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On the Lattice of Convex Sublattices

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ABSTRACT

Let L be a finite lattice. A sublattice K of a lattice L is said to be convex if a,b ϵ K; c ϵ L, a$\leq$ c$\leq$ b imply that c ϵ K. Let CS(L) be the set of convex sublattices of L including the empty set. Then CS(L) partially ordered by inclusion relation, forms an atomic algebric lattice. Let P be a finite graded poset. A finite graded poset P is said to be Eulerian if its MÖbius function assumes the value $\mu(x; y) = (-1)^{l(x;y)}$ for all x$\leq$y in P, where $l(x, y) = \rho(y)-\rho(x)$ and ρ is the rank function on P. In this paper, we prove that the lattice of convex sublattices of a Boolean algebra B_n, of rank n, CS(B_n) with respect to the set inclusion relation is a dual simplicial Eulerian lattice. But the structure of the lattice of convex sublattices of a non- Boolean Eulerian lattice with respect to the set inclusion relation is not yet clear.

Introduction

The study of lattice of convex sublattices of a lattice was started by K.M. Koh [3], in the year 1972. He investigated the internal structure of a lattice L, in relation to CS(L), like so many others for various algebraic structures such as groups, Boolean algebras, directed graphs and so on.

In [3], several basic properties of CS(L) have been studied and proved that "If L is complemented then CS(L) is complemented". Also, the connection of the structure of CS(L) with those of the ideal lattice I(L) and the dual ideal lattice D(L) are examined. K.M. Koh, derived the best lower bound and upper bound for the cardinality of CS(L), where L is finite. In a subsequent paper[1], C.C. Chen, and K.M. Koh, proved that CS(LxK) $\cong$ [(CS(L) - $\emptyset$) X (CS(K) - $\emptyset$)]$\cup$ $\emptyset$ Finally, they proved that when L is a finite lattice and CS(L)$\cong$ CS(M) and if L is relatively complemented (complemented) then M is relatively complemented (complemented). This is true for Eulerian lattices, since an Eulerian lattice is relatively complemented. These results gave motivation for us to look into the connection between L and CS(L) for Eulerian lattices which are a class of lattices not defined by identities. Since a B_n, Boolean algebra of rank of n, for n = 1; 2…, is Eulerian, we start looking into the structure of CS(B_n).

In section 3, we prove that CS(B_n), the lattice of convex sublattices of B_n with respect to the set inclusion relation is a dual simplicial Eulerian lattice. But the structure of the lattice of convex sublattices of non-Boolean Eulerian lattice with respect to the set inclusion relation is not yet clear.

Preliminaries

Throughout this section CS(L) is equipped with the partial order of set inclusion relation.

Definition 2.1 . *A finite graded poset P is said to be Eulerian if its MÖbius function assumes the value $\mu(x, y) = (-1)^{l(x;y)}$ for all x$\leq$y in P, where $l(x, y) = \rho(y)-\rho(x)$ and ρ is the rank function on P.*

An equivalent definition for an Eulerian poset is as follows:

Lemma 2.2 [8] *A finite graded poset P is Eulerian if and only if all intervals [x, y] of length l$\geq$ 1 in P contain an equal number of elements of odd and even rank.*

Example: *Every Boolean algebra of rank n is Eulerian and the lattice C_4 of Figure 1 is an example for a non-modular Eulerian lattice. Also, every C_n is Eulerian for n $\geq$4.*

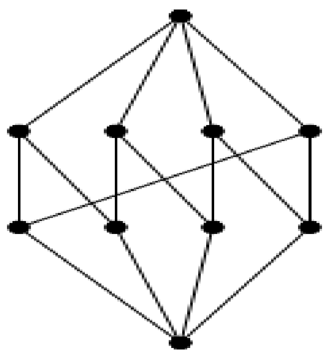

Figure 1

Lemma 2.3 [10] *If L_1 and L_2 are two Eulerian lattices then L_1 X L_2 is also Eulerian.*

We note that any interval of an Eulerian lattice is Eulerian and an Eulerian lattice cannot contain a three element chain as an interval. For more structures of Eulerian lattices, see [12]

Definition 2.4 *A poset P is called Simplicial if for all t $\neq$ 1 ϵ P, [0, t] is a Boolean algebra and P is called Dual Simplicial if for all t $\neq$ 0 ϵ P, [t, 1] is a Boolean algebra.*

The following remark is the example 1.1.17 in the book of R.P.Stanley [9]

Remark 2.5 $\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$

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Theorem 2.6 *The lattice of convex sublattices of a Boolean algebra B_n of rank n, $CS(B_n)$ with respect to the set inclusion relation is a dual simplicial Eulerian lattice.*

Proof: It is clear that rank of $CS(B_n)$ is n + 1.

First we prove that the interval $[\emptyset, B_n]$ is Eulerian.

That is to prove that this interval has same number of elements of odd and even rank.

Let a_i be the number of elements of rank i in $CS(B_n)$.

Since the elements of rank 1 in $CS(B_n)$ are just the singleton subsets of B_n, we have, $a_1 = 2^n$.

The rank two elements in the interval $[\emptyset, B_n]$ are the two-element chains.

We have to determine the total number of two-element chains in B_n. Since there are $\binom{n}{1}$ atoms in B_n, the number of two-element chains containing 0 in B_n is $\binom{n}{1}$ Since there are $\binom{n-1}{1}$ edges emanating from an atom in B_n there are $\binom{n}{1}\binom{n-1}{1}$ two-element chains containing an atom in B_n.

A rank two element is connected by $\binom{n-2}{1}$ edges to some of the rank three elements and since there are $\binom{n}{2}$ rank 2 elements in B_n, we have the number of two-element chains from the rank two elements of B_n are $\binom{n}{2}\binom{n-2}{1}$. Similarly, the total number of two-element chains from the rank three elements are $\binom{n}{3}\binom{n-3}{1}$.

Considering all the elements upto rank n - 1 the total number of two element chains in B_n is

$$a_2 = \binom{n}{1} + \binom{n}{1}\binom{n-1}{1} + \binom{n}{2}\binom{n-2}{1} + \binom{n}{3}\binom{n-3}{1} + \ldots + \binom{n}{n-1}\binom{n-(n-2)}{1}$$

A rank three element in $[\emptyset, B_n]$ is a sublattice B_2 of B_n. There are $\binom{n}{2}$ rank two elements in B_n.

Therefore, the number of B_2's containing 0 is $\binom{n}{2}$

There are $\binom{n}{1}$ atoms in B_n . If a is an atom then $[a, 1] \cong B_{n-1}$.

From 'a' to a rank three element in [a, 1] we have a sublattice B_2 with a as the lowest element.

Since there are $\binom{n-1}{2}$ such rank three elements we have the number of such B_2's is $\binom{n-1}{2}$

In all, the number of B_2's with an atom as the lowest element is $\binom{n}{1}\binom{n-1}{2}$

Similarly, the number of B_2's with a rank two element as the lowest element is $\binom{n}{2}\binom{n-2}{2}$.

Proceeding like this, we get,

$$a_3 = \binom{n}{2} + \binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\binom{n-2}{2} + \binom{n}{3}\binom{n-3}{2} + \ldots + \binom{n}{n-2}\binom{n-(n-2)}{2}$$

Continuing like this, we get,

$$a_4 = \binom{n}{3} + \binom{n}{1}\binom{n-1}{3} + \binom{n}{2}\binom{n-2}{3} + \binom{n}{3}\binom{n-3}{3} + \ldots + \binom{n}{n-3}\binom{n-(n-3)}{3}$$

and so on

$$a_{n-2} = \binom{n}{n-3} + \binom{n}{1}\binom{n-1}{n-3} + \binom{n}{2}\binom{n-2}{n-3} + \binom{n}{3}\binom{n-3}{n-3}$$

$$a_{n-1} = \binom{n}{n-2} + \binom{n}{1}\binom{n-1}{n-2} + \binom{n}{2}\binom{n-2}{n-2}$$

$$a_n = \binom{n}{n-1} + \binom{n}{1}\binom{n-1}{n-1}$$

Case(i) : Suppose n is even.

$a_1 - a_2 + a_3 - \ldots - a_{n-2} + a_{n-1} - a_n$

$$\begin{aligned}
&= 2^n - \{\binom{n}{1} + \binom{n}{1}\binom{n-1}{1} + \binom{n}{2}\binom{n-2}{1} - \binom{n}{3}\binom{n-3}{1} + \cdots + \binom{n}{n-1}\binom{n-(n-2)}{1}\} \\
&\quad + \{\binom{n}{2} + \binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\binom{n-2}{2} + \binom{n}{3}\binom{n-3}{2} + \cdots + \binom{n}{n-2}\binom{n-(n-2)}{2}\} - \\
&\quad \{\binom{n}{3} + \binom{n}{1}\binom{n-1}{3} + \binom{n}{2}\binom{n-2}{3} + \binom{n}{3}\binom{n-3}{3} + \cdots + \binom{n}{n-3}\binom{n-(n-3)}{3}\} \\
&\quad - \ldots - \{\binom{n}{n-3} + \binom{n}{1}\binom{n-1}{n-3} + \binom{n}{2}\binom{n-2}{n-3} + \binom{n}{3}\binom{n-3}{n-3}\} + \{\binom{n}{n-2} + \\
&\quad \binom{n}{1}\binom{n-1}{n-2} + \binom{n}{2}\binom{n-2}{n-2}\} - \{\binom{n}{n-1} + \binom{n}{1}\binom{n-1}{n-1}\}. \\
&= 2^n - [\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \cdots + \binom{n}{n-1}] - \\
&\quad \binom{n}{1}[\binom{n-1}{1} - \binom{n-1}{2} + \binom{n-1}{3} - \cdots - \binom{n-1}{n-2} + \binom{n-1}{n-1}] - \binom{n}{2} \\
&\quad [\binom{n-2}{1} - \binom{n-2}{2} + \binom{n-2}{3} - \cdots - \binom{n-2}{n-3} + \binom{n-2}{n-2}] - \binom{n}{3}[\binom{n-3}{1} - \\
&\quad \binom{n-3}{2} + \binom{n-3}{3} - \cdots - \binom{n-3}{n-4} + \binom{n-3}{n-3}] - \cdots - \binom{n}{n-1}[\binom{n-(n-1)}{1}]
\end{aligned}$$

$= 2^n - [2 - (1-1)^n] - \binom{n}{1}[1-(1-1)^{n-1}] - \binom{n}{2}[1-(1-1)^{n-2}] - \ldots - \binom{n}{n-1}$

$= 2^n - 2 - \binom{n}{1} - \binom{n}{2} - \binom{n}{3} - \binom{n}{4} - \ldots - \binom{n}{n-1}$

$= 2^n - [2 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \ldots + \binom{n}{n-1}$

$= 2^n - (1+1)^n$

$= 2^n - 2^n = 0$

Therefore, $a_1 - a_2 + a_3 - \ldots - a_{n-2} + a_{n-1} - a_n = 0$

Case(i) : Suppose n is odd.

$a_1 - a_2 + a_3 - \ldots - a_{n-2} + a_{n-1} - a_n$

$2^n - [\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \cdots + \binom{n}{n-1}] - \binom{n}{1}[\binom{n-1}{1} - \binom{n-1}{2} + \binom{n-1}{3} - \cdots - \binom{n-1}{n-2} + \binom{n-1}{n-1}] - \binom{n}{2}[\binom{n-2}{1} - \binom{n-2}{2} + \binom{n-2}{3} - \cdots - \binom{n-2}{n-3} + \binom{n-2}{n-2}] - \binom{n}{3}[\binom{n-3}{1} - \binom{n-3}{2} + \binom{n-3}{3} - \cdots - \binom{n-3}{n-4} + \binom{n-3}{n-3}] - \cdots - \binom{n}{n-1}[\binom{n-(n-1)}{1}]$

$= 2^n - [-(1-1)^n] - \binom{n}{1}[1-(1-1)^{n-1}] - \binom{n}{2}[1-(1-1)^{n-2}] - \cdots - \binom{n}{n-1}$

$= 2^n - \binom{n}{1} - \binom{n}{2} - \binom{n}{3} - \binom{n}{4} - \cdots - \binom{n}{n-1}$

$= 2^n - [\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \cdots + \binom{n}{n-1}$

$= 2^n - (1+1)^n - 2]$

$= 2^n - 2^n + 2 = 2$

Therefore, $a_1 - a_2 + a_3 - \ldots - a_{n-2} + a_{n-1} - a_n = 2$

Hence the interval $[\emptyset, B_n]$ has the same number of elements of odd and even rank.

Now we are going to claim that $CS(B_n)$ is dual simplicial:

Let a be any element of rank k in B_n. We have to calculate the number of elements of rank r in $[\{a\}, B_n]$ in the lattice $CS(B_n)$.

The number of atoms of $[\{a\}, B_n]$ is equal to n- k + k = n, since $[0, a] \cong B_k$ and $[a, 1] \cong B_{n-k}$, and so the number of edges containing a in [0,a] and [a,1] are respectively k and n-k.

A rank 2 element in $[\{a\}, B_n]$ is a B_2 containing a.

There are three possibilities, namely, either a may be in the top or a may be in the middle or a may be in the bottom of the B_2.

The B_2's with a in the top is an upper inerval in [0, a]. Therefore, the number of such B_2's are exactly $\binom{k}{2}$.

The top elements in the B_2's with a as a middle element are just the n-k atoms of the interval [a, 1]

We observe that a B_2 with a in the middle has one of the atoms of [a, 1] as the top element and one of the co-atoms of [0; a] as the bottom element.

Now, fix an atom in [a, 1]. Since there are k co-atoms in [0, a] the number of B_2's with this atom as a top element and a in the middle is k.

This is true for every atom in [a,1]. Since there are n - k such atoms the total number of B_2's having a in the middle is $\binom{n-k}{1} \times \binom{k}{1}$

The B_2's with a in the bottom is a lower interval in [a,1]. Therefore, the number of such B_2's are $\binom{n-k}{2}$

Hence the total number of rank 2 elements in $[\{a\}, B_n]$ is $\binom{k}{2} + \binom{n-k}{1} \times \binom{k}{1} + \binom{n-k}{2} = \binom{n}{2}$

The rank 3 elements in $[\{a\}, B_n]$ is isomorphic to B_3 containing a.

There are four possibilities, namely, a may be a top element or may be a rank 1 element or may be a rank 2 element or may be a bottom element of B_3.

The B_3's with a in the top is an upper interval in [0, a]. Therefore, the number of such B_3's are exactly $\binom{k}{3}$

Suppose that a is a rank 2 element of the B_3's, that is, a co-atom of the B_3's.

For a typical such B_3 an atom in [a; 1] is the topmost element.

Below this a there are two atoms of this B_3 which belong to [0,a].

That atoms are just the coatoms of [0, a]. Now, fix an atom in [a, 1].

The number of B_3's with this atom as the topmost element is $\binom{k}{2}$, since there are k coatoms in [0, a]. This is true for every atom in [a, 1].

Therefore, there are exactly $\binom{n-k}{1}\binom{k}{2}$ such B_3's.

Suppose that a is a rank 1 element of the B_3's, that is an atom of the B_3's.

The lowest element of such a typical B_3 is a co-atom in [0, a]. Now, fix a co-atom in [0, a]. The number of B_3's with this co-atom as the lowest element is $\binom{n-k}{2}$, since there are n-k atoms in [a, 1]. This is true for every co-atom in [0; a].

Therefore, the number of such B_3's are $\binom{n-k}{2}\binom{k}{1}$.

The B_3's with a in the bottom is a lower interval in [a,1]. Therefore, the number of such B_3's are exactly $\binom{n-k}{3}$

Thus the total number of rank 3 elements in $[\{a\}, B_n]$ is $\binom{k}{3} + \binom{n-k}{1}\binom{k}{2} + \binom{n-k}{2}\binom{k}{1} + \binom{n-k}{3} = \binom{n}{3}$

Similarly, we can write the total number of elements of rank r in as $\binom{k}{r} + \binom{n-k}{1}\binom{k}{r-1} +$
$\binom{n-k}{2}\binom{k}{r-2} + \binom{n-k}{3}\binom{k}{r-3} + \cdots + \binom{n-k}{r-2}\binom{k}{2} + \binom{n-k}{r-1}\binom{k}{1} + \binom{n-k}{r} = \binom{n}{r}$,
by remark 2.11.

The terms in the number of B_r's are of the form $\binom{n-k}{x}\binom{k}{y}$ where x + y = r.
The B_{r+1}'s containing these B_r's are obtained by moving up or down by one rank.
Therefore, we get a B_{r+1} by adding 1 with x or with y.
Therefore, either (x + 1) + y = r + 1 or x + (y + 1) = r + 1.
Therefore, a typical term in the number of B_{r+1}'s is of the form, $\binom{n-k}{x+1}\binom{k}{y}$ or $\binom{n-k}{x}\binom{k}{y+1}$

The number of elements of rank r + 1 in [$\{a\}$, B_n] is $\binom{k}{r+1} + \binom{n-k}{1}\binom{k}{r} +$
$\binom{n-k}{2}\binom{k}{r-1} + \binom{n-k}{3}\binom{k}{r-2} + \cdots + \binom{n-k}{r}\binom{k}{1} + \binom{n-k}{r+1} = \binom{n}{r+1}$
, by remark 2.11.
Therefore, [$\{a\}$, B_n]is a Boolean lattice of rank n.

If we take any upper interval then it is a subinterval of one of the intervals of the form [$\{a\}$, B_n].

Since [$\{a\}$, B_n] is Boolean any subinterval is also Boolean.

Therefore, every upper interval is Boolean.
Hence , CS(B_n) is a dual simplicial Eulerian Lattice.

Conclusion

A Boolean algebra B_n is a particular case of an Eulerian lattice for which we proved CS(B_n) is a dual simplicial Eulerian lattice under the set inclusion relation. For a non-Boolean Eulerian lattice we can not decide the structure. For lattices of small ranks CS(L) is Eulerian. So, we strongly believe that CS(L) would be Eulerian yet it is still open.

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