



Transport of solute with exponentially function of space flow velocity

Dilip Kumar Jaiswal

Department of Mathematics and Astronomy, Lucknow University, Lucknow-226 007, U.P., India.

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ABSTRACT

Analytical solution of advective–dispersive solute transport through inhomogeneous porous medium is obtained by using Laplace transformation. Pulse type input point source is considered for constant solute dispersion along non-uniform flow through medium. Flow of the medium is considered exponential function of space variable. First order decay term which is directly proportional to flow velocity of the medium is also considered. Result and discussions are given with different graphs.

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Introduction

Deterministic groundwater models generally require the solution of partial differential equations. Exact solutions can be obtained analytically and numerical methods yield approximate solutions to the governing equation through the discretisation of space and time, but analytical models require that the parameters and boundaries be highly idealized. Some deterministic models treat the properties of porous media as lumped parameters, but this precludes the representation of heterogeneous hydraulic properties in the model. Heterogeneity, is characteristic of all geologic systems and is now recognized as playing a key role in influencing solute transport and groundwater flow. Transport equation (advective-dispersive equation) has many applications like groundwater hydrology, chemical engineering bio-sciences, environmental sciences and petroleum engineering. Flow velocities and hydrodynamic dispersion coefficients are key parameters for description of fluid and solute transport in porous media. Transport of pollutant degraded groundwater quality. Degradation of groundwater quality can be caused by either point sources (septic tank, garbage disposal sites, cemeteries, mine spoils and oil spoils) or by line sources of poor quality water (like seepage from polluted streams or intrusion of salt water, from oceans). Some exhaustive list of works are related with solute transport, like Bastian and Lapidus (1956), Ogata (1970), Marino (1974), Al-Niami and Rushton (1977). Most of these works take into account the effects due to adsorption, first order decay, zero order production. Coming nearer to real problems, Shamir and Harleman (1967), Lin (1977), considered the layered porous media and non linear adsorption, Banks and Jerasate (1962), Hunt (1978), Kumar (1983) considered the unsteady/non-uniform porous media flow.

Leij et al.(1993) presented analytical solutions for non-equilibrium solute transport in semi-infinite porous media during steady unidirectional flow. The transport equation incorporates terms accounting for advections, dispersion, zero-order production, and first-order decay. van Kooten (1995) presented a method for predicting the advective-dispersive transport of a

contaminant towards a well in a confined aquifer. Due to (macro-) dispersion, particles carry out random walks through the porous formation. Logan and Zlotnik (1995) obtained solutions of the convection–diffusion equation with decay for periodic boundary conditions on a semi-infinite domain. The boundary conditions take the form of a periodic concentration or a periodic flux, and a transformation is obtained that relates the solutions of the two, pure boundary value problems. The dispersion coefficient tends to increase with the distance of solute concentration observations, this is generally mentioned as the scale effect on the dispersion process (Pachepsky et al.; 2000). Su et al. (2005) presented an analytical solution to advection-dispersion equation with spatially and temporally varying dispersion coefficient for predicting solute transport in a steady, saturated sub-surface flow through homogeneous porous media. Smedt (2006) presented analytical solutions for solute transport in rivers including the effects of transient storage and first order decay. Jaiswal et al. (2009) and Kumar et al. (2010) obtained analytical solutions for temporally (assumptions are based on the observations of Matheron and deMarsily;1980) and spatially dependent solute dispersion in one dimensional semi-infinite media.

Water velocity through a pore depends upon its size. This variation in pore velocity causes some contaminant to move faster and some contaminant to move slower. This variation in contaminant movement (due to variation in pore water velocity) is called hydrodynamic dispersion. In the present study, one-dimensional advective–dispersive equation with variable coefficient is solved with decay term. Constant dispersion coefficient is considered along exponentially function of space flow velocity. Point source is assumed varying pulse type nature in semi-infinite medium in an initially not solute free domain i.e. before introduction of input source the domain is already contaminated.

Mathematical Model and Solution

There are many parameters which cause the subsurface transport of contaminants. The mechanisms of transport of

contaminants in groundwater include advection, dispersion, adsorption and decay, chemical reaction and ion exchange, biological processes. Dispersion is the combined effects of two mass transport processes in porous media namely mechanical dispersion and molecular diffusion and advection represents the movement of a contaminant with the bulk fluid according to the seepage velocity in pore space. One dimensional advective-dispersive equation may be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left(D(x,t) \frac{\partial C}{\partial x} - u(x,t)C \right) \quad (1)$$

where $D(x,t)$ is solute dispersion and it is called dispersion coefficient if it is uniform/ steady, and $u(x,t)$ is the flow velocity of the medium and C is the solute concentration at position x and time t . It is described by principal of conservation of mass and Fick's law of diffusion. If the medium is porous, velocity of the medium satisfies the Darcy law. In the limit of very low fluid velocity, dispersion is determined solely by molecular diffusion and at high fluid velocities, dispersion is purely 'fluid mechanical'.

There are three possibilities for dispersion and velocity occurs in which dispersion is constant with variable velocity, variable dispersion with uniform velocity and lastly both are variable functions. The mass transport phenomenon occurs specially due to heterogeneities in the medium that cause variation in flow velocities and in flow path, which is referred as mechanical dispersion. Molecular diffusion is caused by the inhomogeneous distribution of the pollutant concentration. Analytical solution of hydrodynamic dispersion problems, even in one dimensional may be obtained in limited cases, so numerical solutions of the problems applicable to realistic engineering have been obtained using finite difference and finite element method. But Yates (1990, 1992) developed an analytical solution for describing the transport of dissolved substances in heterogeneous porous media with a distance-dependent dispersion relationship. In present discussion, first possibility i.e., constant dispersion with variable velocity take into account.

Let $D(x,t) = D_0$ and $u(x,t) = u_0 \exp(\pm ax)$ and first order decay term which is directly proportional to flow velocity i.e., $\lambda(x,t) = \lambda_0 \exp(\pm ax)$ where $\lambda_0 = au_0$ in Eq. (1). The parameter a is inhomogeneity factor and have dimension of inverse of space variable. Inhomogeneity of the medium was addressed by dividing the medium into stratified layers and variation in the flow velocity causes by the inhomogeneity of the medium. The

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left(D_0 \frac{\partial C}{\partial x} - u_0 \exp(\pm ax)C \right) - \lambda_0 \exp(\pm ax)C \quad (2)$$

where D_0 , u_0 and λ_0 are constants.

Let us introduce a new independent variable X by following transformation

$$x = m \log X \quad (3)$$

Eq. (2) becomes,

$$\frac{\partial C}{\partial t} = D_0 X^2 \frac{\partial^2 C}{\partial X^2} + D_0 X \frac{\partial C}{\partial X} - u_0 X^{-a+1} \frac{\partial C}{\partial X} \quad (4)$$

The thickness between two layers is so small and transport properties being homogeneous in each layer and differing from other layer. The range of inhomogeneity factor in following discussion taken to be account as $0 < a \leq 0.01$ for each layer

and after this range the transport properties are changed. There is no any fixed range of the inhomogeneity factor for the stratification. It is depend upon structure of medium. Pickens and Grisak (1981) supposed this factor is $0.05 \leq a \leq 0.1$ but Yates (1990) considered the range of inhomogeneity factor is $0 \leq a \leq 2$. Let the inhomogeneity factor is $a = 0.01$. Therefore, X^{-a+1} becomes $X^{-0.01+1} = X^{0.99}$. Now, if 0.99 is considered approximately 1.0. The partial differential equation (4) reduces into that with variable coefficients as,

$$\frac{\partial C}{\partial t} = D_0 X^2 \frac{\partial^2 C}{\partial X^2} - (u_0 - D_0) X \frac{\partial C}{\partial X} \quad (5)$$

Again, introducing a new independent variable Z and then a dependent variable K by the following transformations as,

$$Z = \log X \quad (6a)$$

and

$$C(Z,t) = K(Z,t) \exp \left(\frac{U_0}{2D_0} Z - \frac{U_0^2}{4D_0} t \right), \quad (6b)$$

where $U_0 = (u_0 - D_0)$

The partial differential equation (5) becomes,

$$\frac{\partial K}{\partial t} = D_0 \frac{\partial^2 K}{\partial Z^2} \quad (7)$$

An input concentration is assumed at the origin of the domain. It may be either uniform or varying pulse type. But, due to human and other responsible activities, input source condition may not be uniform. In the presence of the source of pollution, the input concentration may be of increasing nature. As soon as the source of pollution is eliminated, the input concentration starts decreasing instead of becoming zero. Uniform and varying pulse type input source condition are written as,

$$C(x,t) = \begin{cases} C_0, & 0 < t \leq t_0, \quad x = 0 \\ 0 & t > t_0 \end{cases} \quad (8)$$

$$u(x,t)C(x,t) - D(x,t) \frac{\partial C}{\partial x} = \begin{cases} u_0 C_0, & 0 < t \leq t_0, \quad x = 0 \\ 0 & t > t_0 \end{cases} \quad (9)$$

respectively. But, varying pulse type input source is more realistic condition, because if we considered dispersion coefficient $D = 0$ and velocity $u(x,t) = u_0$ in Eq. (9), we can

get condition (8). So third-type boundary condition (9) taken into account in the present study.

Let us assume, the domain is initially not solute free, it means before introduction of input source the domain is already polluted. The second boundary condition is considered of flux type of homogeneous nature. Thus, initial and second boundary condition is as follows,

$$C(x,t) = C_i, \quad t = 0, \quad x \geq 0 \quad (10)$$

$$\frac{\partial C(x,t)}{\partial x} = 0, \quad t \geq 0, \quad x \rightarrow \infty \quad (11)$$

The initial and boundary conditions written in terms of new dependent and independent variables as,

$$K(Z,t) = C_i \exp \left(-\frac{U_0}{2D_0} Z \right), \quad Z \geq 0, \quad t = 0 \quad (12)$$

$$\frac{U_0}{2} K(Z,t) - D_0 \frac{dK}{dZ} = \begin{cases} u_0 C_0 \exp \left(\frac{U_0^2}{4D_0} t \right), & 0 < t \leq t_0, \quad Z = 0 \\ 0 & t > t_0 \end{cases} \quad (13)$$

$$\frac{\partial K(Z,t)}{\partial Z} = 0, \quad t \geq 0, \quad Z \rightarrow \infty \quad (14)$$

Applying Laplace transformation on diffusion equation (7) and initial and boundary conditions (12)-(14), the solution of advective-dispersive solute transport may be written in terms of $C(x,t)$ as,

$$C(x,t) = C_i + (C_0 - C_i)A(x,t), \quad 0 < t \leq t_0 \quad (15a)$$

$$C(x,t) = C_i + (C_0 - C_i)A(x,t) - C_0A(x,t-t_0), \quad t > t_0 \quad (15b)$$

where

$$A(x,t) = \left[\frac{1}{2} \operatorname{erfc} \left(\frac{x-U_0t}{2\sqrt{D_0t}} \right) + \left(\frac{U_0t}{\pi D_0} \right) \exp \left(-\frac{(x-U_0t)^2}{D_0t} \right) - \frac{1}{2} \left(1 + \frac{U_0x}{D_0} + \frac{U_0^2t}{D_0} \right) \exp \left(\frac{U_0x}{D_0} \right) \operatorname{erfc} \left(\frac{x+U_0t}{2\sqrt{D_0t}} \right) \right],$$

and $U_0 = (u_0 - D_0)$.

Particular cases

(i) The solution of advective-dispersive solute transport for uniform pulse type input source concentration may be obtained to taking condition (8) in the place of (9) and proceed further. The solution for uniform pulse type input source concentration is,

$$C(x,t) = C_i + (C_0 - C_i)B(x,t), \quad 0 < t \leq t_0 \quad (16a)$$

$$C(x,t) = C_i + (C_0 - C_i)B(x,t) - C_0B(x,t-t_0), \quad t > t_0 \quad (16b)$$

where

$$B(x,t) = \frac{1}{2} \left[\operatorname{erfc} \left\{ \frac{x-U_0t}{2\sqrt{D_0t}} \right\} - \exp \left(\frac{U_0x}{D_0} \right) \operatorname{erfc} \left\{ \frac{x+U_0t}{2\sqrt{D_0t}} \right\} \right],$$

and $U_0 = (u_0 - D_0)$.

(ii) If the inhomogeneity factor a is zero, then the partial differential equation (2) reduces into constant coefficients without first order decay term as,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left(D_0 \frac{\partial C}{\partial x} - u_0 C \right) \quad (17)$$

Thus the analytical solutions of advective-dispersive solute transport equation with constant coefficients for same initial and boundary conditions (reported by van Genuchten and Alves; 1982) are same as above obtained solutions only differ by $U_0 = u_0$.

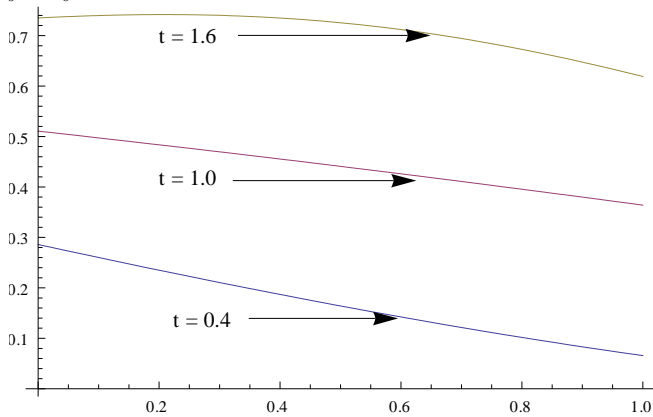


Fig. 1. Distribution of solute for varying pulse type input in presence of source concentration

Result and Discussions

In some cases, such as where there is a fixed amount of radioactive decay material, the assumption of boundary condition (9) may be reasonable. However the source of solute concentration is much more complicated and varies with time

irregularly. Examples of such sources are distribution of solute in aquifers and the distribution of the atmospheric chemical concentration.

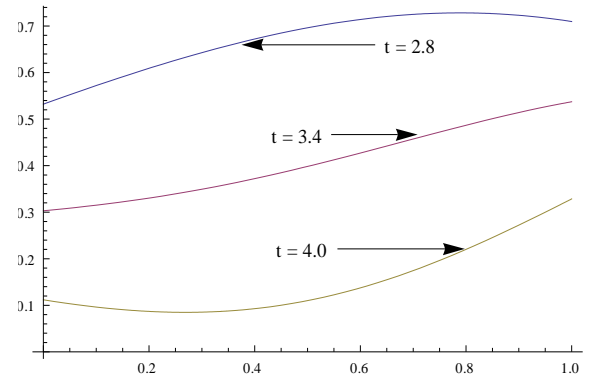


Fig. 2. Distribution of solute for varying pulse type input in absence of source concentration

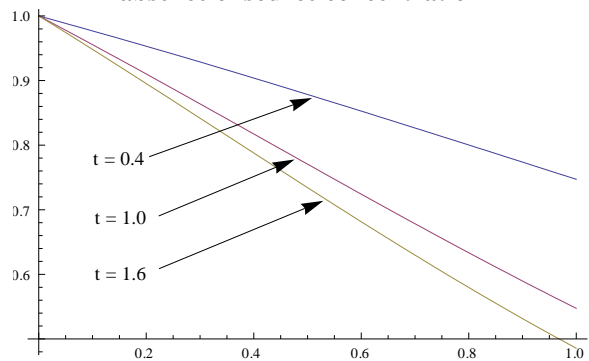


Fig. 3. Distribution of solute for uniform pulse type input in presence of source concentration

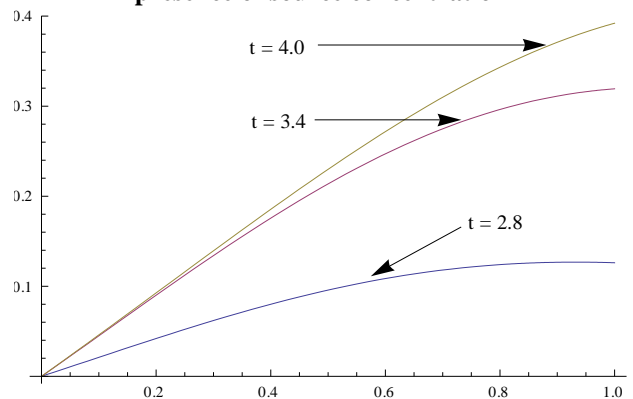


Fig. 4. Distribution of solute for uniform pulse type input in absence of source concentration

The concentration values are evaluated from the analytical solution described by Eq. (15a, b) in the finite domain $0 \leq x \leq 1$ (km) of semi-infinite medium. The other input values are considered as: $C_0 = 1.0$, $C_i = 0.1$, $D_0 = 1.28$ (km²/yr), $u_0 = 1.69$ (km/yr). Fig. (1) and (2) are drawn at different time $t = 0.4, 1.0, 1.6$ in time domain $t \leq t_0$ and $t = 2.8, 3.4, 4.0$ in time domain $t > t_0$, respectively. The time of elimination of source concentration is $t_0 = 1.8$ (yr). The distribution of concentration in both figures is higher for large time i.e., time increases then concentration increases at particular position. The behavior of solute transport for uniform pulse type input source for analytical solution (16a, b) are shown by Figs. (3) and (4). It is observed that from figures, distribution of solute for constant

dispersion and exponentially space variable flow velocity is quite different from constant dispersion with uniform flow velocity. The concentration is decrease with increasing time at different time in time domain $t \leq t_0$ in Fig. (3), while in time domain $t > t_0$, solute concentration is increase with increasing time. The nature of Figs. (3) and (4) is reverse of advective-dispersive equation with constant coefficients.

The advective-dispersive equation describing groundwater flow and transport can be solved mathematically using either analytical solutions or numerical solutions. The analytical solution can be used to characterize differences in the transport process relative to the classical convection-dispersion equation which assumes that the hydrodynamic dispersion in the porous medium remains constant. With dispersion and convection, there will be some early appearance of contaminant and it will take long time to displace all the contaminant from the medium. Higher the degradation rate, less is the total amount of contaminant that will appear in geological medium (groundwater). Higher is the degradation rate less time it will take to displace all contaminant out of the medium.

Conclusion

In the present study, one-dimensional advective-dispersive equation with constant dispersion and variable velocity coefficient is solved through inhomogeneous semi-infinite porous medium with first order decay term. Varying pulse type point source is considered in an initially not solute free domain. Pulse type input condition is useful to rehabilitation of time. Inhomogeneity or variability in aquifer properties is characteristic of all geologic systems and is now recognized as playing a key role in influencing groundwater flow and solute transport.

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