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Discrete Mathematics

Elixir Dis. Math. 48 (2012) 9482-9485

ISSN: 2229-712X

Harmonic mean labeling in the context of duplication of graph elements

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ARTICLE INFO

Article history:

Received: 18 May 2012;

Received in revised form: 28 June 2012;

Accepted: 13 July 2012;

Keywords

Mean Labegling,

Harmonic Mean Labeling,

Duplication.

ABSTRACT

The harmonic mean labeling is a variation of arithmetic mean labeling. We investigate harmonic mean labeling for various graphs resulted from the duplication of graph elements.

Introduction

We begin with simple, finite, connected and undirected graph $G=(V(G),E(G))$. The members of $V(G)$ and $E(G)$ are commonly termed as graph elements. For all standard terminology and notation we follow Chartrand and Lesniak [1]. In order to maintain compactness we provide a brief summary of definitions and existing results.

Definition 1.1 : A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called *a vertex labeling (or an edge labeling)*.

Most of the graph labeling problems have following three common characteristics:

1. a set of numbers for assignment of vertex labels;
2. a rule that assigns a label to each edge;
3. some condition(s) these labels must satisfy.

According to Beineke and Hegde [2] labeling of discrete structure serves as a frontier between graph theory and theory of numbers. A dynamic survey of graph labeling is frequently carried out by Gallian [3].

Definition 1.2 : A function f is called a *mean labeling of graph* G if $f:V(G)\rightarrow\{0,1,2,\ldots,|E(G)|\}$ is injective and the induced edge function $f^*:E(G)\rightarrow\{1,2,\ldots,|E(G)|\}$ defined as follows is bijective.

$$f^*(e=uv)=\begin{cases}\dfrac{f(u)+f(v)}{2}; & \text{if } f(u)+f(v) \text{ is even}\\[2mm] \dfrac{f(u)+f(v)+1}{2}; & \text{if } f(u)+f(v) \text{ is odd}\end{cases}$$

The graph which admits mean labeling is called a *mean graph*.

The mean labeling was introduced by Somasundaram and Ponraj [4] and they proved the graphs P_n, C_n, $P_n\times P_m$, $P_n\times C_m$ etc. admit mean labeling. Mean labeling in the context of some graph operations is discussed by Vaidya and Lekha [5] while the same authors in [6] have investigated some new families of mean graphs.

When arithmetic mean in the definition of mean labeling is replaced by harmonic mean then it is called harmonic mean labeling.

Definition 1.3 : A function f is called a *harmonic mean labeling of graph* G if $f:V(G)\rightarrow\{1,2,\ldots,|E(G)|+1\}$ is injective and the induced edge function $f^*:E(G)\rightarrow\{1,2,\ldots,|E(G)|\}$ defined as $f^*(e=uv)=\left\lceil\dfrac{2f(u)f(v)}{f(u)+f(v)}\right\rceil$ or $\left\lfloor\dfrac{2f(u)f(v)}{f(u)+f(v)}\right\rfloor$ is bijective.

The graph which admits harmonic mean labeling is called a *harmonic mean graph.*

The concept of harmonic mean graph was introduced by Sandhya et al. [7] and they have investigated harmonic mean labeling for some standard graphs. The same authors in [8] proved that polygonal chains, square of a path and dragons admit harmonic mean labeling.

There are three types of problems that can be considered in this area.

1. How the harmonic mean labeling is affected under various graph operations;
2. Construct new families of harmonic mean graphs by investigating suitable labeling;
3. Given a graph theoretic property P, characterise the class of graphs with property P that are harmonic mean graph.

This paper is focused on problems of first type.

Definition 1.5 : Two vertices of a graph which are adjacent are said to be *neighbours*. The set of all neighbours of a vertex v is called the neighbourhood set denoted as $N(v)$.

Definition 1.6 : *Duplication of a vertex* v_k of graph G produces a new graph G' by adding a vertex v'_k with $N(v_k)=N(v'_k)$.

In other words a vertex v'_k is said to be duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k also.

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Definition 1.7 : *Duplication of an edge* $e = uv$ of graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.

Definition 1.8 : *Duplication of a vertex* v_k *by a new edge* $e' = u'v'$ in a graph G produces a new graph G' such that $N(v') = \{v_k, u'\}$ and $N(u') = \{v_k, v'\}$.

Definition 1.9 : Duplication of an edge $e = uv$ by a vertex v' in a graph G produces a new graph G' such that $N(v') = \{u, v\}$.

We immediately observe the followings.

Observation 1.10 : For harmonic mean graph G, the edge label 1 is produced only if one of the incident vertices to the edge has label 1.

Observation 1.11 : For harmonic mean graph G, the edge label $|E(G)|$ is produced only if one of the incident vertices to the edge has label $|E(G)|$.

Observation 1.12 : Since vertex with label 1 will generate edge label as 1 & 2 only. Thus vertex with label 1 must have degree less than 3.

Observation 1.13 : Any k - regular graph ($k > 2$) is not a harmonic mean graph.

Observation 1.14 : Harmonic mean M of n_1 and n_2 is always less than $2n_1$ and $2n_2$. That is $M < 2n_1$ and $M < 2n_2$.

Main Results

Theorem - 2.1 : The graph obtained by duplication of an arbitrary vertex v_k in cycle C_n is a harmonic mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n. Without loss of generality we duplicate the vertex v_1 thus added vertex is v'. Now the resultant graph G will have $n+1$ vertices and $n+2$ edges.

To define $f : V(G) \rightarrow \{1, 2, ..., n+3\}$ we consider following two cases.

Case 1: When $n = 3$.

The graph and its harmonic mean labeling is shown in *Figure 1*.

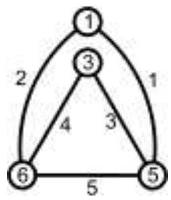

Figure 1

Case 2: When $n \neq 3$.

$f(v') = 1;$

$f(v_1) = 2;$

$f(v_i) = i + 3, \quad 2 \leq i \leq n.$

In view of the above labeling pattern we have distinct edge labels from $\{1, 2, ..., n+2\}$.

Hence from case 1 and case 2 we have the graph obtained by duplication of an arbitrary vertex v_k in cycle C_n is a harmonic mean graph.

Illustration 2.2 : The graph obtained by duplication of a vertex in C_5 and its harmonic mean labeling is shown in *Figure 2*.

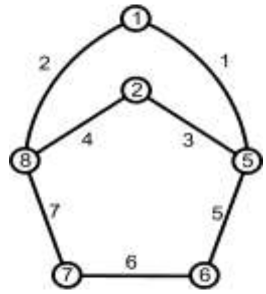

Figure 2

Theorem 2.3 : The graph obtained by duplication of an arbitrary edge e_k in cycle C_n is a harmonic mean graph.

Proof: Let $e_1, e_2, ..., e_n$ be the edges of cycle C_n. Without loss of generality we duplicate the edge e_1 thus added vertices are v_1' and v_2'. Now the resultant graph G will have $n+2$ vertices and $n+3$ edges.

We define $f : V(G) \rightarrow \{1, 2, ..., n+4\}$ as follows.

$f(v_1') = 1;$

$f(v_2') = 6;$

$f(v_1) = 2;$

$f(v_2) = 3;$

$f(v_i) = i + 4, \quad 3 \leq i \leq n.$

In view of the above labeling pattern we have distinct edge labels from $\{1, 2, ..., n+3\}$.

Hence the graph obtained by duplication of an arbitrary edge e_k in cycle C_n is a harmonic mean graph.

Illustation 2.4 : The graph obtained by duplication of an edge in C_6 and its harmonic mean labeling is shown in *Figure 3*.

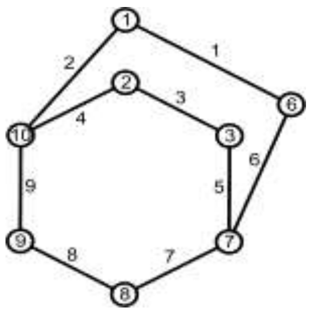

Figure 3

Theorem 2.5 : The graph obtained by duplication of an arbitrary vertex by a new edge in cycle C_n is harmonic mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, ..., e_n$ be the edges of cycle C_n. Without loss of generality we duplicate the vertex v_n by an edge e_{n+1} with end vertices as v_1' and v_2'. The resultant graph G will have $n+2$ vertices and $n+3$ edges.

We define $f : V(G) \rightarrow \{1, 2, ..., n+4\}$ as follows.

$f(v_1') = 1;$

$f(v_2') = 2;$

$f(v_1) = 3;$

$f(v_i) = i + 4, \quad 2 \leq i \leq n.$

In view of the above labeling pattern we have distinct edge labels from $\{1, 2, ..., n+3\}$.

Hence the graph obtained by duplication of an arbitrary vertex by a new edge in cycle C_n is harmonic mean graph.

Illustration 2.6 : The graph obtained by duplication of a vertex by a new edge in cycle C_5 and its harmonic mean labeling is shown in *Figure 4*.

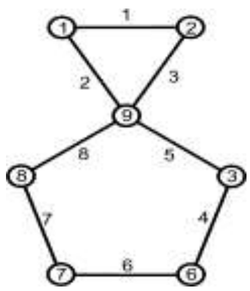

Figure 4

Theorem 2.7 : The graph obtained by duplication of an arbitrary edge by a new vertex in cycle C_n is a harmonic mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, ..., e_n$ be the edges of cycle C_n. Without loss of generality we duplicate the edge $v_n v_1$ by a vertex v'. The resultant graph G will have $n+1$ vertices and $n+2$ edges.

To define $f: V(G) \to \{1, 2, ..., n+3\}$ we consider following two cases.

Case 1: When $n = 3$.

The graph and its harmonic mean labeling is shown in *Figure 5*.

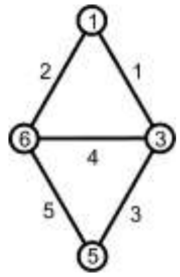

Figure 5

Case 2: When $n \neq 3$.

$f(v') = 1;$

$f(v_1) = 2;$

$f(v_i) = i+3, \quad 2 \leq i \leq n.$

In view of the above labeling pattern we have distinct edge labels from $\{1, 2, ..., n+2\}$.

Hence from case 1 and case 2 we have the graph obtained by duplication of an arbitrary edge by a new vertex in cycle C_n is a harmonic mean graph.

Illustration 2.8 : The graph obtained by duplication of an edge by a new vertex in cycle C_5 and its harmonic mean labeling is shown in *Figure 6*.

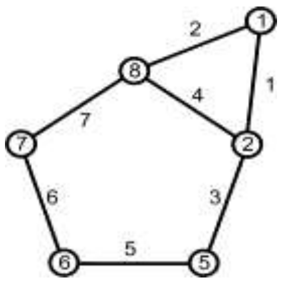

Figure 6

Theorem - 2.9 : The graph obtained by duplication of an arbitrary vertex v_k in path P_n is a harmonic mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of path P_n. We duplicate the vertex v_k thus added vertex is v'.

We consider following two cases.

Case 1: When pendant vertex is duplicated. Then the resultant graph G will have $n+1$ vertices and n edges.

We define $f: V(G) \to \{1, 2, ..., n+1\}$ as follows.

$f(v') = 1;$

$f(v_i) = i+1, \quad 1 \leq i \leq n.$

Case 2: When other then pendant vertex is duplicated. Than the resultant graph G will have $n+1$ vertices and $n+1$ edges.

We define $f: V(G) \to \{1, 2, ..., n+2\}$ as follows.

$f(v') = 1;$

$f(v_n) = 2;$

$f(v_i) = i+3, \quad 1 \leq i \leq n-1.$

In view of the above labeling pattern we have distinct edge labels.

Hence from case 1 and case 2 we have the graph obtained by duplication of an arbitrary vertex v_k in path P_n is a harmonic mean graph.

Illustration 2.10 : The graph obtained by duplication of a pendant vertex in P_5 and its harmonic mean labeling is shown in *Figure 7*. While the graph obtained by duplication of other than pendant vertex in P_5 and its harmonic mean labeling is shown in *Figure 8*.

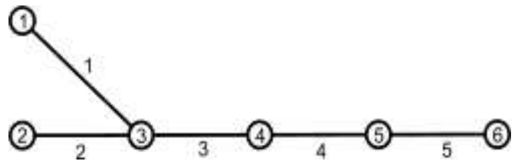

Figure 7

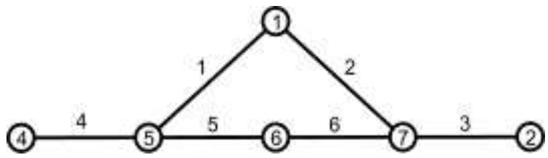

Figure 8

Theorem 2.11 : The graph obtained by duplication of an arbitrary edge e_k in path P_n is a harmonic mean graph.

Proof: Let $e_1, e_2, ..., e_{n-1}$ be the edges of path P_n. We duplicate the edge e_k thus added vertices are v_1' and v_2'. We consider following two cases.

Case 1: When pendant edge is duplicated. Then the resultant graph G will have $n+2$ vertices and $n+1$ edges.

We define $f: V(G) \to \{1, 2, ..., n+2\}$ as follows.

$f(v_1') = 2;$

$f(v_2') = 1;$

$f(v_i) = i+2, \quad 1 \leq i \leq n.$

Case 2: When other then pendant edge is duplicated. Than the resultant graph G will have $n+2$ vertices and $n+2$ edges.

We define $f: V(G) \to \{1, 2, ..., n+3\}$ as follows.

$f(v_1') = 2;$

$f(v_2') = 1;$

$f(v_i) = i+3, \quad 1 \le i \le n.$

In view of the above labeling pattern we have distinct edge labels.

Hence from case 1 and case 2 we have the graph obtained by duplication of an arbitrary edge e_k in path P_n is a harmonic mean graph.

Illustation 2.12 : The graph obtained by duplication of a pendant edge in P_5 and its harmonic mean labeling is shown in *Figure 9*. While the graph obtained by duplication of an edge other than pendant edge in P_5 and its harmonic mean labeling is shown in *Figure 10*.

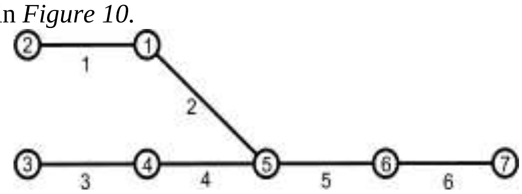

Figure 9

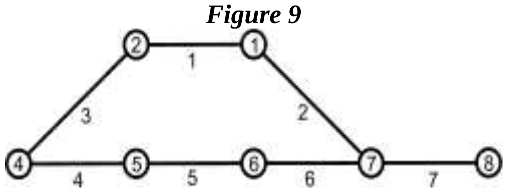

Figure 10

Theorem 2.13 : The graph obtained by duplication of an arbitrary vertex by a new edge in path P_n is a harmonic mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, ..., e_{n-1}$ be the edges of path P_n. We duplicate the vertex v_k by an edge e_n with end vertices as v_1' and v_2'. The resultant graph G will have $n+2$ vertices and $n+2$ edges.

We define $f: V(G) \rightarrow \{1, 2, ..., n+3\}$ as follows.

$f(v_1') = 1;$

$f(v_2') = 2;$

$f(v_i) = i+3, \quad 1 \le i \le n.$

In view of the above labeling pattern we have distinct edge labels from $\{1, 2, ..., n+2\}$.

Hence the graph obtained by duplication of an arbitrary vertex by a new edge in path P_n is a harmonic mean graph.

Illustration 2.14 : The graph obtained by duplication of a vertex by a new edge in path P_5 and its harmonic mean labeling is shown in *Figure 11*.

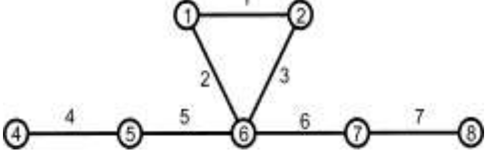

Figure 11

Theorem 2.15 : The graph obtained by duplication of an arbitrary edge by a new vertex in path P_n is a harmonic mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, ..., e_{n-1}$ be the edges of path P_n. We duplicate the edge $v_k v_{k+1}$ by a vertex v'. The resultant graph G will have $n+1$ vertices and $n+1$ edges.

We define $f: V(G) \rightarrow \{1, 2, ..., n+2\}$ as follows.

$f(v') = 1;$

$f(v_i) = i+2, \quad 1 \le i \le n.$

In view of the above labeling pattern we have distinct edge labels from $\{1, 2, ..., n+1\}$.

Hence from case 1 and case 2 we have the graph obtained by duplication of an arbitrary edge by a new vertex in path P_n is a harmonic mean graph.

Illustration 2.16 : The graph obtained by duplication of an edge by a new vertex in path P_5 and its harmonic mean labeling is shown in *Figure 12*.

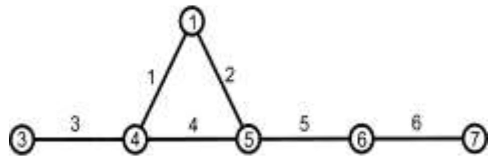

Figure 12

Concluding Remarks

The study of labeled graph is important due to its diversified applications. We discuss here harmonic mean labeling in the context of duplication of graph elements. The derived results are demonstrated by means of sufficient illustrations which provide better understanding. To derive similar results for other graph families is an open area of research.

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