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Some Results Concerned to the Generalized Functions of Fractional Calculus

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ABSTRACT

In the present paper, we introduce two functions namely $\Omega(c,\upsilon,\gamma,p,q,x)$ and $\Omega(c,-\mu,\gamma,p,q,x)$ in terms of K-function introduced recently by Sharma[9] and show their properties by using fractional integrals and derivatives. Results derived in this paper are the extensions of the results derived earlier by Shukla and Prajapati[2].

1. Introduction:

The function which is introduced and studied by Mittag-Leffler[4,5] in terms of the power series given below

$$E_\alpha(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+1)},\quad(\alpha>0) \tag{1.1}$$

A generalization of this series in the following form

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)},\quad(\alpha,\beta>0) \tag{1.2}$$

is given by Wiman[3].

The K-function[9] is given by

$${}_pK_q^{\alpha,\beta;\gamma}(a_1,\ldots,a_p;b_1,\ldots,b_q;x)=$$

$${}_pK_q^{\alpha,\beta;\gamma}(x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n x^n}{n!\,\Gamma(\alpha n+\beta)} \tag{1.3}$$

where $\alpha,\beta,\gamma\in C, R(\alpha)>0$ and $(a_i)_n (i=1,2,\ldots,p)$ and $(b_j)_n\ (j=1,2,\ldots,q)$ are the Pochhammer symbols. Further details of this function are given by [9].

The Riemann-Liouville operator of fractional integral of order υ is given by

$$I_x^\upsilon\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\int_0^x(x-t)^{\upsilon-1}f(t)\,dt \tag{1.4}$$

provided that the integral exists.

The Riemann-Liouville operator of fractional derivative of order υ is defined[1,6,7,8] in the following form

$$D_x^\upsilon\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\frac{d^n}{dx^n}\int_0^x\frac{f(t)}{(x-t)^{\upsilon+n-1}}dt,\ (n-1<\upsilon<n) \tag{1.5}$$

provided that the integral exists.

2. Fractional Calculus Operators and K-Function:

Let

$$f(x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(cx)^n}{(n!)^2} \tag{2.1}$$

where c is an arbitrary constant.

The fractional integral operator of order ν is given by

$$I_x^\upsilon\{f(x)\}=\frac{1}{\Gamma(\upsilon)}\int_0^x(x-\tau)^{\upsilon-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(c\tau)^n}{(n!)^2}d\tau$$

$$=\frac{1}{\Gamma(\upsilon)}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n c^n}{(n!)^2}\int_0^x(x-\tau)^{\upsilon-1}\tau^n\,d\tau$$

$$=x^\upsilon\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(cx)^n}{n!\,\Gamma(\upsilon+n+1)}$$

By using (1.3), the above equation can be written as

$$=x^\upsilon\ {}_pK_q^{1,\upsilon+1;\gamma}(cx) \tag{2.2}$$

We define

$$\Omega(c,\upsilon,\gamma,p,q,x)=x^\upsilon\ {}_pK_q^{1,\upsilon+1;\gamma}(cx) \tag{2.3}$$

Now, the fractional differential operator of order μ is given by

$$D_x^\mu\{f(x)\}=D^k\{I_x^{k-\mu}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(cx)^n}{(n!)^2}\}$$

On simplifying, we arrive at

$$=D^k\{x^{k-\mu}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(cx)^n}{n!\,\Gamma(k-\mu+n+1)}$$

$$=x^{-\mu}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(cx)^n}{n!\,\Gamma(n+1-\mu)}$$

Again, by using(1.3), the above equation can be written as

$$=x^{-\mu}\ {}_pK_q^{1,1-\mu;\gamma}(cx) \tag{2.4}$$

We also define

$$\Omega(c,-\mu,\gamma,p,q,x)=x^{-\mu}\ {}_pK_q^{1,1-\mu;\gamma}(cx) \tag{2.5}$$

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3. Properties of the functions $\Omega(c,\upsilon,\gamma,p,q,x)$ and $\Omega(c,-\mu,p,\gamma,q,x)$:

Theorem 3.1 If c is an arbitrary constant then

$$I_x^{\lambda}\,\Omega(c,\upsilon,\gamma,p,q,x)=\Omega(c,\lambda+\upsilon,\gamma,p,q,x) \qquad (3.1)$$

Proof:
From the definition of the fractional integral (1.4), we have

$$I_x^{\lambda}\Omega(c,\upsilon,\gamma,p,q,x)=\frac{1}{\Gamma(\lambda)}\int_0^x (x-\tau)^{\lambda-1}\,\Omega(c,\upsilon,\gamma,p,q,\tau)\,d\tau \qquad (3.2)$$

Using (2.3), it reduces to

$$=\frac{1}{\Gamma(\lambda)}\int_0^x (x-\tau)^{\lambda-1}\,\tau^{\upsilon}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(c\tau)^n}{n!\Gamma(\upsilon+n+1)}\,d\tau$$

On substituting $\tau = zx$, it yields

$$=\frac{1}{\Gamma(\lambda)}x^{\lambda+\upsilon}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\gamma)_n(cx)^n}{n!\Gamma(\upsilon+n+1)}\int_0^1(1-z)^{\lambda-1}z^{k+\upsilon}dz \qquad (3.3)$$

On simplifying and using (2.3), we arrive at

$$I_x^{\lambda}\,\Omega(c,\upsilon,\gamma,p,q,x)=\Omega(c,\lambda+\upsilon,\gamma,p,q,x) \qquad (3.4)$$

Hence proved.

Theorem 3.2 If c is an arbitrary constant then

$$D_x^{\lambda}\,\Omega(c,\upsilon,\gamma,p,q,x)=\Omega(c,\upsilon-\lambda,\gamma,p,q,x)$$

Proof: By the definition of the fractional derivative (1.5), we get

$$D_x^{\lambda}\,\Omega(c,\upsilon,\gamma,p,q,x) = D^k\{I_x^{k-\lambda}\,K(c,\upsilon,p,q,x)\}$$
$$= D^k\{x^{k\upsilon-\lambda}\;{}_pK_q^{1,k+\upsilon-\lambda+1;\gamma}(cx)$$

Applying(2.3), we arrive at

$$D_x^{\lambda}\,\Omega(c,\upsilon,\gamma,p,q,x)=\Omega(c,\upsilon-\lambda,\gamma,p,q,x)$$

This proves theorem(3.2).

Theorem 3.3 If $\mu\in C,\mathrm{Re}(\mu)>0,c$ is an arbitrary constant then

$$I_x^{\lambda}\,\Omega(c,-\mu,\gamma,p,q,x)=\Omega(c,\lambda-\mu,\gamma,p,q,x)$$

Theorem 3.4 If $\mu\in C,\mathrm{Re}(\mu)>0,c$ is an arbitrary constant then

$$D_x^{\lambda}\,\Omega(c,-\mu,\gamma,p,q,x)=\Omega(c,-\lambda-\mu,\gamma,p,q,x)$$

Remarks: If we take p=q=o in above theorems we will get[2].

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References

[1] A. A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holand Mathematics Studies, 204, Elsevier. New York- London, 2006.

[2]A.K. Shukla and J.C. Prajapati, Some Remarks on Generalized Mittag-Leffler Function, Proyecciones.Vol.28 No.1(2009),27-34.

[3] A. Wiman, Uber die Nullsteliun der Fuctionen $E_{\alpha}(x)$, Acta Math. 29, 217-234.

[4] G. M. Mittag-Leffler, Sur la nuovelle function $E_{\alpha}(x)$. C.R.Acad.Sci. Paris(2) 137(1903), 554-558.

[5] G. M. Mittag-Leffler, Sur la representation analytique de'une branche uniforme une function monogene, Acta. Math. 29(1905), 101-181.

[6] I. Podlubny, Fractional Differential Equations,to Methods of their Solution and Some of their Applications, Mathematics in Science and Engineering, Vol. 198, Acadmic Press, Califonia,1999.

[7] Keith B. Oldham, Jerome Spanier, The Fractional Calculus; Theory and Applications of Differentiation and integration to Arbitrary Order, Academic Press, New York and London(1974). ISBN 0-12-525550-0.

[8] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York etc (1993).

[9]K. Sharma, Application of Fractional Calculus Operators to Related Areas, Gen. Math. Notes. (2011), 33-40.