Available online at www.elixirpublishers.com (Elixir International Journal)

Statistics

Elixir Statistics 49 (2012) 10166-10170

ISSN: 2229-712X

Construction of a Family of $(4 \le v \le 10, b, r, 3, 2 \le \lambda \le 8)$ Balanced Incomplete Block Designs (BIBDs) from Potential Lotto Designs

Alawode, O. A

Department of Statistics, University of Ibadan, Nigeria.

ARTICLE INFO

Article history:

Received: 11 May 2012;
Received in revised form:
16 August 2012;
Accepted: 30 August 2012;

Keywords

Balanced Incomplete Block Designs,
Lotto Designs,
Unreduced Designs.

ABSTRACT

The paper considered the construction of a family of $(4 \le v \le 10,\ b,\ r,\ 3,\ 2 \le \lambda \le 8)$ BIBDs using potential Lotto Designs (LDs) earlier derived from qualifying parent BIBDs. We utilized Li's condition, $\left\lfloor \frac{pr}{t-1} \right\rfloor C(t-1,2) + C\left(pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1), 2\right) < C(p,2)\lambda$, to determine the qualification of a parent (v,b,r,k,λ) BIBD as $LD(v,k,p,t)$ constrained on $v \ge k, v \ge p, t \le \min\{k, p\}$ and then considered the case $k = t$ since t is the smallest number of tickets that can guarantee a win in a lottery. The $(6,20,10,3,4)$ BIBD was used as the parent BIBD for the procedure. This BIBD yielded three potential LDs each of which was completely generated using a Microsoft Office Access database computer program and their properties were studied. The three LDs, after their complete generation, yielded the $(4,4,3,3,2)$, $(5,10,6,3,3)$ and $(6,20,10,3,4)$ BIBDs. These BIBDs follow the generalization $(v+1, b+r+\lambda+1, r+\lambda+1, k, \lambda+1)$ where (v,b,r,k,λ) are the parameters of the $(4,4,3,3,2)$ BIBD. A MATLAB program was used to generate a family of the BIBDs for $(4 \le v \le 10,\ b,\ r,\ 3,\ 2 \le \lambda \le 8)$ with these new set of parameters. All the BIBDs in this family are unreduced designs.

I. Introduction

Balanced Incomplete Block Designs (BIBDs) are statistical experimental designs used where the subjects must be divided into blocks of the same size to receive different treatments, such that each subject is tested the same number of times and every pair of subjects appears in the same number of subsets (Hinkelmann and Kempthorne, 2005). Balanced incomplete block designs have 5 parameters v, b, r, k and λ which are used in describing them. In literature, the (v,b,r,k,λ) BIBD is also referred to as a (v,k,λ) BIBD or 2-(v,k,λ) design (Colbourn and Dinitz, 2006). Stinson (2004) defined a BIBD as follows:

Let v, k, λ be positive integers such that $v > k \ge 2$. A (v,k,λ) BIBD is a design (X, A) such that the following properties are satisfied:

1. $|X| = v$
2. A is a collection of nonempty subsets of X called blocks
3. Each block contains exactly k points
4. Every pair of distinct points is contained in exactly λ blocks

Although BIBDs have applications traditionally in the statistical design and analysis of experiments, they also have applications in other fields such as tournament scheduling, coding theory, threshold schemes among others (Colbourn and Dinitz, 2006). Mathon and Rosa (2006) highlighted the necessary conditions for the existence of a (v,b,r,k,λ) BIBD.

They are:

$$vr = bk, \qquad (i)$$

$$r(k-1) = \lambda(v-1), \qquad (ii)$$

and these imply:

$$b = \frac{\lambda(v^2 - v)}{k^2 - k}, \qquad (iii)$$

$$r = \frac{\lambda(v-1)}{k-1}. \qquad (iv)$$

Since all the five parameters must be integers, these conditions imply that BIBDs with certain parameter sets do not exist. Hanani (1961) proved the existence of BIBDs with positive integers $v, \lambda, k = 3$ or 4 for positive integers b and r satisfying conditions in *(iii)* and *(iv)* above.

Many different techniques have been used in the construction of BIBDs: Bose (1939) used difference sets; Hedayat and Majumdar (1985) used sequential search algorithm to construct Unfinished- BIBDs (U-BIBDs) which were then used in the construction of Virtually-Balanced Incomplete Block Designs; Prestwich (2003) utilized a Constrained Local Search algorithm while Bofill *et al* (2003) utilized simulated and mean field annealing in constructing BIBDs and then gave a comparison of the two approaches. Bayrak and Bulut (2006) constructed Orthogonal BIBDs using initial blocks obtained through difference squares. Kumar (2007) utilized unreduced

Tele: +2348038357957

E-mail addresses: oa.alawode@mail.ui.edu.ng

designs in the construction of Partially BIBDs. The use of tabu search and memetic algorithms for constructing BIBDs can be found in Yokoya and Yamada (2010) and Rueda *et al* (2011) respectively.

Several families of BIBDs exist in literature and are mostly based on the techniques of construction. Mandal *et al* (2008) constructed a complete class of $\beta(7,35,15,3,5)$ BIBDs with 30 repeated blocks, Kang and Lee (2005) developed an explicit formula and algorithm for a class of $(q^2+q+1,\ q+1,\ 1)$ symmetric BIBDs with prime q while Satpati and Parsad (2004) constructed and catalogued nested partially balanced incomplete block designs for $v \le 30$ and $r \le 15$. Morgan *et al* (2001) gave an exhaustive list of nested BIBDs for $v \le 16$ and $r \le 30$. In this paper, we present an approach for constructing a family of unreduced BIBDs from LDs for $(4 \le v \le 10,\ b,\ r,\ 3,\ 2 \le \lambda \le 8)$.

The organization of the rest of this paper is as follows. In Section II, we present an overview of the problem of constructing BIBDs from potential LDs. The concept of LDs and the relationship between LDs and BIBDs is discussed. A brief discussion on unreduced BIBDs is also given. In Section III, we describe the method of construction for BIBDs using LDs. This method only takes care of BIBDs that satisfy some imposed constraints. We present and discuss our results in Section IV while the conclusion is in Section V.

II. Lotto Designs and Balanced Incomplete Block Designs

Balanced Incomplete Block Designs can be constructed from another BIBD (Stinson, 2004). In this section, we present an overview of the problem of constructing BIBDs from potential LDs earlier obtained from a parent BIBD.

Lotto Designs

In a typical lottery game, a person chooses *k* numbers from *v* numbers with a small amount of money. This constitutes the ticket. The sale of tickets is stopped at a certain point and the organisers pick *p* numbers from the *v* numbers randomly. These *p* numbers are called the winning numbers. If any of the tickets sold match *t* or more of the winning numbers, a prize is given to the holder of the matching ticket. The larger the value of *t*, the larger the prize. A historical background of lottery and its various formats around the world are given in Gründlingh (2004).

Li (1999) gives a formal definition of Lotto Designs: "Suppose v, k, p, t are integers and B is a collection of *k*-subsets of a set X of *v* elements (usually *X* is *X(v)*). Then, B is an *(v, k, p, t)* Lotto Design (LD) if an arbitrary *p*-subset of *X(v)* intersects relevant *k*-set of *B* in at least *t* elements. The *k*-sets in *B* are known as the blocks of the Lotto design *B*. The elements of X are known as the varieties of the design". The author also defined potential lotto designs as collections of *k*-sets formed during the construction which may or may not be lotto designs.

In literature, most researchers have been concerned with knowing the minimum number of tickets required to obtain a match of at least *t* numbers. This minimum is usually denoted by $L(v,k,p,t)$. For instance, Bate (1978) wrote a computer program that can be used to construct minimal (v,k,p,t) lotto design. Bate and van Rees (1998) determined the values for $L(v,6,6,2)$ for $v \le 54$. Li (1999) gave several upper bound construction methods for LDs, one of which is the use of BIBDs. Gründlingh (2004) introduced the notion of a lottery graph and obtained closed-form bound formulations for the lottery number.

Relationship between LDs and BIBDs

BIBDs can be used in constructing upper bounds for lotto designs. However, not all BIBDs can produce Lotto Designs. To recognize the qualifying BIBDs, Li (1999) gave the general condition they must satisfy in the following theorem:

Theorem 1: If B is the set of blocks of a (v,b,r,k,λ) BIBD and p,t are positive integers where

$$\left\lfloor \frac{pr}{t-1} \right\rfloor C(t-1,2) + C\left(pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1),\ 2 \right) < C(p,2)\lambda,$$

then B is the set of blocks of an (v,k,p,t) Lotto design. Hence $L(v,k,p,t) \le b$.

Unreduced BIBDs

A (v,b,r,k,λ) BIBD is said to be an unreduced design if $b = C(v,k)$, $r = C(v-1,k-1)$ and $\lambda = C(v-2,k-2)$. They are the simplest type of BIBDs and have been found to be useful in the construction of Partially Balanced Incomplete Block Designs, PBIBDs (Kumar, 2007) and in the theory of resistant designs (Baksalary and Puri, 1990).

III. Methodology

This section presents the computational procedure used for this research work. The parameters *p*, *r*, *t* and λ specified in Li's condition were integers defined such that $3 \le p \le 11$, $3 \le t \le 8$. The ranges of *p* and *t* were selected to fit the range of most lottery formats available (Gründlingh, 2004). Also, $t \le \min\{k, p\}$, and $k = t$. The $(6,20,10,3,4)$ BIBD was selected as the parent BIBD and *v*, *k*, *r* and λ were obtained from this BIBD; $v \ge p, v \ge k$. We consider the case $k = t = 3$ where 3 is chosen as the smallest number of tickets that can guarantee a win in a lottery.

A FORTRAN language program was written to implement the expression for Li's condition (in Theorem 1) so as to determine the (v,k,p,t) potential lotto designs that can be derived from the selected BIBD. All the (v,k,p,t) potential lotto designs that did not satisfy the conditions $v \ge p, v \ge k,\ k = t = 3$ were screened out. A complete generation (that is the intersection of any set of $C(v,p)$ with $C(v,k)$ in at least $t = 3$ elements) of each of the qualifying potential LDs was made using a Microsoft Office Access database computer program which was adapted from Rosen (1991). Any of the p-sets can be used for the generation but the first p-set was used in this work. The potential LDs generated were subjected to tests of compliance to BIBDs. This was achieved by observing whether each of them satisfied the necessary conditions for the existence of a BIBD.

A MATLAB program was used to generate a family of $(v+1, b+r+\lambda+1, r+\lambda+1, k, \lambda+1)$ BIBDs for $(4 \le v \le 10, b, r, 3, 2 \le \lambda \le 8)$. The BIBDs in this family were finally investigated for the unreduced property since all k-sets were involved in the complete generation process.

To illustrate the procedure that we just described for the $(6,20,10,3,4)$ BIBD, we have $r = 10,\ k = 3,\ \lambda = 4$. Furthermore, suppose that $p = 5$ and $t = 3$. Then, Li's condition

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1)}{2} < \binom{p}{2}\lambda$$

would be expressed as
$$\left\lfloor \frac{5*10}{3-1} \right\rfloor \binom{3-1}{2} + \binom{5*10-\left\lfloor \frac{5*10}{3-1} \right\rfloor (3-1)}{2} < \binom{5}{2}4$$
which simplifies to $25<40$. Since $25<40$, the $(6,20,10,3,4)$ BIBD qualified to be a potential LD with lotto parameters $(v,k,p,t)=(6,3,5,3)$ where

- v is the total number of elements in the BIBD and is equivalent to the v numbers available for that particular lottery format
- k is the block size of the BIBD and is equivalent to the k numbers a player chooses out of the v numbers available in the lottery game
- p is a positive integer and it represents the winning numbers selected by organizers of a lottery game
- t is a positive integer and it is the numbers in k that match the winning numbers p.

We needed to generate all possible combinations ${}^{v}C_{k}={}^{6}C_{3}$ and ${}^{v}C_{p}={}^{6}C_{5}$ which are

1 2 3	1 3 5	2 3 4	2 5 6
1 2 4	1 3 6	2 3 5	3 4 5
1 2 5	1 4 5	2 3 6	3 4 6
1 2 6	1 4 6	2 4 5	3 5 6
1 3 4	1 5 6	2 4 6	4 5 6

and

1 2 3 4 5
1 2 3 4 6
1 2 3 5 6
1 2 4 5 6
1 3 4 5 6
2 3 4 5 6

respectively. If the p-set = 1 2 3 4 5 is selected as the winning number, then using $t=3$, the intersection when the p-set = 1 2 3 4 5 is compared with ${}^{v}C_{k}={}^{6}C_{3}$ is the set of blocks

1 2 3	1 2 5	1 3 5	2 3 4	2 4 5
1 2 4	1 3 4	1 4 5	2 3 5	3 4 5

which is a $(5,10,6,3,3)$ BIBD.

IV. Results and Discussion

Table I shows the potential LDs obtained from the implementation of Li's condition for the $(6,20,10,3,4)$ BIBD. However, only those in the first three rows of the table satisfy the imposed constraints $v \ge p, v \ge k$, $k=t=3$. Hence, the $(6,20,10,3,4)$ BIBD was found to qualify as $LD(6,3,4,3)$, $LD(6,3,5,3)$ and $LD(6,3,6,3)$. Tables II, III and IV show the complete generation of each of these LDs. From the properties of the different blocks in these tables, it is observed that all the LDs satisfied the necessary conditions for BIBDs, hence BIBDs can be obtained from them: the $(6,20,10,3,4)$ BIBD led to the production of $(4,4,3,3,2)$, $(5,10,6,3,3)$ and $(6,20,10,3,4)$ BIBDs and these can be generalized as $(v+1, b+r+\lambda+1, r+\lambda+1, k, \lambda+1)$ where (v,b,r,k,λ) are the parameters of the $(4,4,3,3,2)$ BIBD. A family of BIBDs with this new set of parameters for $(4 \le v \le 10, b, r, 3, 2 \le \lambda \le 8)$ deduced from this generalization is presented in Table V. All the BIBDs in this family are unreduced BIBDs since each of them satisfied the conditions $b=C(v,k)$, $r=C(v-1,k-1)$ and $\lambda=C(v-2,k-2)$. This is presented in Table VI.

V. Conclusion

In this paper, we constructed BIBDs from the parent $(6,20,10,3,4)$ BIBD using potential LDs. The LDs were constrained on $v \ge p, v \ge k$, $t \le \min\{k,p\}$ and $k=t$. Three potential LDs were produced and a computer program was used to completely generate the potential LDs. The three LDs yielded the $(4,4,3,3,2)$, $(5,10,6,3,3)$ and $(6,20,10,3,4)$ BIBDs. These BIBDs were generalized as $(v+1, b+r+\lambda+1, r+\lambda+1, k, \lambda+1)$ where (v,b,r,k,λ) are the parameters of the first BIBD produced. A MATLAB program was written to generate a family of BIBDs for $(4 \le v \le 10, b, r, 3, 2 \le \lambda \le 8)$ using this generalization. All the BIBDs in this family are unreduced designs.

References

Bate JA. A generalized covering problem. Ph.D. Thesis, Faculty of Graduate Studies, Department of Computer Science, University of Manitoba, Canada. (1978).

Baksalary JK, Puri PD. Pairwise-balanced, variance-balanced and resistant incomplete block designs revisited. Annals of the Institute of Statistical Mathematics 1990; 42 (1): 163-171.

Bate JA, van Rees GHJ. Lotto Designs. The Journal of Combinatorial Mathematics and Combinatorial Computing 1998; 28: 15-39.

Bayrak H, Bulut H. On the Construction of Orthogonal Balanced Incomplete Block Designs. Hacettepe Journal of Mathematical Statistics 2006; 35 (2): 235-240.

Bofill P, Guimer`a R, Torras C. Comparison of simulated annealing and mean field annealing as applied to the generation of block designs. Neural Networks 2003; 16 (10): 1421-1428.

Bose RC. On the construction of Balanced Incomplete Block Designs. Annals of Eugenics 1939; 9: 353-399.

Colbourn CJ, Dinitz JH, editors. Handbook of combinatorial designs. 2nd ed. Boca Raton: CRC Press; 2006.

Gründlingh WR. Two New Combinatorial Problems involving Dominating sets for Lottery Schemes. PhD. Thesis, Dept. of Applied Mathematics, University of Stellenbosch, South Africa. (2004).

Hanani H. The Existence and Construction of Balanced Incomplete Block Designs. Annals of Mathematical Statistics 1961; 32: 361-386.

Hedayat AS, Majumdar D. Families of A-optimal Block Designs for Comparing Test Treatments with a Control. Annals of Statistics 1985; 13(2): 757-767.

Hinkelmann K, Kempthorne O. Design and analysis of experiments. Volume 2: Advanced Experimental Design. John Wiley & Sons, Inc.; 2005.

Kang S, Lee J. An explicit formula and its fast algorithm for a class of Symmetric Balanced Incomplete Block Designs. Journal of Applied Mathematics & Computing 2005; 19 (1-2): 105-125.

Kumar V. Construction of Partially Balanced Incomplete Block Designs through Unreduced Balanced Incomplete Block Designs. Journal of Indian Society of Agricultural Statistics 2007; 61(1): 38- 41.
Li PC. Some Results on Lotto Designs. PhD. Dissertation, Department of Computer Science, University of Manitoba, Canada. (1999).
Mandal S, Ghosh DK, Sharma RK, Bagui SC. A Complete Class of Balanced Incomplete Block Designs $\beta(7,35,15,3,5)$ with repeated blocks. Statistics and Probability Letters 2008; 78 (18): 3338-3343.
Mathon R, Rosa A. $2-(\boldsymbol{v},\boldsymbol{k},\lambda)$ Designs of Small Order. In: Colbourn CJ, Dinitz JH, editors. Handbook of combinatorial designs. Boca Raton: CRC Press; 2006. p. 25-57.
Morgan JP, Preece DA, Rees DH. Nested Balanced Incomplete Block Designs. Discrete Mathematics 2001; 231 (1-3): 351-389.
Prestwich S. A local search algorithm for Balanced Incomplete Block Designs. Lecture Notes in Computer Science, Springer 2003; 2627: 132-143.
Rosen KH. Discrete Mathematics and its Applications. 2nd Edition, New York: McGraw-Hill; 1991.
Rueda DR, Cotta C, Leiva AJF. A memetic algorithm for designing Balanced Incomplete Blocks. International Journal of Combinatorial Optimization Problems and Informatics 2011; 2(1): 14-22.
Satpati SK, Parsad R. Construction and cataloguing of Nested Partially Balanced Incomplete Block Designs. Ars Combinatorial 2004; 73: 299-309.
Stinson DR. Combinatorial Designs: Constructions and Analysis. New York: Springer-Verlag; 2004.
Yokoya D, Yamada T. A mathematical programming approach to the construction of BIBDs. International Journal of Computing Mathematics 2010: 1-16.

Table I: Potential LDs from the (6, 20, 10, 3, 4) BIBD

v	*k*	*p*	*t*
* 6	3	4	3
* 6	3	5	3
* 6	3	6	3
6	3	7	3
6	3	7	4
6	3	8	3
6	3	8	4
6	3	9	3
6	3	9	4
6	3	9	5
6	3	10	3
6	3	10	4
6	3	10	5
6	3	11	3
6	3	11	4
6	3	11	5

*$satisfy\ v \geq p, v \geq k, k = t = 3$

Table II: Blocks Generated from LD(6, 3, 4, 3) for *p*-set=1 2 3 4 and *t*=3

1	2	3
1	2	4
1	3	4
2	3	4

Table III: Blocks Generated from LD(6, 3, 5, 3) for *p*-set=1 2 3 4 5 and *t*=3

1	2	3
1	2	4
1	2	5
1	3	4
1	3	5
1	4	5
2	3	4
2	3	5
2	4	5
3	4	5

Table IV: Blocks Generated from LD(6, 3, 6, 3) for *p*-set= 1 2 3 4 5 6 and *t*=3

1	2	3
1	2	4
1	2	5
1	2	6
1	3	4
1	3	5
1	3	6
1	4	5
1	4	6
1	5	6
2	3	4
2	3	5
2	3	6
2	4	5
2	4	6
2	5	6
3	4	5
3	4	6
3	5	6
4	5	6

Table V: Family of $(4 \le v \le 10, b, r, 3, 2 \le \lambda \le 8)$ BIBD Constructed

v	b	r	k	λ
4	4	3	3	2
5	10	6	3	3
6	20	10	3	4
7	35	15	3	5
8	56	21	3	6
9	84	28	3	7
10	120	36	3	8

Table VI: Unreduced property of the family of BIBDs generated

BIBD	$b = \binom{v}{k}$	$r = \binom{v-1}{k-1}$	$\lambda = \binom{v-2}{k-2}$
(4,4,3,3,2)	$4 = \binom{4}{3}$	$3 = \binom{4-1}{3-1}$	$2 = \binom{4-2}{3-2}$
(5,10,6,3,3)	$10 = \binom{5}{3}$	$6 = \binom{5-1}{3-1}$	$3 = \binom{5-2}{3-2}$
(6,20,10,3,4)	$20 = \binom{6}{3}$	$10 = \binom{6-1}{3-1}$	$4 = \binom{6-2}{3-2}$
(7,35,15,3,5)	$35 = \binom{7}{3}$	$15 = \binom{7-1}{3-1}$	$5 = \binom{7-2}{3-2}$
(8,56,21,3,6)	$56 = \binom{8}{3}$	$21 = \binom{8-1}{3-1}$	$6 = \binom{8-2}{3-2}$
(9,84,28,3,7)	$84 = \binom{9}{3}$	$28 = \binom{8-1}{3-1}$	$7 = \binom{9-2}{3-2}$
(10,120,36,3,8)	$120 = \binom{10}{3}$	$36 = \binom{10-1}{3-1}$	$8 = \binom{10-2}{3-2}$