^* : \text{Minimize } \mathbb{Z}^0 = K \mathbb{X}^0 + \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p (\mathbb{C}_{ijk}^0 \mathbb{X}_{ijk}^0 + \mathbb{F}_{ijk}^0 \delta_{ijk})$$

$$\text{Subject to } \sum_{i=1}^m \mathbb{X}_{ijk}^0 = \mathbb{A}_{jk}^0$$

$$\sum_{j=1}^n \mathbb{X}_{ijk}^0 = \mathbb{B}_{ki}^0$$

$$\sum_{k=1}^p \mathbb{X}_{ijk}^0 = \mathbb{E}_{ij}^0$$

$$\text{where } \sum_{j=1}^n \mathbb{A}_{ijk}^0 = \sum_{i=1}^m \mathbb{B}_{ki}^0, \sum_{k=1}^p \mathbb{B}_{ki}^0 = \sum_{j=1}^n \mathbb{E}_{ij}^0, \sum_{i=1}^m \mathbb{E}_{ij}^0 = \sum_{k=1}^p \mathbb{A}_{jk}^0.$$

$$\sum_{j=1}^n \sum_{k=1}^p \mathbb{A}_{jk}^0 = \sum_{k=1}^p \sum_{i=1}^m \mathbb{B}_{ki}^0 = \sum_{i=1}^m \sum_{j=1}^n \mathbb{E}_{ij}^0.$$

where $\mathbb{X}^0 = \min \{\mathbb{F}_{ijk}^0\}$ (the smallest fuzzy fixed charge coefficient)

$$\mathbb{F}_{ijk}^0 = \mathbb{F}_{ijk} - \mathbb{X}^0$$

The variable K is a consolidation coefficient equal to the number of non-zero basic variables. In the case of MIFTP, the number of basic variables should be equal to $mnp - (m - 1)(n - 1)(p - 1)$.

The MIFCFTP Heuristic Method

In this section, we outline steps for solving the MIFCFTP, problem FP, with fuzzy cost coefficient \mathbb{C}_{ijk}^0 and fuzzy fixed costs \mathbb{F}_{ijk}^0 .

The Algorithm

Step 1 : Formulate and the problems Max FLP_{ijk} and Min FLP_{ijk} to determine lower and upper bounds \mathbb{X}_{ijk}^{\min} and \mathbb{X}_{ijk}^{\max} for \mathbb{X}_{ijk}^0 , $i = 1, 2, \dots, m; j = 1, 2, \dots, n; k = 1, 2, \dots, p$.

Step 2 : Identify $\mathbb{X}^0 =$ the smallest fuzzy fixed charge coefficient.

Step 3 : Subtract \mathbb{X}^0 from all the fixed charges. Extract fixed cost $K \mathbb{X}^0$. Corresponding to the number of non-zero basic variables.

Step 4 : Formulate the problem FP' with $\mathbb{C}_{ijk}^0 = \mathbb{C}_{ijk} + \mathbb{F}_{ijk} / \mathbb{m}_{ijk}^0$.

Step 5 : Solve the problem FP' using LINDO software and record solution $\{\mathbb{X}_{ijk}^0\}$.

Step 6 : Determine δ'_{ijk} values as outlines in Remark 3.

Step 7 : Step Record the solution for problem FP as $\{\mathbb{X}_{ijk}^0, \delta'_{ijk}\}$.

Numerical Example

Consider a 3 x 3 x 3 fixed charge multi-inter fuzzy transportation problem FP. The fuzzy variable cost C_{ijk}^f are given at the top left corner in Table Ii.

The fuzzy fixed costs are

$$\begin{aligned}
 F_{111}^f &= (8\ 9\ 11\ 12) & F_{121}^f &= (28\ 29\ 31\ 32) & F_{131}^f &= (18\ 19\ 21\ 22) \\
 F_{112}^f &= (18\ 19\ 21\ 22) & F_{122}^f &= (18\ 19\ 21\ 22) & F_{132}^f &= (18\ 19\ 21\ 22) \\
 F_{113}^f &= (28\ 29\ 31\ 32) & F_{123}^f &= (18\ 19\ 21\ 22) & F_{133}^f &= (8\ 9\ 11\ 12) \\
 F_{211}^f &= (8\ 9\ 11\ 12) & F_{221}^f &= (18\ 19\ 21\ 22) & F_{231}^f &= (18\ 19\ 21\ 22) \\
 F_{212}^f &= (8\ 9\ 11\ 12) & F_{222}^f &= (8\ 9\ 11\ 12) & F_{232}^f &= (28\ 29\ 31\ 32) \\
 F_{213}^f &= (38\ 39\ 41\ 42) & F_{223}^f &= (8\ 9\ 11\ 12) & F_{233}^f &= (8\ 9\ 11\ 12) \\
 F_{311}^f &= (8\ 9\ 11\ 12) & F_{321}^f &= (38\ 39\ 41\ 42) & F_{331}^f &= (18\ 19\ 21\ 22) \\
 F_{312}^f &= (18\ 19\ 21\ 22) & F_{322}^f &= (8\ 9\ 11\ 12) & F_{332}^f &= (28\ 29\ 31\ 32) \\
 F_{313}^f &= (18\ 19\ 21\ 22) & F_{323}^f &= (8\ 9\ 11\ 12) & F_{333}^f &= (8\ 9\ 11\ 12)
 \end{aligned}$$

The above MIFCFTP can be formulated as a fuzzy linear programming problem. Using the methodology proposed in Amit Kumar et.al, 2011 to solve the fuzzy linear programming problem and the steps of the algorithm described in 5.3.1, we get the optimal solution to the problem FP' using LINDO is as follows :

$$\begin{aligned}
 X_{113}^f &= (8\ 9\ 11\ 12) & X_{122}^f &= (4\ 5\ 7\ 8) \\
 X_{131}^f &= (4\ 5\ 7\ 8) & X_{132}^f &= (3\ 3\ 3\ 3) \\
 X_{211}^f &= (5\ 5\ 5\ 5) & X_{212}^f &= (12\ 13\ 15\ 16) \\
 X_{213}^f &= (2\ 2\ 2\ 2) & X_{221}^f &= (6\ 7\ 9\ 10) \\
 X_{223}^f &= (1\ 1\ 1\ 1) & X_{233}^f &= (12\ 13\ 15\ 16) \\
 X_{311}^f &= (8\ 9\ 11\ 12) & X_{312}^f &= (3\ 3\ 3\ 3) \\
 X_{313}^f &= (8\ 8\ 8\ 8) & X_{322}^f &= (5\ 5\ 5\ 5) \\
 X_{323}^f &= (6\ 7\ 9\ 10) & X_{331}^f &= (5\ 5\ 5\ 5) \\
 X_{332}^f &= (3\ 4\ 6\ 7) & X_{333}^f &= (2\ 2\ 2\ 2)
 \end{aligned}$$

with $Z_{FP'}$ = (831.24, 976.74, 1280.33, 1441.78)

Conclusion

In this paper, a heuristic algorithm for multi-index fixed charge transportation problem is proposed. The optimal value obtained using the proposed algorithm is 1126.02, where as the already existing exact method proposed by Anu Ahuja, Arora. S.R. ,2001, is 1185 which is more than the optimal value of the proposed method. More over,in the real world applications, the parameters in the transportation problem may not be known precisely due to uncontrollable factors. Therefore this algorithm is extended to solve MIFCFTP. Since the optimal value is expressed as fuzzy numbers rather than crisp value, more information is provided for decision making. The efficiency of this algorithm is proved by comparing the result obtained by using this algorithm with the already existing exact method of solving MIFCFTP.

References

Adlakha, V., Kowalski, K.: On the fixed-charge transportation problem. OMEGA Int. J. Manage. Sci. 27, 381-388 (1999).

Adlakha, V. Kowalski, K.: A heuristic for solving small fixed-charge transportation problems. OMEGA Int. J. Manage. Sci. 33, 205-211 (2003).

Adlakha, V., Kowalski, K.: A simple algorithm for the source induced fixed-charge transportation problem. JORS J. Oper. Res. Soc. 55, 1275-1280 (2004).

Adlakha, V., Kowalski, K., Vemuganti, R.: Heuristic algorithms for the fixed-charge transportation problem. Opsearch 43, 132-151 (2006).

Anu Ahuja, Arora. S.R., Multi-index fixed charge bi-criterion Transportation problem, Indian J.Pure appl.Math.32(5):739-746 (2001)

Balinski, M.L.: Fixed cost transportation problems. Nav. Res. Logistics Q. 8, 41-54 (1961).

Chanas, S., Kuchta, D.: A concept of the optimal solution of the transportation problem with fuzzy cost coefficients. Fuzzy Sets Syst. 82, 299-305 (1996).

Chanas, S., Kuchta, D.: Fuzzy integer transportation problem. Fuzzy Sets Syst. 98,291-298 (1998).

Chen, M., Ishii, H., Wu, C.: Transportation problems on a fuzzy network. Int. J. Innov. Comput. Inf. Control 4, 1105-1109 (2008).

Chiang, J.: The optimal solution of the transportation problem with fuzzy demand and fuzzy product. J. Inf. Sci. Eng. 21, 439-451 (2005).

Cooper, L.: The fixed charge problem—1: a new heuristic method. Comp. Math. Appl. 1, 89-95 (1975).

Cooper, L., Drebes, C.: An approximate algorithm for the fixed charge problem. Nav. Res. Logistics Q. 14,101-113 (1967).

Diaby, M.: Successive linear approximation procedure for generalized fixed-charge transportation problems. J. Oper. Res. Soc. 42, 991-1001 (1991).

Denzler, D.R.: An approximate method for the fixed charge problem. Nav. Res. Logistics Q. 16, 411- 416 (1969).

Efroymsen, M., Ray, T.: A branch-bound algorithm for plant location. Oper. Res. 14, 361-368 (1966).

Gray, R: Exact solution of the fixed-charge transportation problem. Oper. Res. 19, 1529-1538 (1971).

Haley,K.B. (1962), The Solid transportation problem, *Operations Research*, 10 : 448-63

Hirsch, W., Dantzig, G.B. : The fixed charge problem. Nav. Res. Logistics Q. 15, 413~24 (1968).

Junginger, W., On representatives of multi-index transportation problems", *European Journal of Operational Research* 66, 353-371 (1993).

Kuhn, H.W., Baumol, W.J.: An approximation algorithm for the fixed charge transportation problem. Nav. Res. Logistics Q. 9, 1-15 (1962).

Lie, L., Huang, Z., Da, Q., Hu, J.: A new method based on goal programming for solving transportation problem with fuzzy cost. In: International Symposiums on Information Processing, pp. 3-8 (2008).

Lin, F.T.: Solving the transportation problem with fuzzy coefficients using genetic algorithms. In: IEEE International Conference on Fuzzy Systems, pp. 1468-1473 (2009).

Liu, S.T., Kao, C.: Solving fuzzy transportation problems based on extension principle. Eur. J. Oper. Res. 153, 661-674 (2004).

Murty, K.G.: Solving the fixed charge problem by ranking the extreme points. Oper. Res. 16, 268-279 (1968).

Oheigeartaigh, M.: A fuzzy transportation algorithm. Fuzzy Sets Syst. 8,235-243 (1982).

J. Algor. Comput. Math. 2, 65-71 (2009).Palekar, U.S., Karwan, M.H., Zionts, S.: A branch-and-bound method for the fixed charge transportation problem. Manage. Sci. 36, 1092-1105 (1990).
 Rautman C.A., Reid, R.A., and Ryder, E.E., "Scheduling the disposal of nuclear waste material in a geologic repository using the transportation model", *Operations Research* 41, 459-469 (1993).
 Sadagopan, S., Ravindran, A.: A vertex ranking algorithm for the fixed-charge transportation problem. J. Optim. Theory Appl 37, 221-230 (1982).
 Salkin, H.M.: Integer programming, pp.482-504. Addison-Wesley Publishing Company (1975).
 Sandrock, K.: A simple algorithm for solving small fixed-charge transportation problems. J. Oper. Res. Soc. 39, 467-475 (1988).

Steinberg, D.I.: The fixed charge problem. Nav. Res. Logistics Q. 17, 217-235 (1970).
 Stephen Dinagar, S., Palanivel, K.: The transportation problem in fuzzy environment. Int. J. Algor. Comput. Math. 2, 65-71 (2009)
 Sun, M., Aronson, J.E., McKeown, P.G., Drinka, D.: A Tabu search heuristic procedure for the fixed charge transportation problem. Eur. J. Oper. Res. 106, 441-456 (1998).
 Walker, W.E.: A heuristic adjacent extreme point algorithm for the fixed charge problem. Manage. Sci. 22, 587-596 (1976).
 Zadeh, L.A.: Fuzzy sets. Inf. Control 8, 338-353 (1965).
 Zimmermann, H.J.: Fuzzy programming and linear programming with several objective functions. Fuzzy Sets Syst. 1, 45-55 (1978).

Table 1

	j = 1		j = 2		j = 3		B _{ki}	
i = 1	8		5		7		6	
		7		6		3		9
	E ₁₁ = 10 6		E ₁₂ = 6 10		E ₁₃ = 9 11		10	
i = 2	11		9		13		13	
		8		15		7		14
	E ₂₁ = 21 13		E ₂₂ = 9 12		E ₂₃ = 14 8		17	
i = 3	5		8		10		15	
		6		9		6		13
	E ₃₁ = 21 7		E ₃₂ = 13 7		E ₃₃ = 12 11		18	
A _{jk}	15		8		11		34	
		17		11		8		36
	20		9		16		45	

Table 2

	j = 1		j = 2		j = 3		B _{ki}	
i = 1	(6 7 9 10)		(3 4 6 7)		(5 6 8 9)		(4 5 7 8)	
		(5 6 8 9)		(4 5 7 8)		(1 2 4 5)		(7 8 10 11)
	E ₁₁ = (8 9 11 12)		E ₁₂ = (4 5 7 8)		E ₁₃ = (7 8 10 11)		(9 10 12 13)	
i = 2	(9 10 12 13)		(7 8 10 11)		(11 12 14 15)		(11 12 14 15)	
		(6 7 9 10)		(13 14 16 17)		(5 6 8 9)		(12 13 15 16)
	E ₂₁ = (19 20 22 23)		E ₂₂ = (7 8 10 11)		E ₂₃ = (12 13 15 16)		(6 7 9 10)	
i = 3	(3 4 6 7)		(6 7 9 10)		(8 9 11 12)		(13 14 16 17)	
		(4 5 7 8)		(7 8 10 11)		(4 5 7 8)		(11 12 14 15)
	E ₃₁ = (19 20 22 23)		E ₃₂ = (11 12 14 15)		E ₃₃ = (10 11 13 14)		(10 11 13 14)	
A _{jk}	(13 14 16 17)		(6 7 9 10)		(9 10 12 13)		(28 31 37 40)	
		(15 16 18 19)		(9 10 12 13)		(6 7 9 10)		(30 33 39 42)
	(18 19 21 22)		(7 8 10 11)		(14 15 17 18)		(39 42 48 51)	