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The analysis of stress around tunnel in shear stress domain

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ABSTRACT

One of the important models in tunnel design and underground structures are determining the stresses and stress concentrations around them. The method of analysis is usually based the theory of elasticity. Therefore; it has the advantages of accuracy and uniqueness respect to the numerical models. The stress conditions in subsurface or underground domains are usually in the form of both normal and shear stresses. Those are because of the geological features such as bedding, jointing, folding and nonuniformity of petrology. Therefore; the directions of principal stresses are not parallel to the original Cartesian coordinates and they make specific angles to the x and y-axis. The analysis is two-dimensional for circular tunnel and it is applied for plane stress and plane strain conditions. The analysis can be applied for the case of supporting pressure p_i. The radial and tangential deformations could also be determined at the roof and walls of tunnel.

Introduction

One of the most important methods for designing the tunnels and underground structures is the analytical method. In this method the stress and stress concentration around the tunnel and underground structures are obtained. In most cases the analysis is based on the two-dimensional analysis for plane stress and plane strain conditions. The rock mass for these cases are considered homogenous and isotropic media. The stress condition around the underground structure is independent of of the elastic parameters of rock mass and is the same for both plane stress and plane strain situations.

Stress Analysis Method

The detailed explanation the theory of stress and strain around a tunnel is out of discussion of this paper and reader can follow the governing material in Landau and Lifshitz [1], Timoshenko and Goodier [2], Brady and Brown [3], Barber [4], Antman [5] and sad [6] . In the above books a detailed review about stress and strain analysis for elastic material and different structures is brought. The objective of this paper is to determine the stress concentration around circular tunnels in a domain under shear stress situations. That is the most situation that can be possible under the ground surface. The following conditions should be satisfied for each analysis of stress and deformation around underground structure. 1- problem boundary conditions. 2- differential system of equation of equilibrium. 3- constitutive equations, and 4- compatibility equations. For each problem in this situation the boundary conditions are the stress and deformation situations at the internal surface of tunnels also the far field stress conditions where the stress concentration is zero. The stress equilibrium equations in two-dimensional, for body forces equal zero, are explained by Airy [7] and Timoshenko and Goodier [2] as follows,

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=0 \quad , \quad \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0 \qquad (1)$$

For the plane strain condition, isotropic and elastic medium, the normal and shear strains are obtained from Eq. (2).

$$\varepsilon_{xx}=\frac{1-v^2}{E}\left[\sigma_{xx}-\frac{v}{1-v}\sigma_{yy}\right] \quad , \quad \varepsilon_{yy}=\frac{1-v^2}{E}\left[\sigma_{yy}-\frac{v}{1-v}\sigma_{xx}\right]$$

$$\gamma_{xy}=\frac{1}{G}\sigma_{xy}=\frac{2(1+v)}{E}\sigma_{xy} \qquad (2)$$

Where E is modulus of elasticity, v is Poisson's ratio and G is modulus of rigidity. The compatibility equations for two-dimensional analysis is,

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2}+\frac{\partial^2 \varepsilon_{xx}}{\partial y^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad (3)$$

If Eq. (2) is substituted in Eq. (3) then after some simplifications the following equation is obtained.

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2}+\frac{\partial^2 \sigma_{yy}}{\partial x^2}+\frac{\partial^2 \sigma_{xx}}{\partial y^2}+\frac{\partial^2 \sigma_{yy}}{\partial y^2}=0 \quad or \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy})=0 \qquad (4)$$

Eq. (4) shows two-dimensional stress distribution for isotropic and elastic material. The stress distribution is independent of elastic properties of medium. Eq. (4) is Laplacian of the summation of normal stresses for two-dimensional plane stress and plane strain conditions. The equilibrium equations (1) with respect to the boundary conditions should be solved. Airy [7] considered the function $\phi(x,y)$ for simplifying the stress analysis where it satisfies the equilibrium equations as follows

$$\sigma_{xx}=\frac{\partial^2 \phi}{\partial y^2} \quad , \quad \sigma_{yy}=\frac{\partial^2 \phi}{\partial x^2} \quad , \quad \sigma_{xx}=\frac{\partial^2 \phi}{\partial x \partial y} \qquad (5)$$

Applying the Airy partial differential equations (5) into Eq. (4) it results the biharmonic differential equation as,

$$\frac{\partial^4 \phi}{\partial x^4}+\frac{\partial^4 \phi}{\partial y^4}+2\frac{\partial^4 \phi}{\partial x^2 \partial y^2}=\frac{\partial}{\partial x^2}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)+\frac{\partial}{\partial y^2}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=$$

$$\frac{\partial}{\partial x^2}(\nabla^2 \phi)+\frac{\partial}{\partial y^2}(\nabla^2 \phi)=\nabla^2(\nabla^2 \phi)=\nabla^4 \phi=0 \qquad (6)$$

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The biharmonic Eq. (6) which is in cartesian coordinates can be transferred to cylindrical coordinates as Eq. (7).

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \quad , \quad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \quad , \quad \tau_{r\theta} = -\frac{1}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} \quad ,$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) = 0 \tag{7}$$

Solving Eq. (7) by the method of separation variables and using several integration Eq. (8) results,

$$\phi(r,\theta) = Ar^2 + Br^2\ln r + C\ln r + D + (Er^2 + Fr^{-2} + Gr^3 + Hr\ln r)Sin2\theta$$
$$+ (I + Jr^2 + Kr^{-2} + Lr^3 + Mr\ln r)Cos2\theta \tag{8}$$

The stress concentration around tunnel with supporting pressure p_i

The problem of a hole in an infinite plate is of special interest in the rock mechanics field because it corresponds to the problem of a long horizontal tunnel at depth in a uniform and homogenous rock formation. This methods is only applicable to the special geometric cross-sectional shapes for openings; however for nongeometric cross-sectional shapes the methods of photoelastic can be applied. Fig. (1) shows the far-field stress conditions around a circular tunnel with diameter $2a$ and support system pressure of p_i. The boundary conditions for this case is,

$$\sigma_r(r \to \infty) = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)Cos2\theta \quad , \quad \sigma_r(r = a) = p_i$$
$$\tau_{r\theta}(r \to \infty) = \frac{1}{2}(\sigma_x - \sigma_y)Sin2\theta \quad , \quad \tau_{r\theta}(r = a) = 0 \tag{9}$$

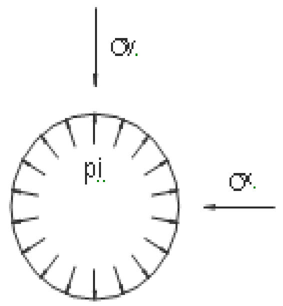


Figure 1 Far-field stress around circular tunnel.

Where σ_r and σ_θ are the radial and tangential stresses at far-field from the centre of tunnels which stress concentration is zero. The radial and tangential stresses on the internal surface of tunnel are p_i and zero; respectively. For the solution of this problem there is no need to apply all components of Eq. (8). They are chosen according to the following equation.

$$\phi(r,\theta) = A\ln r + Br^2 + (Cr^2 + Er^{-2} + F)Cos2\theta \tag{10}$$

The 1st and 2nd partial derivatives of ϕ respect to r and θ are obtained as Eq. (11).

$$\frac{\partial\phi}{\partial r} = \frac{A}{r} + 2Br + (2Cr - 2Er^{-3})Cos2\theta$$
$$\frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + 2B + (2C + 6Er^{-4})Cos2\theta$$
$$\frac{\partial\phi}{\partial\theta} = -2(Cr^2 + Er^{-2} + F)Sin2\theta$$
$$\frac{\partial^2\phi}{\partial\theta^2} = -4(Cr^2 + Er^{-2} + F)Cos2\theta$$
$$\frac{\partial^2\phi}{\partial r\partial\theta} = -2(2Cr - 2Er^{-3})Sin2\theta \tag{11}$$

By substituting Eq. (11) in Equations (7) for σ_r , σ_θ and $\tau_{r\theta}$ the following equations are obtained.

$$\sigma_r = \frac{A}{r^2} + 2B + (-2C - 6Er^{-4} - 4Fr^{-2})Cos2\theta$$
$$\sigma_\theta = -\frac{A}{r^2} + 2B + (2C + 6Er^{-4})Cos2\theta$$
$$\tau_{r\theta} = (-2C + 6Er^{-4} + 2Fr^{-2})Sin2\theta \tag{12}$$

Applying the boundary conditions Eq. (9) in equations (12) the parameters A, B, C, E and F are calculated.

$$\sigma_r(r \to \infty) = 2B - 2CCos2\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)Cos2\theta \Rightarrow$$
$$B = \frac{1}{4}(\sigma_x + \sigma_y) \quad , \quad C = -\frac{1}{4}(\sigma_x - \sigma_y) \tag{13}$$

Also in the same way,

$$\sigma_r(r = a) = \frac{A}{a^2} + 2B + (-2C - 6Ea^{-4} - 4Fa^{-2})Cos2\theta = p_i \Rightarrow$$
$$\frac{A}{a^2} + 2B = p_i \quad , \quad 2C + 6Ea^{-4} + 4Fa^{-2} = 0 \quad , \quad 2C - 6Ea^{-4} - 2Fa^{-2} = 0 \tag{14}$$

Solving the equations (14) results

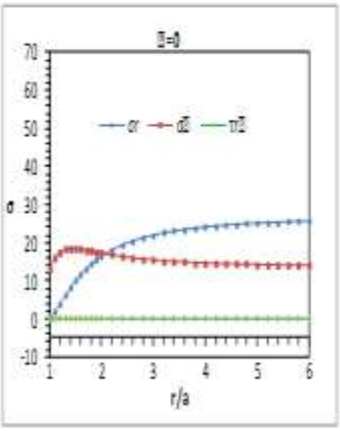
$$A = a^2 p_i - \frac{a^2}{2}(\sigma_x + \sigma_y) \quad , \quad F = \frac{a^2}{2}(\sigma_x - \sigma_y) \quad , \quad E = -\frac{a^4}{2}(\sigma_x - \sigma_y) \tag{15}$$

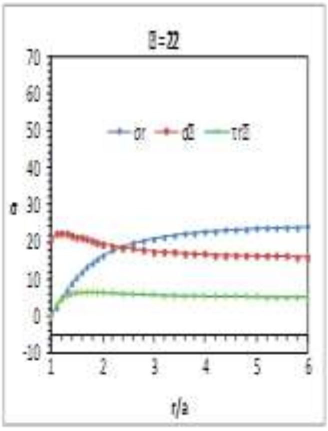
With the substitution of parameters A, B, C, E and F in Eq. (12) the stress components σ_r ,
σ_θ and $\tau_{r\theta}$ around the tunnel are obtained as,

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2}(1 - \frac{a^2}{r^2}) + \frac{\sigma_x - \sigma_y}{2}(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2})Cos2\theta + \frac{a^2 p_i}{r^2}$$
$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2}(1 + \frac{a^2}{r^2}) - \frac{\sigma_x - \sigma_y}{2}(1 + \frac{3a^4}{r^4})Cos2\theta - \frac{a^2 p_i}{r^2}$$
$$\tau_{r\theta} = \frac{\sigma_x - \sigma_y}{2}(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2})Sin2\theta \tag{16}$$

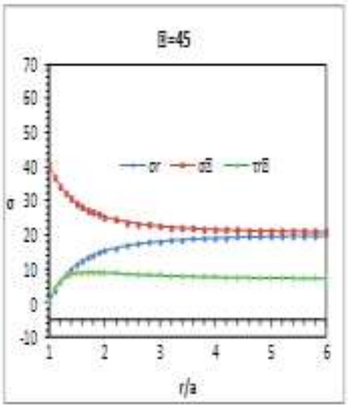
Example 1

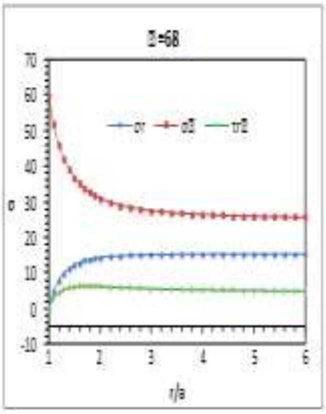
Suppose a tunnel with the diameter 4 m, depth of 500 m in a limestone formation with vertical and horizontal stresses 13.5 and 27 Mpa, is drilled. Figs. (2) show the shear and normal stresses conditions around and up to a distance five times the radius of the tunnel and using Eq. (16). Also in Figs. (3) the stress conditions for this tunnel for r/a constant from 1.0 to 6.0 and Θ variable are shown, where r is the distance of the point which is supposed to calculate the stress conditions and a is the radius of tunnel. Points 1 to 10 on the x-axis of Figs (3) coincide with the points in Fig. (3-a) where point 1 is along the tunnel roof with Θ=90 Degree and point 10 is along the tunnel wall with Θ=0 Degree.



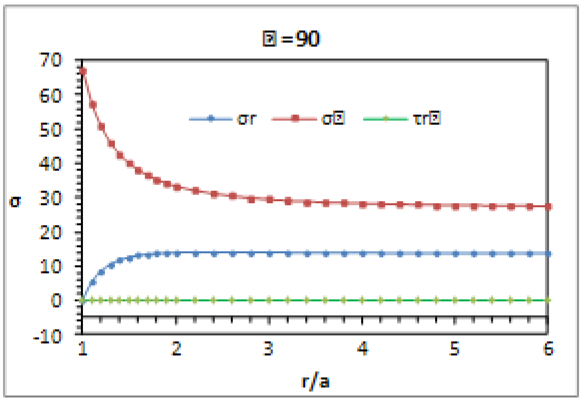


(a) (b)



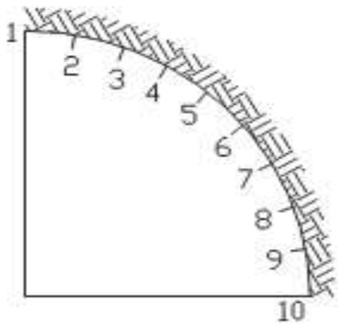


(c) (d)

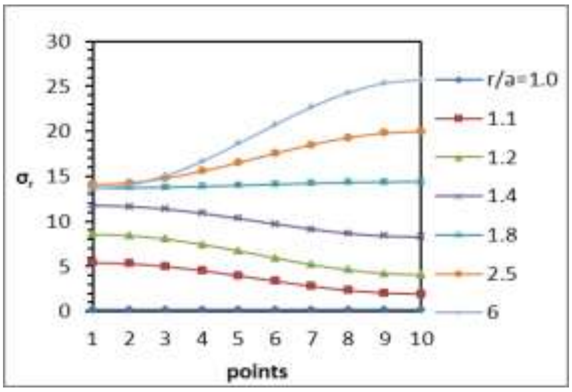


(e)

Figure (2) Stress distribution around tunnel for $\theta = 0, 22, 45, 68$ *and* 90 *D*



(a)



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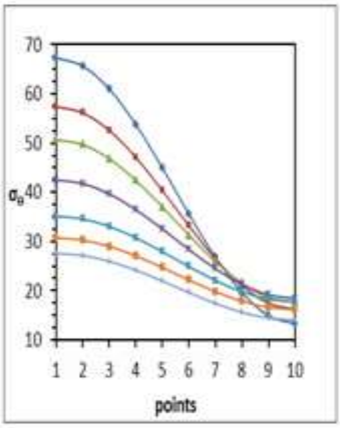
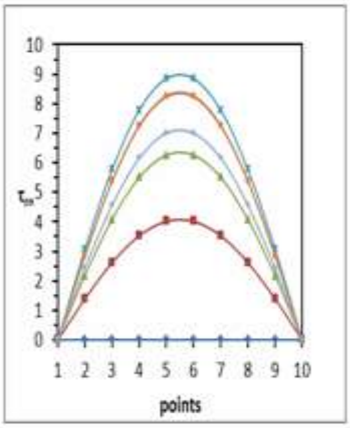




(c) (d)

Fig. (3) Stress distribution around the tunnel respect to ϴ. (a) the points situations on the tunnel surface, (b) radial stress, (c) tangential stress and (d) shear stress

Stress conditions around tunnel for far-field shear stress

In the most cases the stress conditions under the ground surface is in form of nonprincipal stresses because of the heterogeneity, anisotropy, bedding and folding of rock mass formations. Fig. (4) shows the far-field stress conditions around circular tunnel where the supporting pressure which acts on the surface of tunnel is p_i .

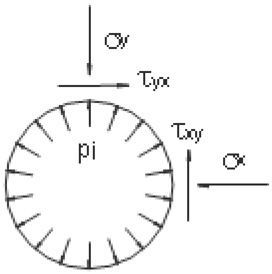


Figure (4) The far-field stress condition around tunnel.

The boundary conditions for this problem are,

$$\sigma_r(r \to \infty) = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) Cos\,2\theta - \tau_{xy} Sin\,2\theta \quad , \quad \sigma_r(r = a) = p_i$$

$$\tau_{r\theta}(r \to \infty) = \frac{1}{2}(\sigma_x - \sigma_y) Sin\;2\theta + \tau_{xy} Cos\,2\theta \quad , \quad \tau_{r\theta}(r = a) = 0$$

(17)

For the above boundary conditions the Airy function is considered as follows,

$$\phi(r,\theta) = A \ln r + Br^2 + (C_1 r^2 + D_1 r^{-2} + E_1)\; Cos\;2\theta +$$
$$(C_2 r^2 + D_2 r^{-2} + E_2) Sin\,2\theta$$

(18)

Obtaining the 1st and 2nd partial derivatives of Eq. (18) respect to r and θ results,

$$\frac{\partial \phi}{\partial r} = \frac{A}{r} + 2Br + (2C_1 r - 2D_1 r^{-3})\, Cos\, 2\theta + (2C_2 r - 2D_2 r^{-3})\, Sin\, 2\theta$$

$$\frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + 2B + (2C_1 + 6D_1 r^{-4})\, Cos\, 2\theta + (2C_2 + 6D_2 r^{-4})\, Sin\, 2\theta$$

$$\frac{\partial \phi}{\partial \theta} = -2(C_1 r^2 + D_1 r^{-2} + E_1)\, Sin\, 2\theta + 2(C_2 r^2 + D_2 r^{-2} + E_2)\, Cos\, 2\theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -4(C_1 r^2 + D_1 r^{-2} + E_1)\, Cos\, 2\theta - 4(C_2 r^2 + D_2 r^{-2} + E_2)\, Sin\, 2\theta$$

$$\frac{\partial^2 \phi}{\partial r \partial \theta} = -2(2C_1 r - 2D_1 r^{-3})\, Sin\, 2\theta + 2(2C_2 r - 2D_2 r^{-3})\, Cos\, 2\theta \qquad (19)$$

By substitution the Eqs. (19) in Eq. (7) it can be written,

$$\sigma_r = \frac{A}{r^2} + 2B + (-2C_1 - 6D_1 r^{-4} - 4E_1 r^{-2})\, Cos\, 2\theta + (-2C_2 - 6D_2 r^{-4} - 4E_2 r^{-2})\, Sin\, 2\theta$$

$$\sigma_\theta = -\frac{A}{r^2} + 2B + (2C_1 + 6D_1 r^{-4})\, Cos\, 2\theta + (2C_2 + 6D_2 r^{-4})\, Sin\, 2\theta$$

$$\tau_{r\theta} = (-2C_1 + 6D_1 r^{-4} + 2E_1 r^{-2})\, Sin\, 2\theta + (2C_2 - 6D_2 r^{-4} - 2E_2 r^{-2})\, Cos\, 2\theta \qquad (20)$$

Applying the first boundary condition Eq. (17) in Eqs. (20) the parameters B, C_1 and C_2 result.

$$\sigma_r(r \to \infty) = 2B - 2C_1 Cos\, 2\theta - 2C_2 Sin\, 2\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) Cos\, 2\theta - \tau_{xy} Sin\, 2\theta \Rightarrow$$

$$B = \frac{1}{4}(\sigma_x + \sigma_y) \;,\; C_1 = -\frac{1}{4}(\sigma_x - \sigma_y) \;,\; C_2 = \frac{1}{2}\tau_{xy} \qquad (21)$$

Also substituting the 2nd and 3rd boundary conditions of Eq. (17) in Eq. (20) results the following two systems of equations.

$$\tau_{r\theta}(r = a) = 0 \Rightarrow \begin{cases} -2C_1 + 6D_1 a^{-4} + 2E_1 a^{-2} = 0 \\ -2C_2 + 6D_2 a^{-4} + 2E_2 a^{-2} = 0 \end{cases}$$

$$\sigma_r(r = a) = p_i \Rightarrow \begin{cases} \frac{A}{a^2} + 2B = p_i \\ -2C_1 - 6D_1 a^{-4} - 4E_1 a^{-2} = 0 \\ -2C_2 - 6D_2 a^{-4} - 4E_2 a^{-2} = 0 \end{cases} \qquad (22)$$

By solving the system of Eqs. (22) and the substitution for the parameters obtained in Eq. (21) the coefficients D_1, D_2, E_1 and E_2 are determined as the following equations.

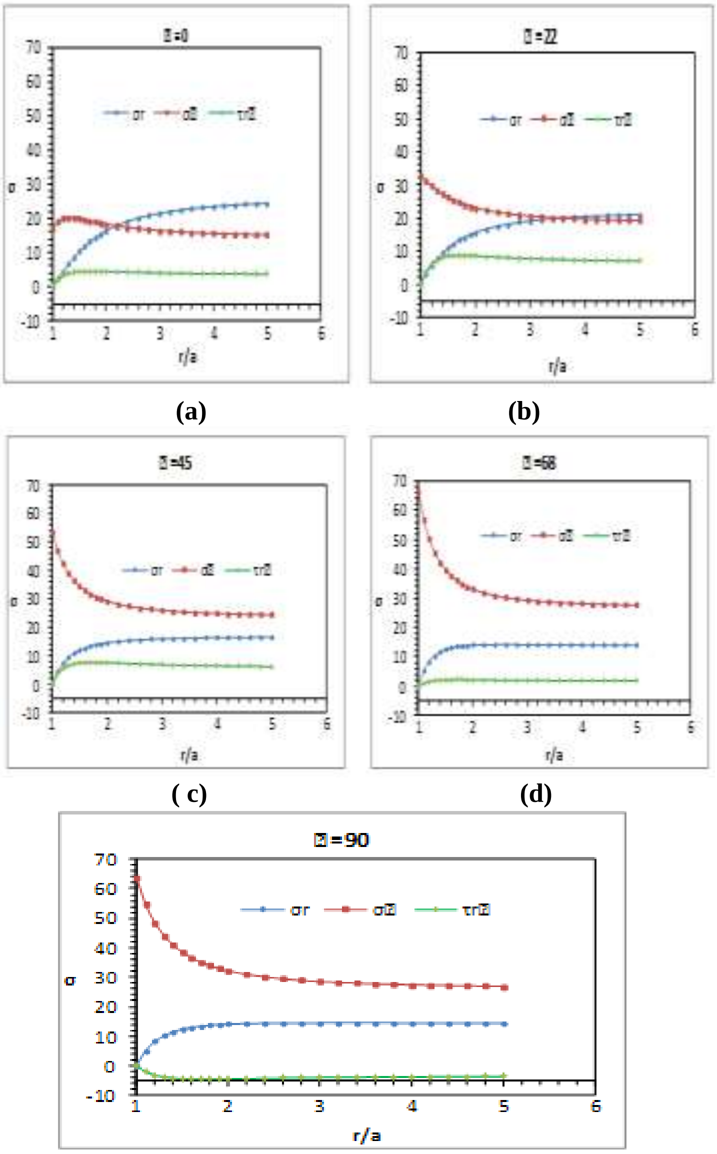
$$D_1 = -\left(\frac{\sigma_x - \sigma_y}{4}\right) a^4 \;,\; E_1 = \left(\frac{\sigma_x - \sigma_y}{2}\right) a^2 \;,\; D_2 = \frac{1}{2} a^4 \tau_{xy} \;,\; E_2 = -a^2 \tau_{xy} \qquad (23)$$

Substituting the all above coefficients in Eqs. (20) the components of stresses conditions around the tunnel are obtained.

$$\begin{cases} \sigma_r = \frac{\sigma_x + \sigma_y}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_x - \sigma_y}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) Cos\, 2\theta - \tau_{xy}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) Sin\, 2\theta + \frac{a^2}{r^2} p_i \\ \sigma_\theta = \frac{\sigma_x + \sigma_y}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_x - \sigma_y}{2}\left(1 + \frac{3a^4}{r^4}\right) Cos\, 2\theta + \tau_{xy}\left(1 + \frac{3a^4}{r^4}\right) Sin\, 2\theta - \frac{a^2}{r^2} p_i \\ \tau_{r\theta} = \left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\left[\frac{\sigma_x - \sigma_y}{2} Sin\, 2\theta + \tau_{xy} Cos\, 2\theta\right] \end{cases} \qquad (24)$$

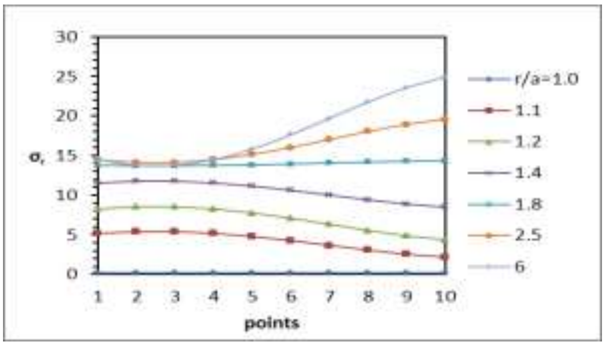
Example 2

If the tunnel in problem 1 be under the stresses conditions of $\sigma_x = 26.1$, $\sigma_y = 14.4$ and $\tau_{xy} = 3.4\ Mpa$ at the depth of 500 m. The normal and shear stresses conditions around the tunnel up to a distance five times the radius of tunnel ($5a$) for ϴ constant and r variable and vice versa are shown in Figs. (5) and (6); respectively.

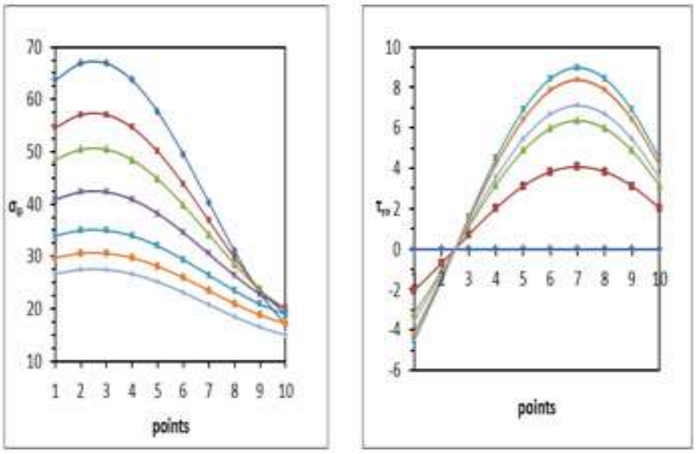


(a) **(b)** **(c)** **(d)** **(e)**

Figures 5. The stress distribution around the tunnel for general state of stress conditions for $\theta = 0, 22, 45, 68\ and\ 90\ D$



(a)



(b) **(c)**

Fig. 6. Stress distribution around the tunnel respect to Ө. (a) radial stress, (b) tangential stress and (c) shear stress

Conclusions

In this research the state of stress around the circular tunnel are obtained for the principal and nonprincipal stress domain by analytical methods. The method is accurate and precise for two-dimensional analysis because it satisfies the equilibrium, the compatibility and the boundary conditions equations. It can be used for checking and testing the available numerical methods which exist in this subject. For noncircular tunnel cross-sectional forms such as the elliptical and rectangular shapes the model can be generated by the conformal mapping transformation. The results show the effective radius of stress concentration which is created and developed by the influence of tunnel is $5a$.

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