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Homomorphism in (Q, L)-fuzzy subgroups of a group

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ABSTRACT

In this paper, we study some of the properties of homomorphism in (Q, L)-fuzzy subgroup of a group and prove some results on these.

Introduction

After the introduction of fuzzy sets by L.A.Zadeh[20], several researchers explored on the generalization of the notion of fuzzy set. Azriel Rosenfeld[4] defined a fuzzy groups. Asok Kumer Ray[3] defined a product of fuzzy subgroups and A.Solairaju and R.Nagarajan[17, 18, 19] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. We introduce the concept of homomorphism in (Q, L)-fuzzy subgroup of a group and established some results.

1.Preliminaries:

1.1 Definition: Let X be a non-empty set and L = (L, ≤) be a lattice with least element 0 and greatest element 1 and Q be a non-empty set. A (Q, L)-fuzzy subset A of X is a function A : XxQ → L.

1.2 Definition: Let (G, +) be a group and Q be a non empty set. A (Q, L)-fuzzy subset A of G is said to be a (Q, L)-**fuzzy subgroup(QLFSG)** of G if the following conditions are satisfied:

(i) A(x+y, q) ≥ A(x, q) ∧ A(y, q),

(ii) A(–x, q) ≥ A(x, q), for all x and y in G and q in Q.

1.3 Definition: Let (G, +) and (G^{l}, +) be any two groups and Q be a non empty set. Let f : G → G^{l} be any function and A be a (Q, L)-fuzzy subgroup in G, V be a (Q, L)-fuzzy subgroup in f (G) =G^{l}, defined by V(y, q)= $\sup_{x\in f^{-1}(y)}$ A(x, q), for all x in G and y in G^{l} and q in Q. Then A is called a pre-image of V under f and is denoted by f^{-1}(V).

1.4 Definition: Let A be a (Q, L)-fuzzy subgroup of a group G. Then A^{0} is defined as A^{0}(x, q) = A(x, q) / A(0, q), for all x in G and q in Q, where 0 is the identity of G.

1.5 Definition: Let (G, +) be a group and Q be a non empty set. A (Q, L)-fuzzy subset A of G is said to be a (Q, L)-**anti-fuzzy subgroup(QLAFSG)** of G if the following conditions are satisfied:

(i) A(x+y, q) ≤ A(x, q) ∨ A(y, q),

(ii) A(–x, q) ≤ A(x, q), for all x and y in G and q in Q.

1.6 Definition: Let A be a (Q, L)-fuzzy subgroup of a group (G, +) and a in G. Then the **pseudo (Q, L)-fuzzy coset** $(aA)^{p}$ is defined by ($(aA)^{p}$)(x, q) = p(a)A(x, q), for every x in G and for some p in P and q in Q.

1.7 Definition: Let A be a (Q, L)-fuzzy subgroup of a group (G, +). For any a in G, a+A defined by (a+A)(x, q) = A(x–a, q), for every x in G and q in Q, is called a **(Q, L)-fuzzy coset** of G.

2. Some properties of (Q, L)-fuzzy subgroup of a group under homomorphism and antihomomorphism:

2.1 Theorem: Let (G, +) and (G^{l}, +) be any two groups and Q be a non-empty set. The homomorphic image of a (Q, L)-fuzzy subgroup of G is a (Q, L)-fuzzy subgroup of G^{l}.

Proof: Let (G, +) and (G^{l}, +) be any two groups and Q be a non-empty set and f : G→G^{l} be a homomorphism. That is f(x+y) = f(x)+f(y), for all x and y in G. Let A be a (Q, L)-fuzzy subgroup of G. Let V be the homomorphic image of A under f. We have to prove that V is a (Q, L)-fuzzy subgroup of f(G) = G^{l}. Now, for f(x) and f(y) in G^{l}, we have V(f(x)+f(y), q) = V(f(x+y), q) ≥ A(x+y, q) ≥ A(x, q) ∧ A(y, q), which implies that V(f(x)+f(y), q) ≥ V(f(x), q)∧V(f(y), q). For f(x) in G^{l}, we have V(–f(x), q) = V(f(–x), q) ≥ A(–x, q) ≥ A(x, q), which implies that V(–f(x), q) ≥ V(f(x), q). Hence V is a (Q, L)-fuzzy subgroup of a group G^{l}.

2.2 Theorem: Let (G, +) and (G^{l}, +) be any two groups and Q be a non-empty set. The homomorphic pre-image of a (Q, L)-fuzzy subgroup of G^{l} is a (Q, L)-fuzzy subgroup of G.

Proof: Let (G, +) and (G^{l}, +) be any two groups and Q be a non-empty set and f : G→G^{l} be a homomorphism. That is f(x+y) = f(x)+f(y), for all x and y in G. Let V be a (Q, L)-fuzzy subgroup of f(G) = G^{l}. Let A be the pre-image of V under f. We have to prove that A is a (Q, L)-fuzzy subgroup of G. Let x and y in G and q in Q. Then, A(x+y, q) = V(f(x+y), q) = V(f(x)+f(y), q) ≥ V(f(x), q)∧V(f(y), q) = A(x, q) ∧ A(y, q), which implies that A(x+y, q) ≥ A(x, q) ∧ A(y, q), for x and y in G and q in Q. And A(–x, q) = V(f(–x), q) = V(–f(x), q) ≥ V(f(x), q)

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= A(x, q), which implies that $A(-x, q) \geq A(x, q)$, for x in G and q in Q. Hence A is a (Q, L)-fuzzy subgroup of the group G.

2.3 Theorem: Let (G, +) and (G^I, +) be any two groups and Q be a non-empty set. The anti-homomorphism image of a (Q, L)-fuzzy subgroup of G is a (Q, L)-fuzzy subgroup of G^I.

Proof: Let (G, +) and (G^I, +) be any two groups and Q be a non-empty set and $f : G \rightarrow G^I$ be a anti-homomorphism. That is f(x+y) = f(y)+f(x), for all x and y in G and q in Q. Let A be a (Q, L)-fuzzy subgroup of G. Let V be the homomorphism image of A under f. We have to prove that V is a (Q, L)-fuzzy subgroup of $f(G) = G^I$. Now, let f(x) and f(y) in G^I, we have V(f(x)+f(y), q) =V(f(y+x), q) $\geq$ A(y + x, q) $\geq$ A(x, q) $\wedge$ A(y, q), which implies that V(f(x) + f(y), q) $\geq$ V(f(x), q) $\wedge$ V(f(y), q). For x in G and q in Q, V(–f(x), q) =V(f(–x), q) $\geq$ A(–x, q) $\geq$ A(x, q), which implies that V(–f(x), q) $\geq$ V(f(x), q), for x in G and q in Q. Hence V is a (Q, L)-fuzzy subgroup of G^I.

2.4 Theorem: Let (G, +) and (G^I, +) be any two groups and Q be a non-empty set. The anti-homomorphism pre-image of a (Q, L)-fuzzy subgroup of G^I is a (Q, L)-fuzzy subgroup of G.

Proof: Let (G, +) and (G^I, +) be any two groups and Q be a non-empty set and $f : G \rightarrow G^I$ be a anti-homomorphism. That is f(x + y) = f(y) + f(x), for all x and y in G and q in Q. Let V be a (Q, L)-fuzzy subgroup of $f(G) = G^I$. Let A be the pre-image of V under f. We have to prove that A is a (Q, L)-fuzzy subgroup of G. Let x and y in G and q in Q. Now, A(x + y, q) = V(f(x + y), q) = V(f(y)+f(x), q) $\geq$ V(f(x), q)$\wedge$ V(f(y), q) = A(x, q) $\wedge$ A(y, q), which implies that A(x + y, q) $\geq$ A(x, q)$\wedge$A(y, q), for all x and y in G and q in Q. And, A(–x, q) = V(f(–x), q) = V(– f(x), q) $\geq$ V(f(x), q) = A(x, q), which implies that A(–x, q) $\geq$ A(x, q) , for x in G and q in Q. Hence A is a (Q, L)-fuzzy subgroup of the group G.

In the following Theorem ◦ is the composition operation of functions:

2.5 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group H and f is an isomorphism from a group G onto H. Then A◦f is a (Q, L)-fuzzy subgroup of G.

Proof: Let x and y in G and A be a (Q,L)-fuzzy subgroup of the group H and Q be a non-empty set. Then we have, (A◦f)(x–y, q) = A(f(x–y), q) = A (f(x) – f(y), q) $\geq$ A(f(x), q)$\wedge$ A(f(y), q)$\geq$ (A◦f)(x, q)$\wedge$(A◦f)(y, q), which implies that (A◦f)(x– y, q) $\geq$ (A◦f)(x, q)$\wedge$(A◦f)(y, q). Therefore (A◦f) is a (Q, L)-fuzzy subgroup of a group G.

2.6 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group H and f is an anti-isomorphism from a group G onto H. Then A◦f is a (Q, L)-fuzzy subgroup of G.

Proof: Let x and y in G and A be a (Q, L)-fuzzy subgroup of the group H and Q be a non-empty set. Then we have, (A◦f)(x–y, q) = A(f(x–y), q) = A (f(y) – f(x), q) $\geq$ A(f(x), q)$\wedge$A(f(y), q) $\geq$ (A◦f)(x, q)$\wedge$(A◦f)(y, q), which implies that (A◦f)(x– y, q) $\geq$ (A◦f)(x, q)$\wedge$(A◦f)(y, q). Therefore (A◦f) is a (Q, L)-fuzzy subgroup of a group G.

2.7 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group G and Q be a non-empty set, A^+ be a (Q, L)- fuzzy set in G defined by $A^+(x, q)$ = A(x, q) +1 – A(0, q), for all x in G and q in Q, where 0 is the identity element. Then A^+ is a (Q, L)-fuzzy subgroup of the group G.

Proof: Let x and y in G and q in Q. We have, $A^+(x-y, q)$ = A(x–y, q) +1– A(0, q) $\geq$ {A(x, q) $\wedge$ A(y, q)}+1– A(0, q) = {A(x, q) +1– A(0, q)} $\wedge${ A(y, q) +1– A(0, q)}= $A^+(x, q)\wedge A^+(y, q)$. Therefore, $A^+(x-y, q)\geq A^+(x, q)\wedge A^+(y, q)$, for all x and y in G and q in Q. Hence A^+ is a (Q, L)-fuzzy subgroup of the group G.

2.8 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group G and Q be a non-empty set, A^+ be a (Q, L)-fuzzy set in G defined by $A^+(x, q)$ = A(x, q) +1 – A(0, q), for all x in G and q in Q, where 0 is the identity element. Then there exists 0 in G such that A(0, q) = 1 if and only if $A^+(x, q)$ = A(x, q).

Proof: It is trivial.

2.9 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group G, A^+ be a (Q, L)-fuzzy set in G defined by $A^+(x, q)$ = A(x, q) +1 – A(0, q), for all x in G and q in Q, where 0 is the identity element. Then there exists x in G such that $A^+(x, q)$ = 1 if and only if x = 0.

Proof: It is trivial.

2.10 Theorem: Let A be a (Q,L)-fuzzy subgroup of a group G, A^+ be a (Q, L)-fuzzy set in G defined by $A^+(x, q)$ = A(x, q) +1 – A(0, q), for all x in G and q in Q, where 0 is the identity element. Then $(A^+)^+ = A^+$.

Proof: Let x and y in G and q in Q. We have, $(A^+)^+(x, q) = A^+(x, q) +1– A^+(0, q)$ = {A(x, q) +1–A(0, q) }+1–{A(0, q) +1–A(0, q) }= A(x, q) +1–A(0, q)= $A^+(x, q)$. Hence $(A^+)^+ = A^+$.

2.11 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group G. Then A^0 is a (Q, L)-fuzzy subgroup of the group G.

Proof: For any x and y in G and q in Q, we have $A^0(x-y, q)$ = A(x–y, q) / A(0, q) $\geq$ [1 / A(0, q)]{A(x, q)$\wedge$ A(y, q)}= [A(x, q) / A(0, q)] $\wedge$ [A(y, q) / A(0, q)] = $A^0(x, q) \wedge A^0(y, q)$. That is $A^0(x-y, q) \geq A^0(x, q) \wedge A^0(y, q)$, for all x and y in G and q in Q. Hence A^0 is a (Q, L)-fuzzy subgroup of the group G.

2.12 Theorem: Let (G, +) be a group and Q be a non-empty set. A is a (Q, L)-fuzzy subgroup of G if and only if A^c is a (Q, L)-anti-fuzzy subgroup of G.

Proof: Suppose A is a (Q, L)-fuzzy subgroup of G. For all x and y in G and q in Q, we have A(x – y, q) $\geq$ A(x, q) $\wedge$ A(y, q), which implies that $1- A^c(x - y, q) \geq \{1- A^c(x, q)\} \wedge \{1- A^c(y, q)\}$, which implies that $A^c(x-y, q) \leq 1-\{1-A^c(x, q)\wedge 1-A^c(y, q)\}$, which implies that $A^c(x-y, q) \leq A^c(x) \vee A^c(y)$. Thus A^c is a (Q, L)-anti-fuzzy subgroup of G. Converse also can be proved similarly.

2.13 Theorem: Let A be a (Q, L)-fuzzy subgroup of a group G, then the pseudo (Q, L)-fuzzy coset $(aA)^p$ is a (Q, L)-fuzzy subgroup of the group G, for every a in G.

Proof: Let A be a (Q, L)-fuzzy subgroup of the group G. For every x and y in G and q in Q, we have, $((aA)^p)(x-y , q)$ = p(a)A(x–y, q) $\geq$ p(a){A(x, q) $\wedge$A(y, q)}= p(a)A(x, q) $\wedge$ p(a)A(y, q) =$((aA)^p)(x, q)\wedge((aA)^p)(y, q)$. Therefore, $((aA)^p)(x-y, q)\geq((aA)^p)(x, q)\wedge ((aA)^p)(y, q)$, for x, y in G and q in Q. Hence $(aA)^p$ is a (Q, L)-fuzzy subgroup of the group G.

2.14 Theorem: Let (G, +) be a group and Q be a non-empty set. If A is a (Q, L)-fuzzy subgroup of G, then x + A = y + A if and only if A(x –y, q) = A(0, q), where 0 is the identity element. In that case A(x, q) = A(y, q).

Proof: Given A is a (Q, L)-fuzzy subgroup of G. Suppose that x + A = y + A, which implies that (x + A)(x, q) = (y + A)(x, q), which implies that A(x–x, q) = A(x – y, q),
which implies that A(0, q) =A(x – y, q). Conversely, assume that A(x – y, q) = A(0, q), then (x + A)(z, q) = A(z – x, q) = A(z – x + y – y, q) $\geq$ A(z–y, q)$\wedge$A(0, q) = A(z–y, q) = (y + A)(z, q), which implies that (x + A)(z, q) $\geq$ (y + A)(z, q) ------------ (1). Now, (y + A)(z, q) = A(z – y, q) = A(z – y + x – x, q) $\geq$ A(z – x, q)$\wedge$ A(0, q) = A(z – x, q) = (x + A)(z, q), which implies

that (y + A)(z, q) ≥ (x + A)(z, q)--------------------(2). From (1) and (2) we get, x + A = y + A.

2.15 Theorem: Let (G, +) be a group and Q be a non-empty set. Let A is a (Q, L)-fuzzy subgroup of G and x, y, u, and v be any elements in G, if x + A = u + A and y + A = v + A, then (x + y) + A = (u + v) + A.

Proof: Given A is a (Q, L)-fuzzy subgroup of G. By Theorem 2.14, A(x–u, q)=A(y–v, q) = A(0, q). We get, A(x + y – u – v, q) = A(x – u + y – v, q) ≥ A(x – u, q) ∧A(y – v, q) = A(0, q). Again, by Theorem 2.14, (x + y) + A = (u + v) + A.

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