Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 47 (2012) 9054-9057

ISSN: 2229-712X

# Some results on *k*- even sequential harmonious labeling of graphs

B.Gayathri[1] and D.Muthuramakrishnan[2]

[1]Department of Mathematics, Periyar E.V.R. College, Tiruchirappalli – 620 023, India.

[2]Department of Mathematics, National College, Tiruchirappalli – 620 001, India.

**ARTICLE INFO**

**Article history:**

Received: 26 April 2012;

Received in revised form: 15 June 2012;

Accepted: 20 June 2012;

**Keywords**

Label,

Graphs,

Harmonious.

**ABSTRACT**

Graham and Sloane [7] introduced the harmonious graphs and Singh & Varkey [11] introduced the odd sequential graphs. Gayathri and Hemalatha ( [2], [1]) introduced even sequential harmonious labeling of graphs. In [3], we extend this notion to k-even sequential harmonious labeling of graphs and further studied in [4-5]. Also we have introduced k-odd sequential harmonious labeling of graphs in [6]. Here, we investigate some results on k-even sequential harmonious labeling of graphs.



## Introduction

All graphs in this paper are finite, simple and undirected. The symbols V (G) and E(G) denote the vertex set and the edge set of a graph G.

The cardinality of the vertex set is called the order of G. The cardinality of the edge set is called the size of G. A graph with p vertices and q edges is called a (p, q) graph.

In [2], Gayathri and Hemalatha say that a labeling is an even sequential harmonious labeling if there exists an injection $f$ from the vertex set V to {0,1,2,…,2q} such that the induced mapping $f^+$ from the edge set E to $\{2,4,6,\dots,2q\}$ defined by

$$f^+(uv)=\begin{cases} f(u)+f(v), \text{if } f(u)+f(v) \text{ is even} \\ f(u)+f(v)+1, \text{if } f(u)+f(v) \text{ is odd}\end{cases}$$ are distinct.

A graph G is said to be an even sequential harmonious graph if it admits an even sequential harmonious labeling.

Here, we have introduced k-even sequential harmonious labeling by extending the above definition for any integer $k \geq 1$. We say that a labeling is an even sequential harmonious labeling if there exists an injection $f$ from the vertex set V to{k-1, k, k+1,..,k+2q-1} such that the induced mapping $f^+$ from the edge set E to $\{2k,2k+2,2k+4,\dots,2k+2q-2\}$defined by

$$f^+(uv)=\begin{cases} f(u)+f(v), \text{if } f(u)+f(v) \text{ is even} \\ f(u)+f(v)+1, \text{if } f(u)+f(v) \text{ is odd}\end{cases}$$ are distinct.

A graph G is said to be an k-even sequential harmonious graph if it admits an k-even sequential harmonious labeling. In this paper, we investigate some results on k-even sequential harmonious labeling of graphs.

Throughout this paper, *k* denote any positive integer ≥ 1. For brevity, we use *k*-ESHL for *k*-even sequential harmonious labeling.

## Main Results

**Definition:**

By a graph $P_n^2$ we mean the graph obtained from $P_n$ by joining each pair of vertices at distance 2 in $P_n$.

**Theorem:**

The graph $P_n^2$ ,$(n \geq 3)$ is a k-even sequential harmonious graph for any k.

**Proof:**

Let the vertices of $P_n^2$ be {vi :1≤ i ≤ n } and the edges of $P_n^2$ be $\{v_iv_{i+1}: 1\leq i \leq n\}$ $U\{ v_iv_{i+2}: 1\leq i \leq n-2 \}$ which are denoted as in Fig. 2.1.

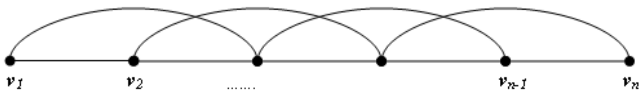


**Fig. 2. 1: $P_n^2$ with ordinary labeling**

We first, label the vertices of $P_n^2$ as follows,

Define $f: V(P_n^2) \rightarrow$ {k-1, k, k+1,..,k+2q-1} by

$f(v_i) = 2(i-1)+k-1 \quad 1\leq i \leq n$

Then the induced edge labels are

$f^+(v_iv_{i+1}) = 4i-4+2k \quad 1\leq i \leq n$

$f^+(v_iv_{i+2}) = 4i+2k-2 \quad 1\leq i \leq n-2$

Clearly, the edge labels are even and distinct, $f^+(E)=\{2k,2k+2,2k+4,.,2k+2q-2\}$.

Hence, the graph $P_n^2$ ,$(n \geq 3)$ is a k-even sequential harmonious graph for any k.

2-ESHL of $P_6^2$ is shown in Fig. 2.2.

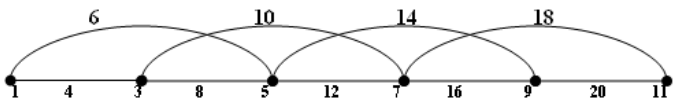


**Fig. 2. 2: 2-ESHL of $P_6^2$ .**

**Theorem:**

The graph $P_n * K_2^C$ $(n \geq 2)$ is a k-even sequential harmonious graph for any k where * denote the wreath product.

Proof:

Let the vertices of $P_n * K_2^C$ be $\{ u_i , v_i : 1\leq i \leq n \}$ and the edges of $P_n * K_2^C$ be

**Tele:**

**E-mail addresses: maduraigayathri@gmail.com, muthunct@gmail.com**

$\{u_iu_{i+1}: 1\le i \le n\text{-}1\} \cup \{ v_iv_{i+1} : 1\le i \le n\text{-}1 \} \cup \{u_iv_{i+1}: 1\le i \le n\text{-}1\} \cup \{ u_{i+1}v_i : 1\le i \le n\text{-}1\}$ which are denoted as in Fig. 2.3.

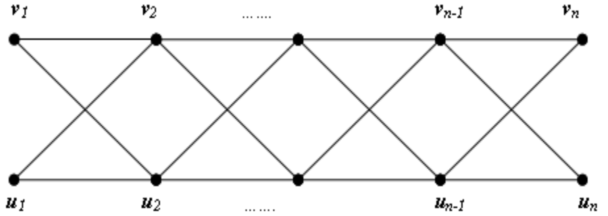


**Fig. 2. 3: $P_n* K_2^C$ with ordinary labeling**

We first, label the vertices of $P_n* K_2^C$ as follows:

Define $f : V(P_n* K_2^C) \rightarrow \{k\text{-}1, k, k+1,..,k+2q\text{-}1\}$ by

$1\le i \le n$

$$f(u_i)=\begin{cases}4(i-1)+k-1 \text{ when } i \text{ is odd}\\ 4i-8+kl \quad \text{ when } i \text{ is even}\end{cases}$$

$$f(v_i)=\begin{cases}4i-2+k \quad \text{when } i \text{ is odd.}\\ 4i-4+k \quad \text{when } i \text{ is even.}\end{cases}$$

Then the induced edge labels are:

$f^+ (u_iu_{i+1}) = 8i\text{-}8+2k \quad 1\le i \le n\text{-}1$

$f^+ (v_iv_{i+1}) = 8i+2k\text{-}2 \quad 1\le i \le n\text{-}1$

$$f(u_iv_{i+1})=\begin{cases}8i-4+2k \quad ;\text{when } i \text{ is odd}\\ 8i-6+2k \quad ;\text{when } i \text{ is even}\end{cases}$$

$$f(u_{i+1}v_i)=\begin{cases}8i-6+2k \quad ;\text{when } i \text{ is odd}\\ 8i-4+2k \quad ;\text{when } i \text{ is even}\end{cases}$$

Clearly, the edge labels are even and distinct, $f^+(E)=\{2k,2k+2,2k+4,.,2k+2q-2\}$.

Hence, the graph $P_n* K_2^C$ $(n \ge 2)$ is a k-even sequential harmonious graph for any k.

4-ESHL of $P_5* K_2^C$ is shown in Fig. 2. 4.

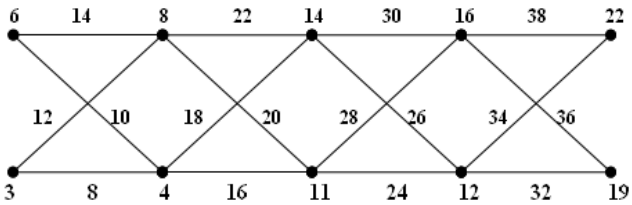


**Fig. 2. 4: 4-ESHL of $P_5* K_2^C$**

**Definition:**

The graph sparklers $P_m@ K_{1,n}$ , $(m, n \ge 2)$ obtained by joining an end vertex of a path to the center of a star.

**Theorem:**

The graph sparklers $P_m@ K_{1,n}$ , $(m, n \ge 2)$ is a k-even sequential harmonious graph for any k.

**Proof:**

Let the vertices of sparklers $P_m@ K_{1,n}$ , be $\{ u_i : 1\le i \le m+n \}$ and the edges of sparklers $P_m@ K_{1,n}$ ,be $\{u_iu_{i+1}: 1\le i \le m\text{-}1\} \cup \{ u_mu_{i+1} : m\le i \le m+n\text{-}1\}$ which are denoted as in Fig. 2. 5..

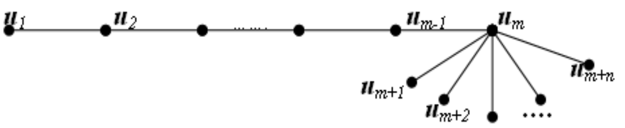


**Fig. 2. 5: $P_m@ K_{1,n}$ with ordinary labeling**

We first, label the vertices as follows:

Define $f : V(P_m@ K_{1,n}) \rightarrow \{k\text{-}1, k, k+1,..,k+2q\text{-}1\}$ by

$f(u_i) = k+ i\text{-}2 \quad 1 \le i \le m$

$f(u_i) = 2i+k\text{-}m\text{-}2 \quad m+1 \le i \le m+n$

Then the induced edge labels are:

$f^+ (u_iu_{i+1}) = 2i +2k\text{-}2 \quad 1\le i \le m\text{-}1$

$f^+ (u_mu_{i+1}) = 2i+2k\text{-}2 \quad m\le i \le m+n\text{-}1.$

Clearly, the edge labels are even and distinct, $f^+(E)=\{2k,2k+2,2k+4,.,2k+2q-2\}$.

Hence, the graph sparklers $P_m@ K_{1,n}$ , $(m, n \ge 2)$ is a $k$-even sequential harmonious graph.

5-ESHL of $P_6 @ K_{1,5}$ is shown in Fig. 2. 6.

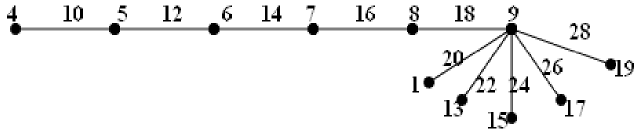


**Fig. 2. 6: 5-ESHL OF $P_6 @ K_{1,5}$**

**Definition:**

A twig $TW(P_n)$ ,$(n \ge 3)$ is a graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.

**Theorem:**

The graph twig $TW(P_n)$ ,$(n \ge 3)$ is a $k$-even sequential harmonious graph for any k.

**Proof:**

Let the vertices of TW($P_n$) be $\{ u_i , v_j ,w_j : 1\le i \le n \ \& \ 1\le j \le n\text{-}2 \}$ and the edges of TW($P_n$) be $\{ (u_iu_{i+1}) , (v_iu_{i+1}) , (w_iu_{i+1}): 1\le i \le n\text{-}2\}$ which are denoted as in Fig. 2. 7.

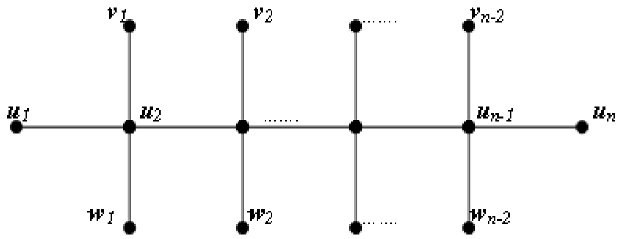


**Fig. 2. 7: TW($P_n$) with ordinary labeling**

We first, label the vertices as follows:

Define $f : V(TW(P_n)) \rightarrow \{k\text{-}1, k, k+1,..,k+2q\text{-}1\}$ by

$f(u_i) = k+ i\text{-}2 \quad 1 \le i \le n$

$f(v_j) = 2n \text{-}2+k + 3(j\text{-}1) \quad 1 \le j \le n\text{-}2$

$f(w_j) = 2n +k + 3(j\text{-}1) \quad 1 \le j \le n\text{-}2$

Then the induced edge labels are

$f^+ (u_iu_{i+1}) = 2i+2k\text{-}2 \quad 1\le i \le n\text{-}1$

$f^+ (v_iu_{i+1}) = 2n + 4(i\text{-}1)+2k\text{-}2 \quad 1\le i \le n\text{-}2$

$f^+ (w_iu_{i+1}) = 2(n+1) +4(i\text{-}1)+2k\text{-}2 \quad 1\le i \le n\text{-}2$

Clearly, the edge labels are even and distinct, $f^+(E)=\{2k,2k+2,2k+4,.,2k+2q-2\}$

Hence, the graph twig $TW(P_n)$ ,$(n \ge 3)$ is a $k$-even sequential harmonious graph for any $k$.

3-ESHL of $TW(P_5)$ is shown in Fig. 2.8.

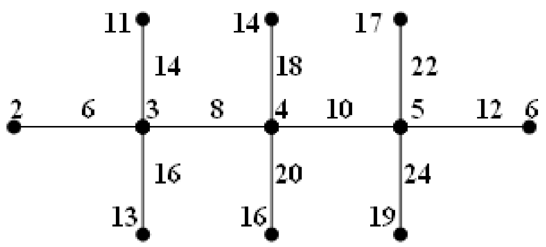


**Fig. 2. 8: 3-ESHL of $TW(P_5)$**

**Theorem:**

The graph $C_3$ e $nK_1$, $(n \geq 2)$ is a $k$-even sequential harmonious graph for any $k$.

**Proof:**

Let the vertices of $C_3$ e $nK_1$ be $\{u_i , v_{ij} ; 1\leq i \leq 3 \ \& \ 1\leq j \leq n\}$ and $\{ e_i ; 1\leq i \leq 3 \ \& \ 1\leq j \leq n \} \cup \{(u_iv_{ij}) ; 1\leq i \leq 3 \ \& \ 1\leq j \leq n\}$ be the edges of $C_3$ e $nK_1$ which are denoted as in Fig. 2.9.

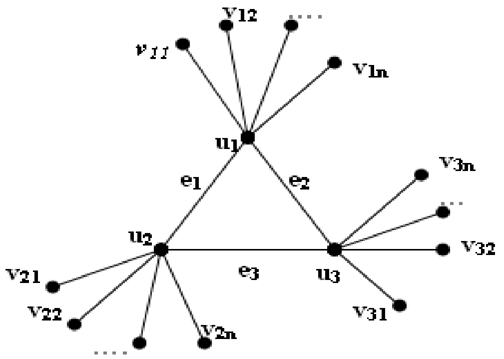


**Fig. 2. 9:** $C_3$ e $nK_1$ **with ordinary labeling**

We first, label the vertices of $C_3$ e $nK_1$ as follows:

Define $f : V(C_3$ e $nK_1) \rightarrow \{k-1, k, k+1,..,k+2q-1\}$by

$f(u_1) = k-1$ ; $f(u_2) = k+1$ ; $f(u_3) = k+2$ ;

$f(v_{ij}) = k+2(j-1)+(2n-1)(i-1)+6 \quad 1\leq i \leq 3 \ \& \ 1\leq j \leq n$

Then the induced edge labels are:

$f^+(e_i) = 2i+2k-2 \quad 1\leq i \leq 3$

$f^+(u_iv_{ij}) = 2k+2(j-1)+2n(i-1)+6 \quad 1\leq i \leq 3 \ \& \ 1\leq j \leq n$

Clearly, the edge labels are even and distinct, $f^+(E) = \{2k, 2k+2, 2k+4,.,2k+2q-2\}$.

Hence, the graph $C_3$ e $nK_1$, $(n \geq 2)$ is a k-even sequential harmonious graph.

4-ESHL of $C_3$ e $4K_1$ is shown in Fig. 2. 10.

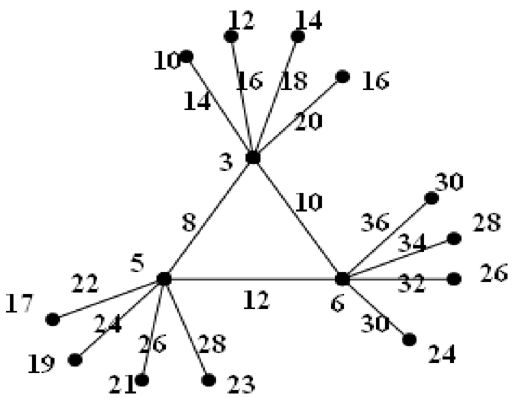


**Fig. 2. 10: 4-ESHL of** $C_3$ e $4K_1$**.**

**Theorem:**

The graph $T_{n,m,t}$ , $(n,m,t \geq 2)$ is a k-even sequential harmonious graph.

**Proof:**

Let the vertices of $T_{n,m,t}$ be $\{u_i , v_j , w_k ; 1\leq i \leq t , 1\leq j \leq n \ \& \ 1\leq l \leq m\}$and $\{(u_iu_{i+1}) , (u_1v_j) , (u_tw_k) ; 1\leq i \leq t-1 , 1\leq j \leq n \ \& \ 1\leq l \leq m\}$be the edges of $T_{n,m,t}$ which are denoted as in Fig. 2. 11.

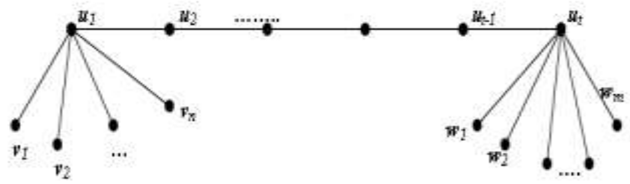


**Fig. 2. 11:** $T_{n,m,t}$**. with ordinary labeling**

We first, label the vertices of $T_{n,m,t}$ as follows:

Define $f : V(T_{n,m,t}) \rightarrow \{k-1, k, k+1,..,k+2q-1\}$by

$f(u_i) = k+ i-2 \quad 1\leq i \leq t$

$f(v_j) = 2t+2j+k-4 \quad 1\leq j \leq n$

For $1\leq l \leq m$,

$$f(w_l) = \begin{cases} 2n+t+2(l-1)+k-1 & \text{when } t \text{ is even} \\ 2n+t+2l+k-3 & \text{when } t \text{ is odd} \end{cases}$$

Then the induced edge labels are:

$f^+(u_iu_{i+1}) = 2i+2k-2 \quad 1\leq i \leq t-1$

$f^+(u_iv_j) = 2(t+j-1)+2k-2 \quad 1\leq j \leq n$

$f^+(u_tw_l) = 2(t+n+l-1)+2k-2 \quad 1\leq l \leq m$

Clearly, the edge labels are even and distinct, $f^+(E) = \{2k, 2k+2, 2k+4,.,2k+2q-2\}$.

Hence, the graph $T_{n,m,t}$ , $(n,m,t \geq 2)$ is a $k$-even sequential harmonious graph for any $k$.

6-ESHL of $T_{3,5,6}$ is shown in Fig. 2. 12.

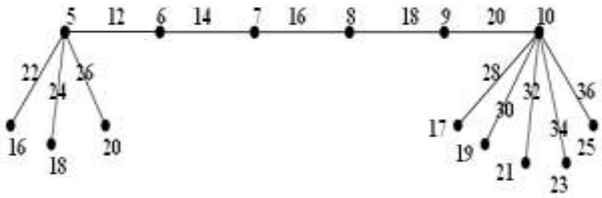


**Fig. 2. 12: 6-ESHL of** $T_{3,5,6}$

**Theorem:**

The graph $CH_n$ , $(n \geq 3)$ is a $k$-even sequential harmonious graph for any $k$ where $CH_n$ is a closed helm without central vertex.

**Proof:**

Let the vertices of $CH_n$ be $\{u_{ij} ; 1\leq i \leq n \ \& \ 1\leq j \leq n\}$and the edges of $CH_n$ be

$\{(u_{ij}u_{i+1j}) ; 1\leq i \leq n \ \& \ 1\leq j \leq n\} \cup \{ e_j^i = (u_{ij} , u_{ij+1}) ; 1\leq i \leq n \ \& \ 1\leq j \leq n-1\} \cup$

$\{ e_n^i = (u_{in} , u_{i1}) ; 1\leq i \leq n \}$ which are denoted as in Fig. 2. 13.

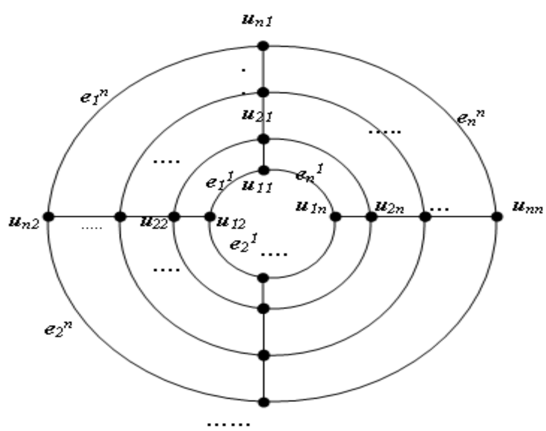


**Fig. 2. 13:** $CH_n$ **with ordinary labeling**

First we label the vertices of $CH_n$ as follows:

Define $f : V(CH_n) \rightarrow \{k-1, k, k+1,..,k+2q-1\}$by

Case (i) : When $i$ is odd.

The vertices labels are as follows,

$f(u_{ij}) = 2(j\text{-}1)+2n(i\text{-}1)+k\text{-}1 \quad 1\le j \le (n+1)/2$ , if $n$ is odd
& $1\le j \le (n+2)/2$ , if $n$ is even.
$f(u_{i\,(n+3)/2}) = f(u_{i\,(n+1)/2}) +k \quad n$ is odd.
$f(u_{i\,(n+4)/2}) = f(u_{i\,(n+2)/2}) +k\text{-}2 \quad n$ is even.
$f(u_{ij}) = f(u_{i\,j\text{-}1})+k\text{-}3$ ; $(n+3)/2 < j \le n$ , *if n is odd*
& $(n+4)/2 < j \le n$ , *if n is even.*

Then the induced edge labels are

$$f^+(e^i_j)=\begin{cases} 4j-4+4n(i-1)+2k & 1\le j\le (n+1)/2, \text{ if } n \text{ is odd} \\ & 1\le j\le n/2, \text{ if } n \text{ is even} \\ f^+(e^i_{(n+1)/2})+2k-4 & j=(n+3)/2, \text{if } n \text{ is odd} \\ f^+(e^i_{n/2})+2k & j=(n+2)/2, \text{if } n \text{ is even} \\ f^+(e^i_{j-1})+2k-6 & (n+3)/2< j\le n, \text{if } n \text{ is odd} \\ & (n+2)/2< j< n, \text{if } n \text{ is even} \end{cases}$$

Case (ii) : When $i$ is even .
The vertex labels are as follows:
$f(u_{ij}) = 2(n+1)+2(j\text{-}1)+2n(i\text{-}2)+k\text{-}1 \quad 1\le j \le (n\text{-}1)/2$ , if $n$ is odd.
$1\le j \le n/2$ , if $n$ is even.
$f(u_{i\,(n+1)/2}) = f(u_{i\,(n-1)/2}) + k \quad n$ is odd.
$f(u_{i\,(n+2)/2}) = f(u_{in/2})+k-2 \quad n$ is even.
$f(u_{ij}) = f(u_{i\,j\text{-}1})+k\text{-}3 \quad (n+3)/2 < j \le n\text{-}1$, if $n$ is odd.
$(n+2)/2 < j \le n\text{-}1$ , if $n$ is even.
$f(u_{in}) = f(u_{i\,n\text{-}1})+k\text{-}4$ .

Then the induced edge labels are:

$$f^+(e^i_j)=\begin{cases} 4n+4j+3+4n(i-2)+2k & 1\le j\le (n-1)/2, \text{if } n \text{ is odd} \\ & 1\le j\le (n-1)/2, \text{if } n \text{ is even} \\ f^+(e^i_{(n-1)/2})+2k-4\,; & j=(n+1)/2, \text{if } n \text{ is odd} \\ f^+(e^i_{n/2})+2k\,; & j=(n+2)/2, \text{if } n \text{ is even} \\ f^+(e^i_{j-1})+2k-6 & (n+1)/2< j\le n-1, \text{if } n \text{ is odd} \\ & (n+2)/2< j\le n-1, \text{if } n \text{ is even} \\ f^+(e^i_n)+2k-4 & \end{cases}$$

Clearly, the edge labels are even and distinct,
$f^+(E)=\{2k, 2k+2, 2k+4,.,, 2k+2q-2\}$ Hence, the graph $CH_n$ , $(n\ge 3)$ is a k- even sequential harmonious graph.
2-ESHL of $CH_4$ is shown in Fig. 2. 14.

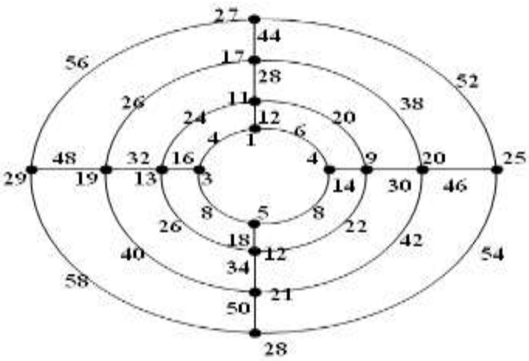


**Fig. 2. 14: 2-ESHL of $CH_4$**


**References**

[1]. J. A. Gallian , A dynamic survey of graph labeling - The Electronic Journal of Combinatorics 15 (2008), #DS6.

[2] B. Gayathri and V. Hemalatha, Even sequential harmonious labeling of some Graphs - Presented in National Conference held at Govt. College for Women, Pudukkottai-28 & 29 th March-2008.

[3] B. Gayathri and D. Muthuramakrishnan, *k*–even sequential harmonious labeling of some tree related graphs - communicated.

[4] B. Gayathri and D. Muthuramakrishnan, *k*–even sequential harmonious labeling of some cycle related graphs - communicated.

[5] B. Gayathri and D. Muthuramakrishnan *k*–even sequential harmonious labeling of some graphs - communicated.

[6] B. Gayathri and D. Muthuramakrishnan *k*–odd sequential harmonious labeling of some graphs - communicated.

[7]. R.L.Graham and N.J.A.Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Math., 1 (1980) 382-404.

[8]. M. Seond, A. E. I. Abdel Maqsoud, and J. sheehan, Harmonious graphs, Util. Math., 47 (1995)225- 233.

[9]. M. A. Seoud and M. Z. Yossef, Families of harmonious and nonharmonious graphs, J.Egyptian Math.Soc., 7(1999)117-125.

[10]. S. C. Shee, On harmonious and related graphs, Ars Combin., 23(1987) A,237-247.

[11]. G. S. Singh and T. K. M.Varkey, On odd sequential and bisequential graphs, preprint.

[12]. R. G. Sun, harmonious and sequential labelings of the book graphs $B_m$, Gaoxiao yingyong shuxue xuebao ser. A, 9(1994)335-337.

[13]. J. Yuan and W. Zhu, Some results on harmonious labelings of graphs, J. Zhengzhou Univ. Nat. Sci. Ed., 30 (1998)7-12.

[14]. S. Zhou and J. Yuan , On construction of sequential graphs, Mathematica Applicata, 6 (supplementry)(1993) 104-108.