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A 'useful' information measure and its mean codewords length

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ABSTRACT

A parametric mean length is defined as the quantity

$$L_{\alpha(u)} = \frac{1}{\alpha-1}\left[1-\left(\sum P_i\left(\frac{u_i}{\sum u_i \; p_i}\right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right)^{\alpha}\right],$$

Where $\alpha > 0\ (\neq 1),\ u_i > 0,\ \sum P_I = 1$. This being the useful mean length of codewords weighted by utilities, u_i. Lower and upper bounds for $L_{\alpha(u)}$ are derived in terms of 'useful' information measure.

Introduction

Consider the following model for a random experiment S,

$$S_N = [E;\ P;\ U]$$

where $E = (E_1, E_2, ..., E_N)$ is a finite system of events happening with respective probabilities $P = (P_1, P_2, ..., P_n)$, $p_i \geq 0$, and $\sum p_i = 1$ and credited with utilities $U = (u_1, u_2, ..., u_N)$, $u_i > 0,\ i = 1, 2, ..., N$. Denote the model by S_N, where

$$S_N = \begin{bmatrix} E_1, E_2, ...E_N \\ P_1, P_2, ...P_N \\ u_1, u_2, ...u_N \end{bmatrix} \qquad ....(1.1)$$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [4] proposed a measure of information called 'useful information' for this scheme, given by

$$H(P;U) = -\sum u_i\ p_i \log p_i\ , \qquad ....(1.2)$$

where $H(P\ ;\ U)$ reduces to Shannon's [13] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each i. Throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base $D(D > 1)$.

Guiasu and Picard [6] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1, w_2, ..., w_N$ having lengths $n_1, n_2, ..., n_N$ and satisfying Kraft's inequality [5] .

$$\sum_{i=1}^{N} D^{-n_i} \leq 1. \qquad ....(1.3)$$

Where D is the size of the code alphabet. The useful mean length L_u of code was defined as

$$L_u = \frac{\sum u_i\, n_i\, p_i}{\sum u_i p_i}, \qquad ....(1.4)$$

and the authors obtained bounds for it in terms of $H(P;\ U)$. Longo [10], Gurdial and Pessoa [7], Khan and Autar [2] , Singh and Rajeev [12], Khan and Bhat [9] have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under condition (1.3) of unique decipherability.

In this paper, we study some coding theorems by considering a new function depending on the parameters α and a utility function. Our motivation for studying this new function is that it generalizes some entropy functions already existing in the literature Havrda-Charvat [8] and Tsallis entropy [15] which is used in physics.

Coding Theorems

In this section, we define 'useful' information measure as :

$$H_\alpha\ (P;\ U) = \frac{1}{(\alpha-1)}\left[1-\left\{\sum \frac{p_i^\alpha u_i}{\sum p_i u_i}\right\}\right], \qquad ....(2.1)$$

where $\alpha > 0\ (\neq 1), u_i > 0,\ p_i \geq 0,\ i = 1, 2, ..., N$ and $\sum p_i = 1$.

(i). When $u_i = 1$, (2.1) reduces to Havrda-Charvat [8] and C.Tsallis

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entropy [15]. i.e. $_{\alpha}H(P)=\frac{1}{\alpha-1}\left[1-\sum p_i^{\alpha}\right]$ …….(2.2)

(ii). When $\alpha \to 1$, (2.1) reduces to a measure of 'useful' information for the incomplete distribution due to Bhaker and Hooda [3].

(iii). When $u_i = 1$ for each i, i.e. when the utility aspect is ignored, and $\alpha \to 1$, the measure (2.1) reduces to Shannon's [13] entropy .

i.e. $H(P) = -\sum p_i \log p_i$. ………(2.3)

Further consider

Definition: The 'useful' mean length $L_{\alpha(u)}$ with respect to 'useful' information measure is defined as:

$$L_{\alpha(u)} = \frac{1}{\alpha-1}\left[1-\left(\sum P_i\left(\frac{u_i}{\sum u_i\, p_i}\right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right)^{\alpha}\right] \quad , \text{….(2.4)}$$

where $\alpha > 0\ (\neq 1), u_i > 0, \ \sum p_i = 1$.

(i). When $u_i = 1$ for each i, and $\alpha \to 1$, $_{\alpha}L_u$ becomes the optimal code length defined by Shannon [13].

(ii). When $u_i = 1$, (2.4) reduced to a new mean codeword length

$$\text{i.e.}\quad {}_{\alpha}L = \frac{1}{\alpha-1}\left[1-\left(\sum_{i=1}^{n} p_i\, D^{-n_i\frac{(\alpha-1)}{\alpha}}\right)^{\alpha}\right], \text{….(2.5)}$$

We establish a result, that in a sense, provides a characterization of $H_{\alpha}(P;U)$ under the condition of unique decipherability.

Theorem 2.1. For all integers $D > 1$

$$L_{\alpha(u)} \geq H_{\alpha}(P;U) \quad \text{………..(2.6)}$$

under the condition (1.3). Equality holds if and only if

$$n_i = -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right). \quad \text{………..(2.7)}$$

Proof. We use Holder's [14] inequality

$$\sum x_i y_i \geq \left(\sum x_i^p\right)^{\frac{1}{p}}\left(\sum y_i^q\right)^{\frac{1}{q}} \quad \text{………..(2.8)}$$

for all $x_i \geq 0,\ y_i \geq 0, i = 1, 2, ..., N$ when $P < 1\ (\neq 1)$ and $p^{-1}+q^{-1}=1$, with equality if and only if there exists a positive number c such that

$$x_i^p = c y_i^q . \quad \text{……..(2.9)}$$

Setting

$$x_i = p_i^{\frac{\alpha}{\alpha-1}}\left(\frac{u_i}{\sum u_i\, p_i}\right)^{\frac{1}{\alpha-1}} D^{-n_i},$$

$$y_i = p_i^{\frac{\alpha}{1-\alpha}}\left(\frac{u_i}{\sum u_i\, p_i}\right)^{\frac{1}{1-\alpha}},$$

$p = 1-\frac{1}{\alpha}$ and $q = 1-\alpha$ in (2.8) and using (1.3) we obtain the result (2.6) after simplification for $\frac{1}{\alpha-1} > 0$ as $\alpha > 1$.

Theorem 2.2. For every code with lengths $\{n_i\},\ i = 1, 2, ..., N,\ _{\alpha}L_u$ can be made to satisfy,

$$L_{\alpha(u)} < H_{\alpha}(P;U)D^{1-\alpha} + \frac{1}{\alpha-1}\left[1-D^{1-\alpha}\right]. \quad \text{………..(2.10)}$$

Proof. Let n_i be the positive integer satisfying, the inequality

$$-\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right) \leq n_i < -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right)+1 \text{………..(2.11)}$$

Consider the intervals

$$\delta_i = \left[-\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right), -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right)+1\right] \quad \text{…..(2.12)}$$

of length 1. In every δ_i, there lies exactly one positive number n_i such that

$$0 < -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right) \leq n_i < -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right) +1 \text{…..(2.13)}$$

It can be shown that the sequence $\{n_i\},\ i = 1, 2, ..., N$ thus defined, satisfies (1.3). From (2.13) we have

$$n_i < -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right)+1$$

$$\Rightarrow D^{-n_i} > \left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right) D^{-1}$$

$$\Rightarrow D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} > \left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right)^{\frac{\alpha-1}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \quad \text{……..(2.14)}$$

Multiplying both sides of (2.14) by $p_i\left(\frac{u_i}{\sum u_i\, p_i}\right)^{\frac{1}{\alpha}}$, summing over $i = 1, 2, ..., N$ and simplifying for $\frac{1}{\alpha-1} > 0$ as $\alpha > 1$ gives (2.10).

Theorem 2.3. For every code with lengths $\{n_i\}$, $i = 1, 2, ..., N$, of Theorem 2.1, $_{\alpha}L_u$ can be made to satisfy.

$$_{\alpha}L_u \geq {}_{\alpha}H\ (U;\ P) > {}_{\alpha}H\ (U;\ P)D + \frac{1}{\alpha-1}\ (1-D) \text{……(2.15)}$$

Proof. Suppose

$$\overline{n}_i = -\log_D\left(\frac{u_i P_i^{\alpha}}{\sum u_i p_i^{\alpha}}\right). \quad \text{………..(2.16)}$$

Clearly $\overline{n}_i$ and $\overline{n}_i$ + 1 satisfy 'equality' in Holder's inequality (2.8). Moreover, $\overline{n}_i$ satisfies Kraft's inequality (1.3).

Suppose n_i is the unique integer between $\overline{n}_i$ and $\overline{n}_i$ + 1, then obviously, n_i satisfies (1.3).

Since $\alpha > 0\ (\neq 1)$, we have

$$\left(\sum P_i \left(\frac{u_i}{\sum u_i \ p_i} \right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} \right)^{\alpha}$$

$$\leq \left(\sum p_i \left(\frac{u_i}{\sum u_i p_i} \right)^{\frac{1}{\alpha}} D^{-\bar{n}_i \frac{(\alpha-1)}{\alpha}} \right)^{\alpha}$$

$$< D \left(\sum p_i \left(\frac{u_i}{\sum u_i p_i} \right)^{\frac{1}{\alpha}} D^{-\bar{n}_i \frac{(\alpha-1)}{\alpha}} \right)^{\alpha} \quad \text{……..(2.17)}$$

Since, $\left(\sum p_i \left(\frac{u_i}{\sum u_i p_i} \right)^{\frac{1}{\alpha}} D^{-\bar{n}_i \frac{(\alpha-1)}{\alpha}} \right)^{\alpha} = \sum \frac{u_i p_i^{\alpha}}{\sum u_i p_i}$.

Hence, (2.17) becomes

$$\left(\sum P_i \left(\frac{u_i}{\sum u_i \quad p_i} \right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} \right)^{\alpha} \leq \sum \frac{u_i p_i^{\alpha}}{\sum u_i p_i} < D \left(\sum \frac{u_i p_i^{\alpha}}{\sum u_i p_i} \right).$$

which gives (2.15).

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