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Similarity of a Singular Sturm-Liouville Operator with Indefinite Weight

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ABSTRACT

We consider a singular Sturm-Liouville differential expression with an indefinite weigh and present a sufficient condition for similarity of indefinite Sturm-Liouville operators to self-adjoint operators. Using this result, we construct a example of operator and prove that it is similar to a self-adjoint one.

Introduction

Consider the singular Sturm-Liouville expression

$$a(y) := \frac{1}{r(x)}((-py')' + qy), \qquad (1)$$

where the weight function r changes its sign. We assume that (1) is in the limit point case at both $-\infty$ and $+\infty$ and that the functions p, q, r are real, $r \neq 0$ a.e., $x \in \mathrm{R}$. Then the maximal operator A associated to (1) is self-adjoint in the Krein space $L_r^2(\mathrm{R})$, where the indefinite inner product is defined by

$$[f, g] := \int_{\mathrm{R}} f(x)\overline{g(x)}r(x)dx,\ f, g \in L_r^2(\mathrm{R}).$$

Two closed operators T_1 and T_2 in a Hilbert space H are called similar if there exist a bounded operator S with the bounded inverse S^{-1} in H such that $S\mathrm{dom}(T_1) = \mathrm{dom}(T_2)$ and $T_2 = ST_1S^{-1}$.

In general, if the operator (1) is in the limit circle case at both $-\infty$ and $+\infty$, we can consider its Riesz basis property. Here the operator (1) considered on $L_r^2(\mathrm{R})$ has continuous spectrum. In the case, one considers the property of similarity either to a normal or to a self-adjoint operator.

Using the Krein-Langer technique of definitizable operators in Krein spaces, Curgus and Langer [1] have obtained the first result in the direction. In particular, their result yields that the J-selfadjoint operator with $r(x) = \mathrm{sgn}\, x$ is similar to a self-adjoint if L is a uniformly positive operator. Next Curgus and Najman [2] showed that the operator $(\mathrm{sgn}\, x)\frac{d^2}{dx^2}$ is similar to a self-adjoint one. In the paper [3], similarity of $\mathrm{sgn}\, x(-\frac{d^2}{dx^2} + c\delta)$ type operators to normal and self-adjoint operators were described. In [4-6] several necessary similarity conditions in terms of Weyl functions were obtained. Based on the concept of boundary triplet and the resolvent similarity criterion, references [7-8] investigate the main spectral properties of quasi-self-adjoint extensions of corresponding operator.

Here we are interested in more general indefinite differential expression of the form (1) and our main goal is to show that a sufficient condition for the operator A to be similar to a self adjoint operator in Hilbert space. Using this result, we construct a example of operator and prove that this operator is similar to a self-adjoint one.

Throughout the article (2) we use the following notations: Let T be a linear operator in a Hilbert space $(H, (\bullet, \bullet))$. In what follows $\mathrm{dom}(T)$, $\ker(T)$, $\mathrm{ran}(T)$ are the domain, kernel, range of T, respectively. We denote by $\rho(T)$, $\sigma(T)$ and $\sigma_p(T)$ the resolvent set, the spectrum and the point spectrum of T. $R_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$ is the resolvent of T. We set $\mathrm{C}_{\pm} := \{\lambda \in \mathrm{C} : \pm\mathrm{Im}\lambda > 0\}$.

Preliminaries.

Indefinite Sturm-Liouville Operators in $L_r^2(\mathrm{R})$

Consider the differential expression

$$a(y) := \frac{1}{r(x)}((-py')' + qy),$$

where p^{-1}, q, $r \in L_{loc}^1(\mathrm{R})$ are assumed to be real valued functions such that $p > 0$ and $r \neq 0$ for a.e., $x \in \mathrm{R}$. Here we assume that the following condition holds:

There exist $a, b \in \mathrm{R}, a < b$, such that the restrictions $r_+ := r|(b, +\infty)$ and $r_- := r|(-\infty, a)$

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satisfy $r_+(x)>0$ for a.e., $x\in(b,+\infty)$ and $r_-(x)<0$ for a.e., $x\in(-\infty,a)$.

By $L_r^2(\mathrm{R})$ we denote the Krein space of all equivalence of measurable functions f defined on R for $\int_{-\infty}^{+\infty}|f(x)|^2|r(x)|dx<+\infty$ which.the indefinite and definite inner products on $L_r^2(\mathrm{R})$ are

$[f,g]:=\int_{-\infty}^{+\infty} f\bar{g}rdx$ and $(f,g):=\int_{-\infty}^{+\infty} f\bar{g}\,|r|dx$.

Evidently, the operator J

$$(Jf)(x)=(\operatorname{sgn} r(x))f(x),\ x\in\mathrm{R} \qquad (5)$$

is the fundamental symmetry connecting the inner products in (4). By $L_{|r|}^2(\mathrm{R})$ we denote the Hilbert space $L_r^2(\mathrm{R},(\bullet,\bullet))$.

Let us assume that the Sturm-Liouville differential expression

$$l(y):=\frac{1}{|r|}((-py')'+qy),$$

is in the limit point case at both singular endpoints $-\infty$ and $+\infty$. Then it is well known that the operator $Ly=l(y)$ defined on the usual maximal domain

$$D_{\max}=\{y\in L_{|r|}^2(\mathrm{R}): y, py'\in AC_{loc}(\mathrm{R}), ly\in L_{|r|}^2(\mathrm{R})\}$$

is self-adjoint in the Hilbert space $L_{|r|}^2(\mathrm{R})$.

In the following we set

$$Ay:=JLy=\frac{1}{r}((-py')'+qy), \operatorname{dom}A=\operatorname{dom}JL=D_{\max}.$$

The operator A is self-adjoint in the Krein space $L_r^2(\mathrm{R})$. We shall interpret the operator A as a finite rank perturbation in resolvent sense of the direct sum of three differential operators A_-, A_{ab} and A_+ defined in the sequel. We identify function $f\in L_r^2(\mathrm{R})$ with $f=f_-+f_{ab}+f_+$, where $f_-\in L_{r_-}^2((-\infty,a))$, $f_{ab}\in L_{r_{ab}}^2((a,b))$ and $f_+\in L_{r_+}^2((b,+\infty))$, respectively. Similarly we denote the restrictions of p and q onto the intervals $(-\infty,a)$ and $(b,+\infty)$ by p_-, p_+ and q_-,q_+, respectively. Moreover we denote the restriction of r, p and q onto the interval (a,b) by r_{ab}, p_{ab} and q_{ab}. Besides the differential expression l in (6) we set

$$l_-(f_-):=\frac{1}{r_-}((p_-f_-')'-q_-f_-),$$

$$l_+(f_+):=\frac{1}{r_+}(-(p_+f_+')'+q_+f_+),$$

and

$$l_{ab}(f_{ab}):=\frac{1}{|r_{ab}|}(-(p_-f_-')'+q_{ab}f_{ab}),$$

respectively, and operators associated to them. Note that l_- and l_+ are in the limit point case at $-\infty$ and $+\infty$ and regular at the endpoints a and b, respectively, whereas l_{ab} is regular at both endpoints a and b. By $D_{\max}^-$ ($D_{\max}^+$ and $D_{\max}^{ab}$) we denote the set in (7) if r, R and l are replaced by r_-, $(-\infty,a)$ and l_- (resp. r_+, $(b,+\infty)$, l_+ and r_{ab}, (a,b), l_{ab}. Therefore the operators

$$A_{\min -}(f_-):=\frac{1}{r_-}(-(p_-f_-')'+q_-f_-),$$

$$A_{\min +}(f_+):=\frac{1}{r_+}(-(p_+f_+')'+q_+f_+) \qquad (4)$$

and

$$S_{ab}(f_{ab}):=\frac{1}{r_{ab}}(-(p_-f_-')'+q_{ab}f_{ab}),$$

defined on

$$\operatorname{dom}(A_{\min -})=\{f_-\in D_{\max}^-: f_-(a)=(p_-f_-')(a)=0\}, \qquad (6)$$

$$\operatorname{dom}(A_{\min +})=\{f_+\in D_{\max}^+: f_+(b)=(p_+f_+')(b)=0\}$$

and

$$\operatorname{dom}(S_{ab})=\{f_{ab}\in D_{\max}^{ab}: f_{ab}(a)=(p_{ab}f_{ab}')(a)=f_{ab}(b)=(p_{ab}f_{ab}')(b)=0\}$$

With

$$D_{\max}^-=\{f_-\in L_{|r_-|}^2((-\infty,a)): f_-, p_-f_-'\in AC_{loc}((-\infty,a)), l(f_-)\in L_{|r_-|}^2((-\infty,a))\}, \qquad (7)$$

$$D_{\max}^+=\{f_+\in L_{r_+}^2((b,+\infty)): f_+, p_+f_+'\in AC_{loc}((b,+\infty)), l(f_+)\in L_{|r_+|}^2((b,+\infty))\}$$

and

$$D_{\max}^{ab}=\{f_{ab}\in L_{|r_{ab}|}^2((a,b)): f_{ab}, p_{ab}f_{ab}'\in AC_{loc}((a,b)), l(f_{ab})\in L_{|r_{ab}|}^2((a,b))\}$$

are closed symmetric operators in the anti-Hilbert space $L_{r_-}^2((-\infty,a))$, Hilbert space $L_{r_+}^2((b,+\infty))$ and Krein space $L_{r_{ab}}^2((a,b))$, respectively. The adjoint operators $A_{\min -}^*$, $A_{\min +}^*$ and S_{ab}^* are the usual maximal operators defined on $D_{\max}^-$, $D_{\max}^+$ and $D_{\max}^{ab}$, respectively.

Let

$\operatorname{dom}(S)=\operatorname{dom}(A_{\min -})\oplus\operatorname{dom}(S_{ab})\oplus\operatorname{dom}(A_{\min +})$ and let the operator S be defined on $\operatorname{dom}(S)$,

$$S=\begin{pmatrix} A_{\min -} & 0 & 0\\ 0 & S_{ab} & 0\\ 0 & 0 & A_{\min +}\end{pmatrix}$$

with respect to the decomposition

$L_r^2(\mathrm{R})=L_{r_-}^2((-\infty,a))\oplus L_{r_{ab}}^2(a,b)\oplus L_{r_+}^2(b,+\infty)$. Then S is a closed symmetric operator in the krein space $L_r^2(\mathrm{R})$ with finite defect 4. Moreover, we have $S=A|_{\operatorname{dom}(S)}$, $A=S^*|_D$, where

$$D=\operatorname{dom}(A)=\{f\in(\operatorname{dom}(A_{\min -}^*)\oplus\operatorname{dom}(S_{ab}^*)\oplus\operatorname{dom}(A_{\min +}^*):$$
$$f_-(a)=f_{ab}(a), (p_-f_-')(a)=(p_{ab}f_{ab}')(a), f_+(b)=f_{ab}(b), (p_+f_+')(b)=(p_{ab}f_{ab}')(b)\}. \qquad (9)$$

Theorem 1. ([15]) Let the operator L be nonnegative and $A=JL$ be self-adjoint operator in the krein space $L_r^2(\mathrm{R})$

with the nonempty resolvent set $\rho(A) \neq \emptyset$. Then the spectrum of A is real, $\sigma(A) \subset R$.

Theorem 2. If the operator L is semibounded from below, then $\rho(A) \neq \emptyset$. (see Theorem 4.5 in [16])

Theorem 3. If the operator L is nonnegative, then the spectrum of A is real, $\sigma(A) \subset R$.

Proof Since $L \neq 0$, Theorem 2 implies that $\rho(A) \neq \emptyset$. Theorem 1 completes the proof.

2.2. Weyl-Titchmarsh m-coefficients

Let $c(x,\lambda)$ and $s(x,\lambda)$ denote the linearly independent solutions of equation (6) satisfying the following initial conditions at a

$$c(a,\lambda) = (ps')(a,\lambda) = 1, (pc')(a,\lambda) = s(a,\lambda) = 0.$$

Since equation (6) is limit point at $+\infty$, the Weyl-Titchmarsh theorem (see [9]) states that there exists a unique holomorphic function $m_+(\cdot) \in C \setminus R \to C$, such that the function $s_+(x,\lambda) - m_+(\lambda)c_+(x,\lambda)$ belongs to $L^2_{r_+}((b,+\infty))$. Similarly, the limit point case at $-\infty$ yields the fact that there exists a unique holomorphic function $m_-(\cdot) \in C \setminus R \to C$, such that $s_-(x,\lambda) - m_-(\lambda)c_-(x,\lambda)$ belongs to $L^2_{-r_-}((-\infty,a))$.

The functions m_+ and m_- are called the Weyl-Titchmarsh m-coefficients for (6) on $(b,+\infty)$ and on $(-\infty,a)$, respectively. We put

$$M_\pm(\lambda) := \pm m_\pm(\pm\lambda), \quad (10)$$

$$\psi_+(x,\lambda) = s_+(x,\lambda) - m_+(\lambda)c_+(x,\lambda),$$

$$\psi_-(x,\lambda) = s_-(x,\lambda) - m_-(\lambda)c_-(x,\lambda). \quad (11)$$

By the definition of $m_\pm$, the functions $\psi_+(x,\lambda)$ and $\psi_-(x,\lambda)$ belong to $L^2_{r_+}((b,+\infty))$ and $L^2_{-r_-}((-\infty,a))$ for all $\lambda \in C \setminus R$, respectively. Besides, $\tau[\psi_\pm(x,\lambda)] = \lambda\psi_\pm(x,\lambda)$. The function $M_+(\cdot)$ ($M_-(\cdot)$) is said to be the Weyl-Titchmarsh m-coefficient for equation (3) on $(b,+\infty)$ (on $(-\infty,a)$).

Definition 1. The class (R) consists of all holomorphic functions $G: C_+ \cup C_- \to C$ such that $G(\bar{\lambda}) = \overline{G(\lambda)}$, and $\mathrm{Im}\lambda \cdot \mathrm{Im}G(\lambda) \geq 0$ for $\lambda \in C_+ \cup C_-$ (see [10]).

It is well known that

$$\int_b^{+\infty} |\psi_+(x,\lambda)|^2\, r(x)dx = \frac{\mathrm{Im}M_+(\lambda)}{\mathrm{Im}\lambda},$$

$$\int_{-\infty}^{a} |\psi_-(x,\lambda)|^2 |r(x)|\, dx = \frac{\mathrm{Im}M_-(\lambda)}{\mathrm{Im}\lambda} \quad (12)$$

for all $\lambda \in C \setminus R$ (see [9]). These formulae imply that the functions M_+ and M_- (as well as m_+ and m_- belong to the class (R). Moreover (see [11]) the functions M_+ and M_- admit the following integral representation

$$M_\pm(\lambda) = \int_{-\infty}^{+\infty} \frac{d\tau_\pm(s)}{s-\lambda}, \lambda \in C \setminus R. \quad (13)$$

Here $\tau_\pm : R \to R$ are nondecreasing functions on R with the following properties:

$$\int_{-\infty}^{+\infty} \frac{d\tau_\pm(s)}{1+|s|} < +\infty, \tau_+(b) = \tau_-(a) = 0,$$

$$\tau_\pm(s) = \tau_\pm(s-0).$$

Notice that the functions τ_+ and τ_- are uniquely determined by the Stieltjes inversion formulae

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_b^s \mathrm{Im}M_+(t+i\varepsilon)dt = \frac{\tau_+(s+0)+\tau_+(s-0)}{2},$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_a^s \mathrm{Im}M_-(t+i\varepsilon)dt = \frac{\tau_-(s+0)+\tau_-(s-0)}{2}.$$

The functions functions τ_+ and τ_- are called spectral functions of the operators

$$A_{0-} := A^*_{\min-} \,|\, \{y \in \mathrm{dom}(A^*_{\min-}) : (p_- y'_-)(a) = 0\} \quad (14)$$

and

$$A_{0+} := A^*_{\min+} \,|\, \{y \in \mathrm{dom}(A^*_{\min+}) : (p_+ y'_+)(b) = 0\}, \quad (15)$$

respectively.

Boundary Triplets and Abstract Weyl Functions

Let K be a Krein space and let H be a separable Hilbert space. Let S be a closed symmetric operator in K with equal and finite deficiency indices $n_+(s) = n_-(s) < \infty$.

Recall the concepts of boundary triplets and abstract Weyl functions (see [12, 13]).

Definition 2. A triplet $\Pi = \{H, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for S^* if the following two conditions:

(i)

$$(S^* f, g)_K - (f, S^* g)_K = (\Gamma_1 f, \Gamma_0 g)_H - (\Gamma_0 f, \Gamma_1 g)_H,$$

$f, g \in \mathrm{dom}(S^*)$;

(ii) the linear

Mapping $\Gamma = \{\Gamma_0 f, \Gamma_1 f\} : \mathrm{dom}(S^*) \to H \oplus H$ is surjective.

The mappings Γ_0 and Γ_1 naturally induce two extensions S_0 and S_1 of S given by

$$S_j := S^* \,|\, \mathrm{dom}(S_j), \mathrm{dom}(S_j) = \mathrm{Ker}\Gamma_j, (j = 0,1).$$

The $\gamma - field$ of the operator S corresponding to the boundary triplet Π is the operator function $\gamma(\cdot) : \rho(S_0) \to [H, N_\lambda(S)]^{-1}$ defined by $\gamma(\lambda) := (\Gamma_0 \,|\, N_\lambda(S))^{-1}$, where $N_\lambda(S) := \mathrm{Ker}(S^* - \lambda I)$.

The function γ is well-defined and holomorphic on $\rho(S_0)$.

Definition 3. Let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator S^*. The operator valued function $M(\cdot) : \rho(S_0) \to [H]$ defined by $M(\lambda) := \Gamma_1 \gamma(\lambda), \lambda \in \rho(S_0)$ is called the Weyl function of S corresponding to the boundary triplet Π.

Let $C, D \in [H]$. Considering the following extension $\tilde{S}$ of $S, S \subset \tilde{S}$,

$\tilde{S} = S_{C,D} := S^* \mid \mathrm{dom}(S_{C,D})$,

$\mathrm{dom}(S_{C,D}) = \{ f \in \mathrm{dom}(S^*) : C\Gamma_1 f + D\Gamma_0 f = 0\}$. (16)

Notice that each proper extension $\tilde{S}$ of S has the form (16), i.e., if $S \subset \tilde{S} \subset S^*$, then there exist $C, D \in [H]$ such that $\tilde{S} = S_{C,D}$.

Theorem 4. Suppose $\Pi = \{H, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^*, $M(\cdot)$ is the corresponding Weyl function, and $\tilde{S} = S_{C,D}$, where $S_{C,D}$ is defined by (16). Assume also that $C, D, (CC^* + DD^*)^{-1} \in [H]$. Then $\lambda \in \rho(S_0) \cap \rho(\tilde{S})$ if and only if $0 \in \rho(D + CM(\lambda))$.

In particular, if C is invertible, let $B = C^{-1}D \in [H]$, then

$\tilde{S} = S_B := S^* \mid \mathrm{dom}(S_B)$,

$\mathrm{dom}(S_B) = \{ f \in \mathrm{dom}(S^*) : \Gamma_1 f - B\Gamma_0 f = 0\}$. (17)

Theorem 5. Suppose $\Pi = \{H, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^*, $M(\cdot)$ is the corresponding Weyl function and S_B is defined by (17). Assume also that $B \in [H]$.Then $\lambda \in \rho(S_0) \cap \rho(S_B)$ if and only if $0 \in \rho(B - M(\lambda))$.

3. Boundary Triplets for Sturm-Liouville Operator. A

1. Let $A_{\min+}$ and $A_{\min-}$ be the operators defined in Subsection 2.1. Since equation (1) is in the limit point case at $-\infty$ and $+\infty$, then the deficiency indices of the symmetric operator are (1,1) and for all $f_\pm, g_\pm \in \mathrm{dom}(A^*_{\min\pm})$ we have

$(A^*_{\min+} f_+, g_+) - (f_+, A^*_{\min+} g) = (p_+ f_+')(b)\overline{g_+(b)} - f_+(b)\overline{(p_+ g')(b)}$ (18)

$(A^*_{\min-} f_-, g_-) - (f_-, A^*_{\min-} g) = (p_- f_-')(a)\overline{g_-(a)} - f_-(a)\overline{(p_- g')(a)}$. (19)

Hence the triplets $\Pi^+ = \{\mathrm{C}, \Gamma_0^+, \Gamma_1^+\}$ and $\Pi^- = \{\mathrm{C}, \Gamma_0^-, \Gamma_1^-\}$, where

$\Gamma_0^+ f_+ = (p_+ f_+')(b), \Gamma_1^+ f_+ = -f_+(b), f_+ \in \mathrm{dom}(A^*_{\min+})$,

$\Gamma_0^- f_- = (p_- f_-')(a), \Gamma_1^- f_- = -f_-(a), f_- \in \mathrm{dom}(A^*_{\min-})$,

are the boundary triplets for $A^*_{\min+}$ and $A^*_{\min-}$, respectively. By the definition of the functions $\psi_+(\cdot,\lambda)$ and $\psi_-(\cdot,\lambda)$ (see Subsection 2.2), we obtain

$N_\lambda(A_{\min\pm}) = \mathrm{Ker}(A^*_{\min\pm} - \lambda) = \{c\psi_\pm(\cdot,\lambda), c \in \mathrm{C}\}$, $\lambda \in \mathrm{C} \setminus \mathrm{R}$. (20)

Denote by γ^+ and γ^- the $\gamma - \mathit{field}$ corresponding to the boundary triplets $\Pi^+ = \{\mathrm{C}, \Gamma_0^+, \Gamma_1^+\}$ and Π^-. By (11) and (20), we get

$\gamma^\pm(\lambda)c = (\Gamma_0^\pm \mid N_\lambda(A_{\min\pm}))^{-1} c = \mathrm{Ker}(A^*_{\min\pm} - \lambda) = c \cdot \psi_\pm(\cdot,\lambda)$, $c \in \mathrm{C}, \lambda \in \mathrm{C} \setminus \mathrm{R}$. (21)

Further, the self-adjoint extension $A^*_{\min\pm} \mid \ker(\Gamma_0^\pm)$ of $A_{\min\pm}$ coincides with the operator $A^0_\pm$. The Weyl function $\tilde{M}_\pm(\cdot)$ of $A_{\min\pm}$ corresponding to the boundary triplets $\Pi^\pm$ is defined by

$\tilde{M}_\pm(\lambda) := \Gamma_1^\pm \gamma^\pm(\lambda), \lambda \in \rho(A^0_\pm)$.

Combining (21) with (10) and (11), one obtains $\tilde{M}_\pm(\lambda)c = cM_\pm(\lambda)$, $c \in \mathrm{C}, \lambda \in \mathrm{C} \setminus \mathrm{R}$. Note that, $\tilde{M}_\pm(\cdot)$ is a holomorphic continuation of $M_\pm(\cdot)$ to the domain $\rho(A^0_\pm)$. In the sequel we will write $M_\pm$ instead of $\tilde{M}_\pm(\cdot)$.

2 Let us consider the regular indefinite Strum-Liouville operator S_{ab}, S_{ab} is a densely defined closed symmetric operator in the Krein space $L^2_{r_{ab}}((a,b))$ and has defect two, its adjoint S^*_{ab} is given by

$$S_{ab}(f_{ab}) := \frac{1}{r_{ab}}(-(p_{ab} f'_{ab})' + q_{ab} f_{ab}),$$

$$\mathrm{dom}(S^*_{ab}) = D^{ab}_{\max}.$$

For $f, g \in \mathrm{dom}(S^*_{ab})$, we have

$$(S^*_{ab} f, g) - (f, S^*_{ab} g) = \begin{pmatrix} f(a)\overline{(p_{ab} g')(a)} + (p_{ab} f')(a)\overline{g(a)} \\ f(b)\overline{(p_{ab} g'(b))} - (p_{ab} f')(b)\overline{g(b)} \end{pmatrix}$$

Hence $\Pi^{ab} = \{\mathrm{C}^2, \Gamma_0^{ab}, \Gamma_1^{ab}\}$ is a boundary triplet for S^*_{ab}, where

$$\Gamma_0^{ab} f_{ab} = \begin{pmatrix} -(p_{ab} f'_{ab})(a) \\ (p_{ab} f'_{ab})(b) \end{pmatrix}, \Gamma_1^{ab} f_{ab} = \begin{pmatrix} f_{ab}(a) \\ f_{ab}(b) \end{pmatrix}.$$

Let $\varphi_\lambda, \psi_\lambda \in L^2_{r_{ab}}((a,b))$ be the fundamental solutions of $-(p_{ab} h')' + q_{ab} h = \lambda r_{ab} h$, $\lambda \in \mathrm{C}$, satisfying the initial conditions

$\varphi_\lambda(a) = 1, (p_{ab}\varphi'_\lambda)(a) = 0$ and $\psi_\lambda(a) = 0$, $(p_{ab}\psi'_\lambda)(a) = 1$.

Since

$N_\lambda(S_{ab}) = \mathrm{Ker}(S_{ab} - \lambda) = \mathrm{sp}\{\varphi_\lambda, \psi_\lambda\}, \lambda \in \mathrm{C} \setminus \mathrm{R}$. (22)

Denote by γ^{ab} the $\gamma - \mathit{field}$ corresponding to the boundary triplets Π^{ab}. By (22), we get $\gamma^{ab}(\lambda)c = (\Gamma_0^{ab} \mid N_\lambda(S_{ab}))^{-1} c = \mathrm{sp}(\varphi_\lambda, \psi_\lambda)$. Furthermore $x \to \varphi_\lambda(x)(p_{ab}\psi'_\lambda)(x) - (p_{ab}\varphi'_\lambda)(x)\psi_\lambda(x) = 1$,

we find that the Weyl function m_{ab} (see [12]) is given by

$$m_{ab}(\lambda) = \frac{1}{(p_{ab}\varphi'_\lambda)(b)} \begin{pmatrix} (p_{ab}\psi'_\lambda)(b) & 1 \\ 1 & \varphi_\lambda(b) \end{pmatrix}, \lambda \in \mathrm{C} \setminus \mathrm{R}.$$

3. The operator $S = A_{\min-} \oplus S_{ab} \oplus A_{\min+}$ is a closed densely defined symmetric operator of defect 4 in the Krein space $L^2_{r_-}((-\infty,a)) \oplus L^2_{r_{ab}}(a,b) \oplus L^2_{r_+}(b,+\infty)$ and it is straightforward to check that $\{\mathrm{C}^4, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f = \begin{pmatrix} \Gamma_0^- f_- \\ \Gamma_0^+ f_+ \\ \Gamma_0^{ab} f_{ab} \end{pmatrix}, \Gamma_1 f = \begin{pmatrix} \Gamma_1^- f_- \\ \Gamma_1^+ f_+ \\ \Gamma_1^{ab} f_{ab} \end{pmatrix}, \qquad (23)$$

$\{f_-, f_+, f_{ab}\} \in (\mathrm{dom}(A^*_{\min -}) \oplus \mathrm{dom}(A^*_+) \oplus \mathrm{dom}(S^*_{ab})$ is a boundary triple for the adjoint operator $A^*_{\min -} \oplus A^*_+ \oplus S^*_{ab}$. Further, we put

$$S_0 := S^* \mid \ker(\Gamma_0) = A^0_- \oplus A^0_+ \oplus A^0_{ab}, \qquad (24)$$

where

$A_{0ab} := S^*_{ab} \mid \{y \in \mathrm{dom}(S^*_{ab}) : (p_{ab} y'_{ab})(a) = (p_{ab} y'_{ab})(b) = 0\}$.

Therefore, the operator function $\gamma(\cdot) : \rho(S_0) \to [C^4, N_\lambda(S)]$ defined by

$$\gamma(\lambda)\begin{pmatrix} c_+ \\ c_- \\ c_{ab} \end{pmatrix} = c_+\psi_+ + c_-\psi_- + c_1\varphi_\lambda + c_2\psi_\lambda$$

is the $\gamma - field$ corresponding to the boundary triplet $\Pi = \{C^4, \Gamma_0, \Gamma_1\}$. Moreover, the operator Weyl function (see [12]) has the following form:

$$M(\lambda) := \begin{pmatrix} M_-(\lambda) & 0 & 0 & 0 \\ 0 & M_+(\lambda) & 0 & 0 \\ 0 & 0 & \dfrac{(p_{ab}\psi'_\lambda)(b)}{(p_{ab}\varphi'_\lambda)(b)} & \dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} \\ 0 & 0 & \dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} & \dfrac{\varphi_\lambda(b)}{(p_{ab}\varphi'_\lambda)(b)} \end{pmatrix},$$

$\lambda \in \rho(S_0)$.

Theorem 6. Let A be the operator associated with equation (3) and let the operator S_0 be defined by (24).Then $\sigma(A) \cap \rho(S_0) = \{\lambda \in \rho(S_0) : \Delta = 0\}$, where

$\Delta = (p_{ab}\psi'_\lambda)(b)M_+(\lambda) - \varphi_\lambda(b)M_-(\lambda) - (p_{ab}\varphi'_\lambda)(b)M_-(\lambda)M_+(\lambda) + \psi_\lambda(b)$

Proof Let us rewrite (8) as follows $\mathrm{dom}(A) = \{f \in \mathrm{dom}(S^*) : C\Gamma_1 f + D\Gamma_0 f = 0\}$, where

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

By Theorem 4, $\lambda \in \rho(A) \cap \rho(S_0)$ if and only if $0 \in \rho(D + CM(\lambda))$. Since

$$\det(D + CM(\lambda)) = \begin{vmatrix} \dfrac{(p_{ab}\psi'_\lambda)(b)}{(p_{ab}\varphi'_\lambda)(b)} - M_-(\lambda) & \dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} \\ \dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} & \dfrac{\varphi_\lambda(b)}{(p_{ab}\varphi'_\lambda)(b)} + M_+(\lambda) \end{vmatrix} = \frac{\Delta}{(p_{ab}\varphi'_\lambda)(b)}$$

.

We see that $\lambda \in \rho(A) \cap \rho(S_0)$ exactly when $\Delta \neq 0$.

Theorem 7. Let S be a symmetric operator defined by (8) and let $M_\pm(\cdot)$ be defined by (10). Then

(i) $\Pi = \{C^4, \Gamma_0, \Gamma_1\}$ defined by

$$\tilde{\Gamma}_0, \tilde{\Gamma}_1 : \mathrm{dom}(S^*) \to C^4, \ \tilde{\Gamma}_0 f := \begin{pmatrix} f_-(a) \\ -f_+(b) \\ -(p_{ab} f'_{ab})(a) \\ (p_{ab} f'_{ab})(b) \end{pmatrix},$$

$$\tilde{\Gamma}_1 f := \begin{pmatrix} (p_- f'_-)(a) \\ (p_+ f'_+)(b) \\ f_{ab}(a) \\ f_{ab}(b) \end{pmatrix}$$

forms a boundary triplet for the operator S^*.

(ii) The corresponding Weyl function is

$$\tilde{M}(\lambda) := \begin{pmatrix} -M_-^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & M_+^{-1}(\lambda) & 0 & 0 \\ 0 & 0 & \dfrac{(p_{ab}\psi'_\lambda)(b)}{(p_{ab}\varphi'_\lambda)(b)} & \dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} \\ 0 & 0 & \dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} & \dfrac{\varphi_\lambda(b)}{(p_{ab}\varphi'_\lambda)(b)} \end{pmatrix},$$

$\lambda \in \rho(\tilde{S}_0)$.

(iii) The operator $A = JL$ is a self-adjoint extension of S and it is determined by.

$A := S^* \mid \mathrm{dom}(A), \mathrm{dom}(A) = \ker(\tilde{\Gamma}_1 - B\tilde{\Gamma}_0)$,where

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(iv) If $\sigma(A^0_{ab}) \subset R$, then $\lambda \in \rho(A) \cap C_\pm$ if and only if $\Delta \neq 0$.

(v) The sets $\sigma_p(A) \cap C_\pm$ are at most countable with possible limit points belonging to $R \cup \{\infty\}$. Moreover, if $\sigma(A^0_{ab}) \subset R$, then $\lambda \in \sigma(A) \cap C_\pm$ if and only if $\Delta = 0$.

(vi) The spectrum $\sigma(A)$ is symmetric with respect to the real line, that is $\lambda_0 \in \sigma_p(A) \Leftrightarrow \overline{\lambda_0} \in \sigma_p(A)$.

Proof (i)-(iii) These statements are obvious.

(iv) By Theorem 3, $\lambda \in \rho(A)$ if and only if $0 \in \rho(B - M(\lambda))$, that is

$$\det(B - \tilde{M}(\lambda)) := \det\begin{pmatrix} M_-^{-1}(\lambda) & 0 & -1 & 0 \\ 0 & -M_+^{-1}(\lambda) & 0 & 1 \\ 1 & 0 & -\dfrac{(p_{ab}\psi'_\lambda)(b)}{(p_{ab}\varphi'_\lambda)(b)} & -\dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} \\ 0 & -1 & -\dfrac{1}{(p_{ab}\varphi'_\lambda)(b)} & -\dfrac{\varphi_\lambda(b)}{(p_{ab}\varphi'_\lambda)(b)} \end{pmatrix} \neq 0$$

$, \lambda \in \rho(\tilde{S}_0)$.

(v) By Theorem 5, $\sigma(A)\cap C_{\pm}$ coincides with the set of zeros of the determinant $\det(B-\widetilde{M}(\lambda))$. Due to Proof of (iv), so $\lambda\in\sigma(A)\cap C_{\pm}$ if and only if $\Delta=0$.

(vi) Note that $\Delta(\lambda_0)=0$ yields $\Delta(\overline{\lambda_0})=\overline{\Delta(\lambda_0)}=0$.Similar implication is valid for jth derivative. This completes the proof.

Similarity conditions

Let S be a symmetric operator in a Krein space K with finite deficiency indices (n,n), $n\in N$. Let T be a quasi-self-adjoint extension of S. Then there exists a boundary triplet $\{H,\Gamma_0,\Gamma_1\}$ for $T^*_{\min}$ such that $\mathrm{dom}(T)=\ker(\Gamma_1 - B\Gamma_0)$ with some $B\in[H]$, that is $T=S_B$. Let $M(\cdot)$ be the Weyl function associated with the boundary triplet $\{H,\Gamma_0,\Gamma_1\}$.

Theorem 8. ([14]) Let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^*, $M(\cdot)$ the corresponding Weyl function, $B\in[H]$ and E an auxiliary Hilbert space. Then for any faracterization $B_I:=\dfrac{B-B^*}{2i}=KJK^*$ of B_I with $P\in[E,H]$ and $J=J^*=J^{-1}\in[E]$, the characteristic function $\theta(\lambda)=\theta_{A_B}(\lambda)$ of the extension A_B, $\mathrm{dom}(S_B)=\ker(\Gamma_1-B\Gamma_0)$, admits the representation $\theta_T(\lambda)=I+2iK^*(B^*-M(\lambda))^{-1}KJ$. Obviously, if $\ker(B-B^*)=0$, then $\theta_T(\lambda)=(B-M(\lambda))(B^*-M(\lambda))^{-1}$.

Theorem 9. ([4]) Let T be a quasi-selfadjoint extension of S and the spectrum $\sigma(T)$ is real. If T is a self-adjoint operator in Krein space K and $\sup\limits_{\lambda\in C_+\cup C_-}\|\theta_T(\lambda)\|<\infty$, then T is similar to a self-adjoint operator T_0. Moreover, if T is completely non-selfadjoint, then T_0 has purely absolutely continuous spectrum.

By Theorem 8 and simple calculation, we can obtain the following theorem:

Theorem 10. Let S be a symmetric operator defined by (8) and $A=JL$. Suppose that conditions of Theorem 7 are satisfied and

$$B=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}.\ \text{Then}$$

(i) $B_I=\begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&i&0&0\end{pmatrix}:=J$ and the characteristic function $\theta_A(\cdot)$ of the operator A admits the following representation:

$$\theta_A(\lambda)=I+\frac{2}{\Delta}\begin{pmatrix}k_1M_- & M_+ & k_1 & 1\\ M_- & k_2M_+ & 1 & k_2\\ M_-(\Delta+k_1M_-) & M_+M_- & k_1M_- & M_-\\ M_+M_- & M_+(\Delta+k_2M_+) & M_+ & k_2M_+\end{pmatrix}, \qquad (25)$$

where $k_1=M_+(p_{ab}\varphi'_\lambda)(b)+\varphi_\lambda(b)$, $k_2=M_-(p_{ab}\varphi'_\lambda)(b)-\psi'_\lambda(b)$.

(iii) The determinant $\det\theta_A(\lambda)$ defined originally on $\rho(A^*)$, admits holomorphic continuation to the complex plane C and $\det\theta_A(\lambda)=1$, $\lambda\in C\setminus R$.

Combining Theorem 9 with formula (25) we conclude that the condition:

Theorem 11. The following condition

$$\max\{\sup_{\lambda\in C_+}\frac{1}{\Delta},\sup_{\lambda\in C_+}\frac{|k_1|}{\Delta},\sup_{\lambda\in C_+}\frac{|k_2|}{\Delta},\sup_{\lambda\in C_+}|M_\pm|\}<\infty \qquad (26)$$

is sufficient for the operator A to be similar to a self-adjoint operator with absolutely continuous spectrum.

5. An example

The main object of this subsection is the following operator

$$(Ay)(x)=-\frac{1}{\mathrm{sgn}(x-\mathrm{sgn}\,x)}(-y''),\ \mathrm{dom}(A)=L^2_{|r|}(R). \qquad (27)$$

Lemma 1. The differential equation

$$-y''(x)=\lambda\,\mathrm{sgn}(x-\mathrm{sgn}\,x)y(x),\ x>1 \qquad (28)$$

is in the limit point case at $+\infty$. Moreover, the function

$$m(\lambda)=\frac{1}{\sqrt{-\lambda}}-\frac{2}{\pi}\frac{1}{\sqrt{-\lambda}}\arctan\frac{1}{\sqrt{-\lambda}},\ \lambda\notin R_+$$

is the Weyl-Titchmarsh m-coefficient for (28).

By Lemma 1, we obviously obtain

$$M_+(\lambda)=\frac{1}{\sqrt{-\lambda}}-\frac{2}{\pi}\frac{1}{\sqrt{-\lambda}}\arctan\frac{1}{\sqrt{-\lambda}},$$

$$M_-(\lambda)=-\frac{1}{\sqrt{\lambda}}+\frac{2}{\pi}\frac{1}{\sqrt{\lambda}}\arctan\frac{1}{\sqrt{\lambda}},\ \lambda\in C\setminus R.$$

Let $\varphi_\lambda,\psi_\lambda\in L^2_r((-1,1))$ be the fundamental solutions of $-y''(x)=\lambda\,\mathrm{sgn}(x-\mathrm{sgn}\,x)y(x)$, $-1<x<1$, satisfying the initial conditions $\varphi_\lambda(-1)=1$, $\varphi'_\lambda(-1)=0$ and $\psi_\lambda(-1)=0$, $\psi'_\lambda(-1)=1$, then

$$\varphi_\lambda(x)=\begin{cases}\dfrac{e^{\sqrt{-\lambda}(x+1)}+e^{-\sqrt{-\lambda}(x+1)}}{2}, & -1\le x\le 0,\\[2mm] \dfrac{e^{\sqrt{\lambda}(x+1)}+e^{-\sqrt{\lambda}(x+1)}}{2}, & 0\le x\le 1\end{cases},$$

$$\psi_\lambda(x)=\begin{cases}\dfrac{e^{\sqrt{-\lambda}(x+1)}-e^{-\sqrt{-\lambda}(x+1)}}{2}, & -1\le x\le 0,\\[2mm] \dfrac{e^{\sqrt{\lambda}(x+1)}-e^{-\sqrt{\lambda}(x+1)}}{2}, & 0\le x\le 1\end{cases}.$$

Theorem 12. Let A be the operator of the form (27). Then

(i) A has a real spectrum, $\sigma(A)\subset R$;

(ii) A is similar to a self-adjoint operator.

Proof (i) By Lemma 1, differential expression (27) is in the limit point case at both $+\infty$ and $-\infty$. Hence the operator A is self-adjoint in Krein space $L_r^2(\mathrm{R})$. Evidently, the operator L is nonnegative. It follows from Theorem 3 that the spectrum of A is real, $\sigma(A)\subset \mathrm{R}$.

To prove (ii) we use Theorem 11. It suffices to prove that if $\sup_{\lambda\in \mathrm{C}_+}\|\theta_A(\lambda)\|<\infty$, then A is similar to a self-adjoint operator.

Let $\lambda=i\varepsilon$, $\varepsilon>0$. If $\varepsilon<+\infty$, i.e., ε is bounded. After the appropriate calculations, we can obtain the relation (26).

If $\varepsilon\to+\infty$, i.e., ε is unbounded. Simple calculation show that

$$M_+(i\varepsilon)=O(\varepsilon^{-\frac{1}{2}}),\ M_-(i\varepsilon)=O(\varepsilon^{-\frac{1}{2}}),\ \Delta=O(e^{\sqrt{2\varepsilon}}),$$

$$\frac{k_1}{\Delta}=O(1),\ \frac{k_2}{\Delta}=O(1).$$

So $\lim_{\varepsilon\to+\infty}M_\pm(i\varepsilon)=0$, $\lim_{\varepsilon\to+\infty}\frac{1}{\Delta}=0$. The relation (26) holds.

By Theorem 11, we get that A is similar to a self-adjoint operator.

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